

Site-specific Gordian distances of spatial graphs ①

Kouki Taniguchi

G : finite graph , $\mathcal{E} := E(G) \times E(G) / \sim$

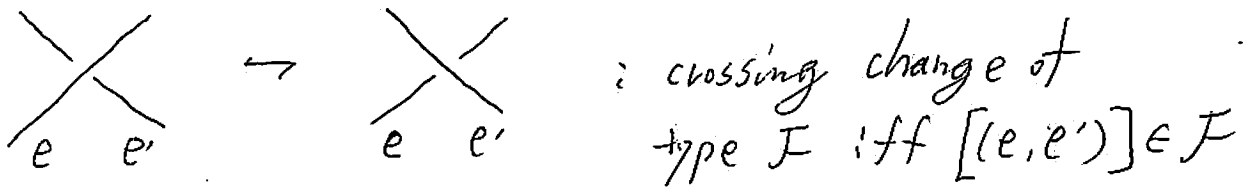
$\mathcal{F} \subseteq \mathcal{E}$: a subset

$$(e_1, e_2) \sim (e_2, e_1)$$

$f, g \in SE(G)$ $SE(G) := \{f: G \rightarrow \mathbb{S}^3 : \text{embedding}\}$

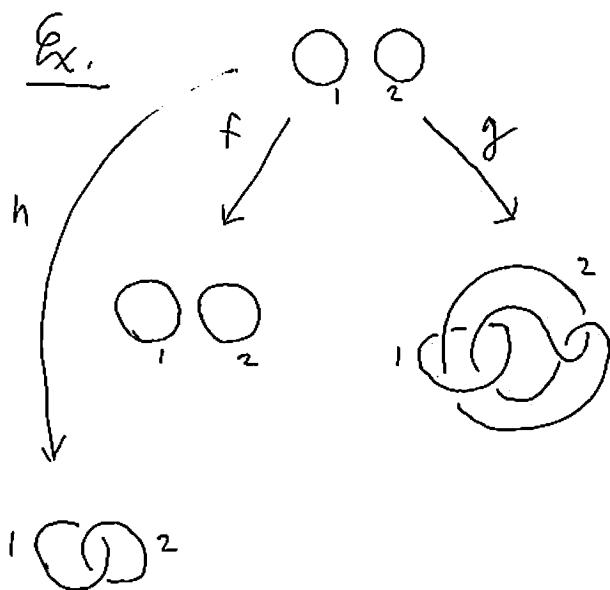
$d_{\mathcal{F}}(f, g)$: $\mathcal{F}$ -Gordian distance

= the minimal number of crossing changes of type $\mathcal{F}$ from f to g

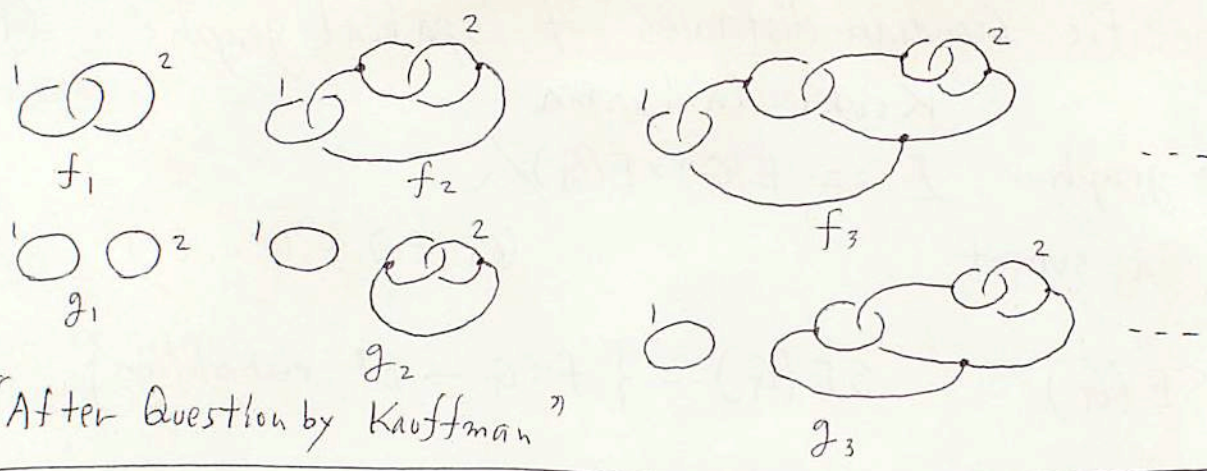
 : crossing change of type $\mathcal{F}$ iff $[(e, e')] \in \mathcal{F}$

$$E(G) = \{e_1, e_2, \dots, e_n\} , \mathcal{F}_{ij} := \{[(e_i, e_j)]\}$$

$$d_{\mathcal{F}_{ij}} = d_{ij}$$



$$\left\{ \begin{array}{l} d_{11}(f, g) = d_{22}(f, g) = 1 \\ d_{12}(f, g) = 2 \\ \\ d_{11}(f, h) = d_{22}(f, h) = \infty \\ d_{11}(g, h) = d_{22}(g, h) = \infty \\ d_{12}(f, h) = d_{12}(g, h) = 1 \end{array} \right.$$

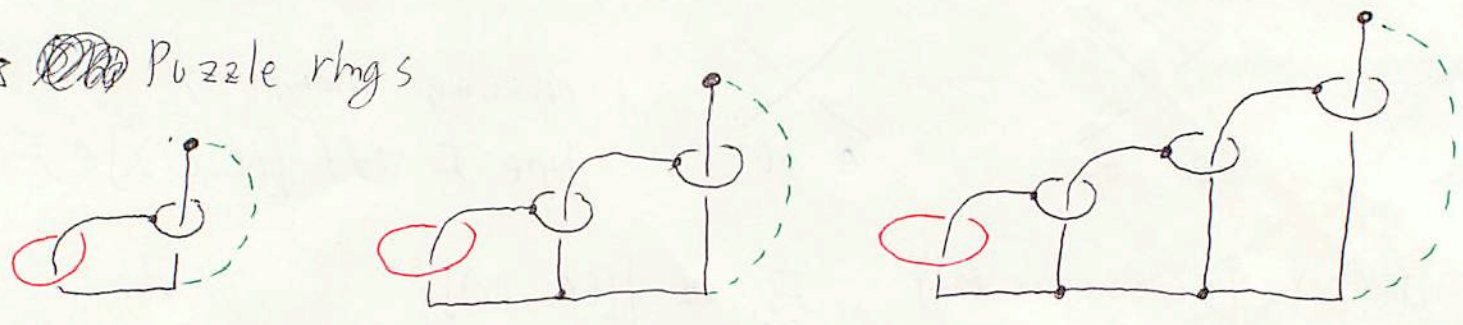


"After Question by Kauffman"

Thm 1 | Przytycki - Sikora "Topological insights from Chinese rings" Proc. AMS 2001

$$d_{12}(f_n, g_n) = 2^{n-1}$$

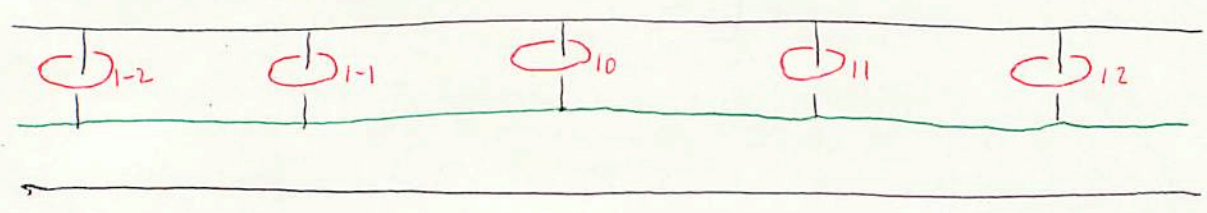
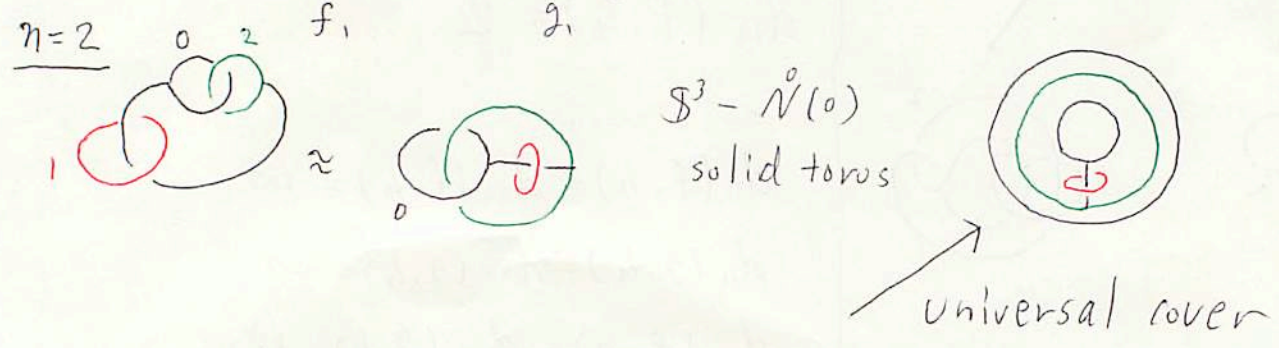
★ ~~Puzzle~~ Puzzle rings



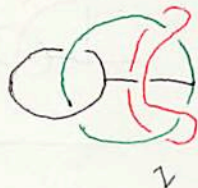
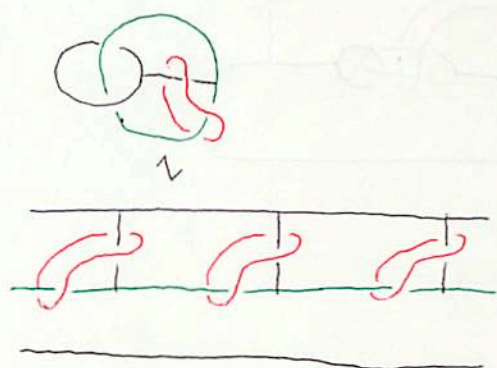
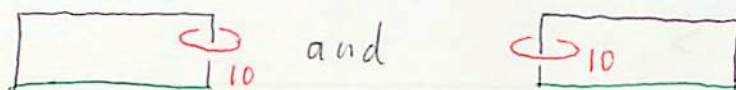
New proof of Thm 1.

Induction on n .

$n=1$ $d_{12}(f_1, g_1) = 1 = 2^0 = 2^{n-1}$ o.k.

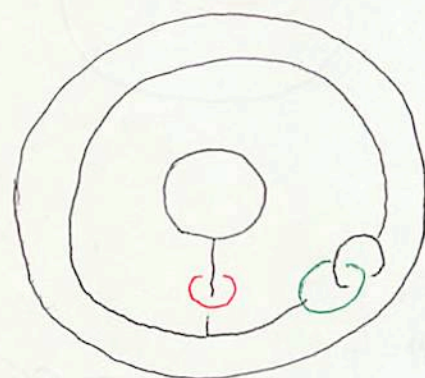
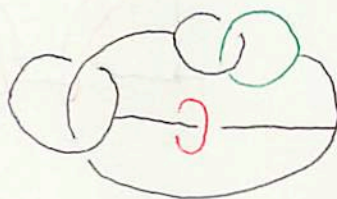


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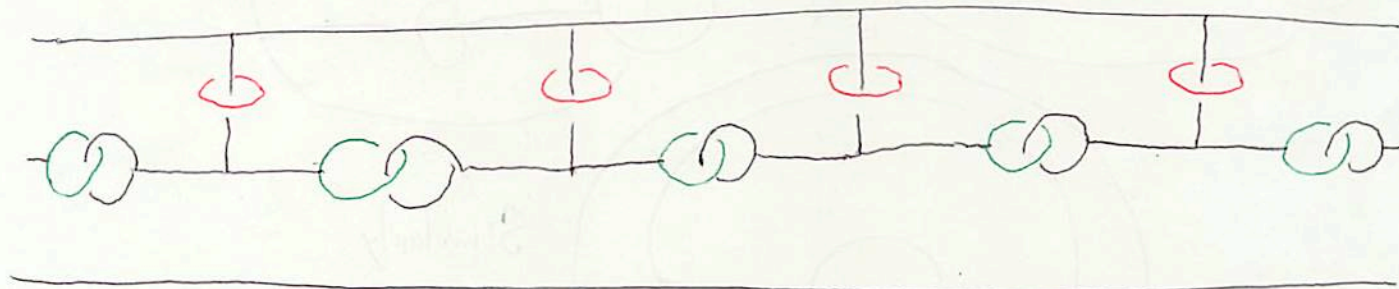


$$\begin{aligned} \therefore d_{12}(f_2, g_1) &\geq 2 d_{12}(f_1, g_1) = 2 \\ \therefore d_{12}(f_2, g_2) &= 2 \end{aligned}$$

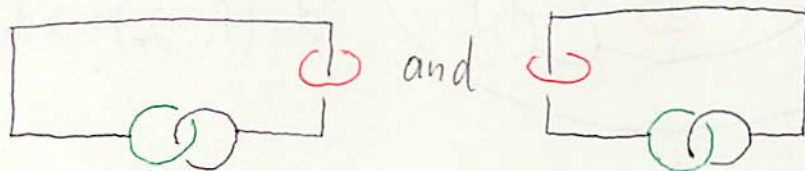
$n=3$



univ. cover

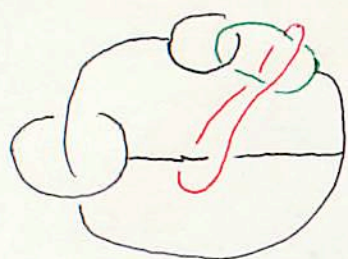


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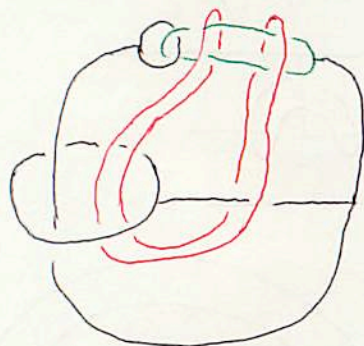
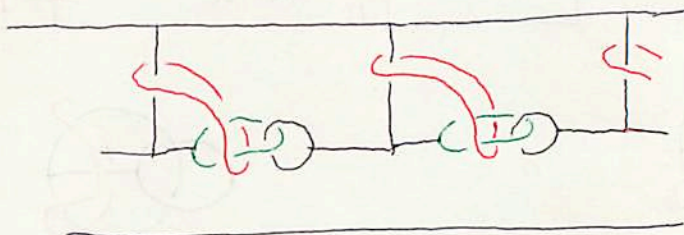


$$\therefore d_{12}(f_3, g_3) \geq 2 d_{12}(f_2, g_2) = 4$$

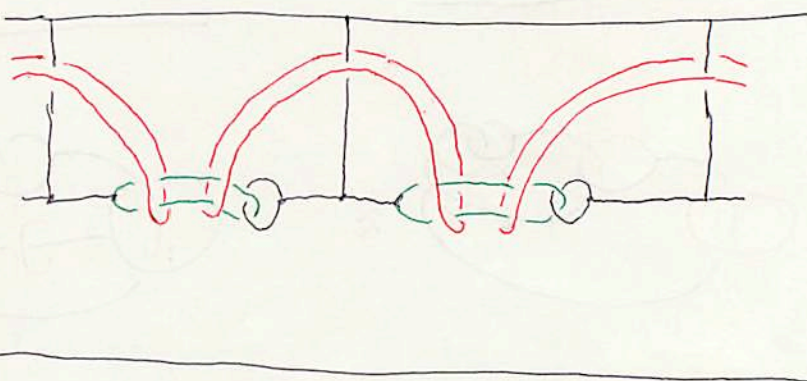
④



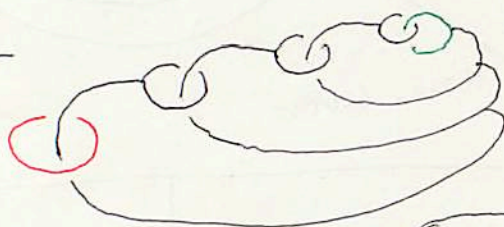
$\hookrightarrow$



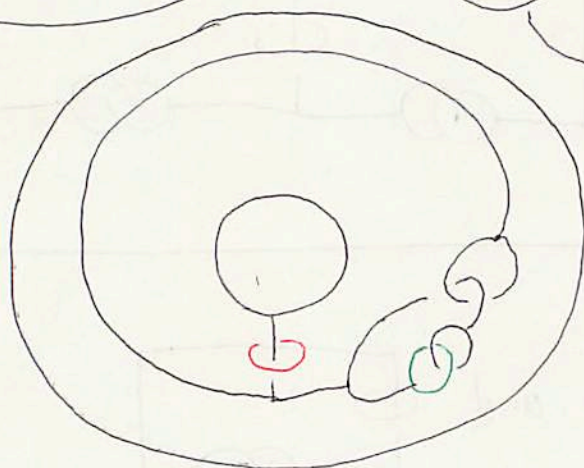
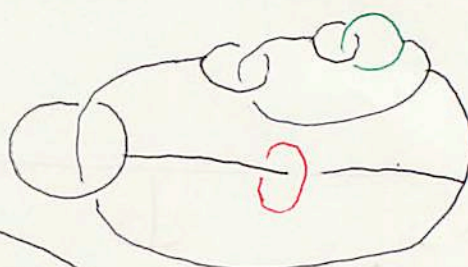
$\hookrightarrow$



$n=4$



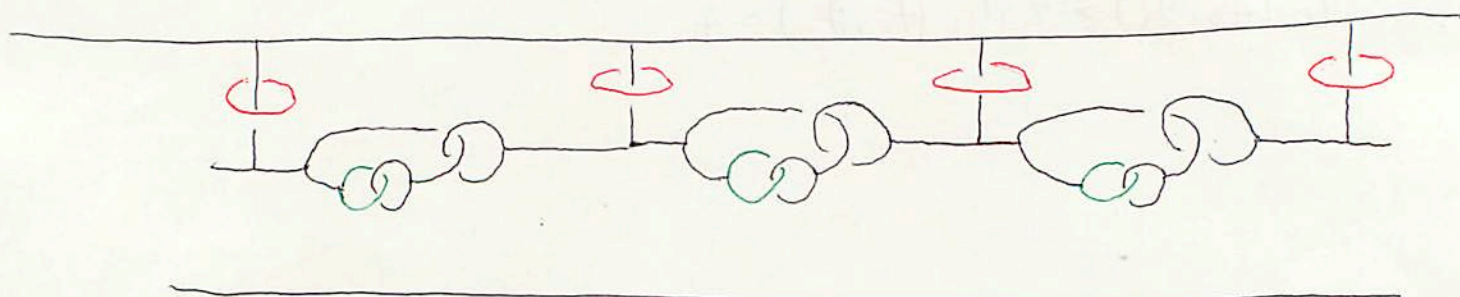
$\approx$



$\uparrow$ univ. cover

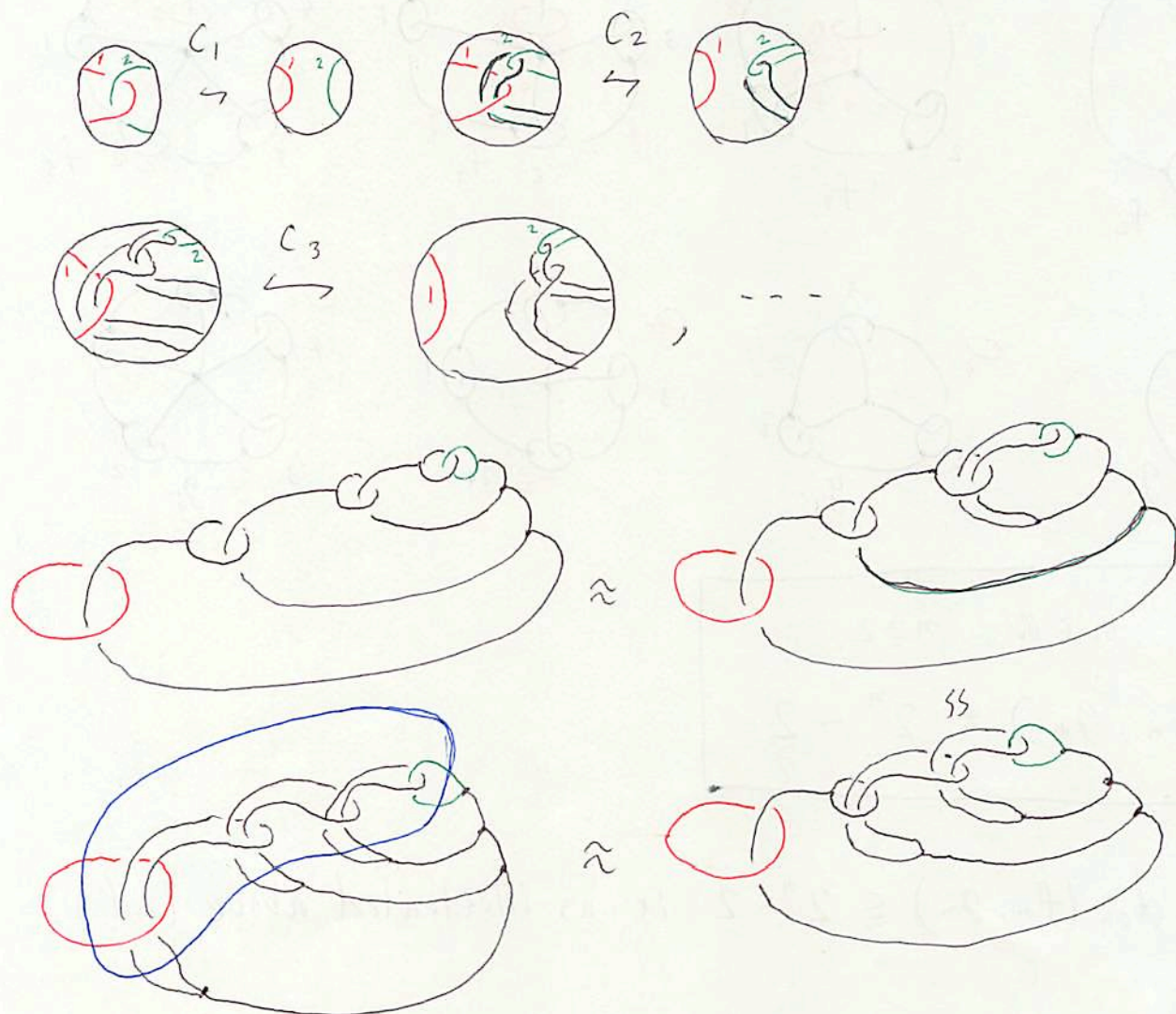
Similarly
we have

$$d_{12}(f_4, g_4) \geq 2d_{12}(f_3, g_3)$$

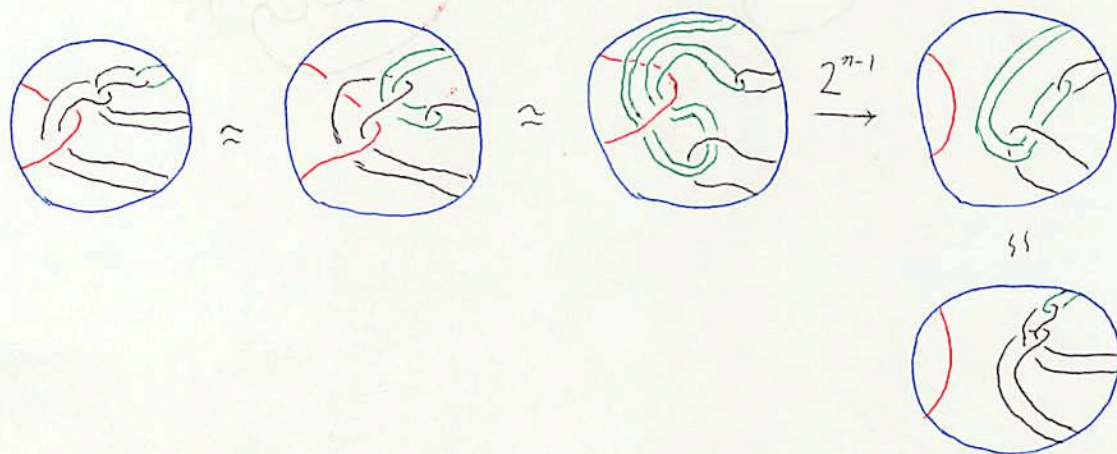


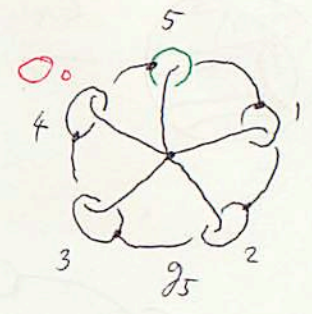
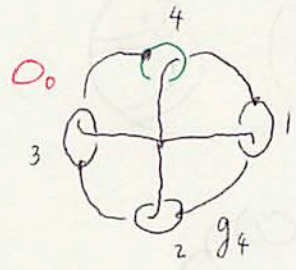
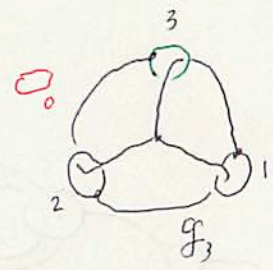
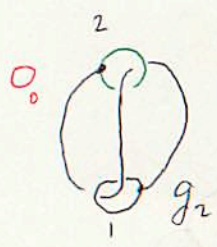
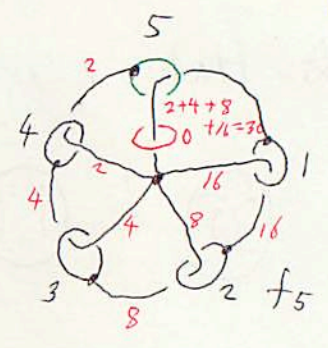
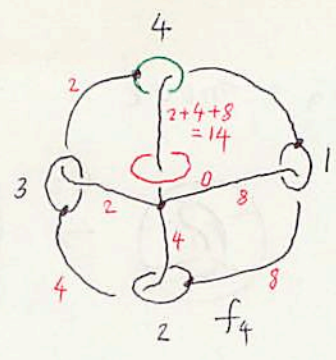
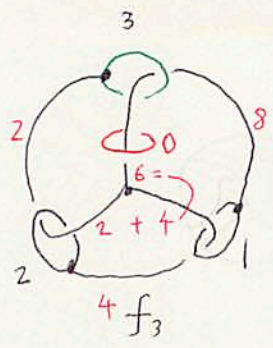
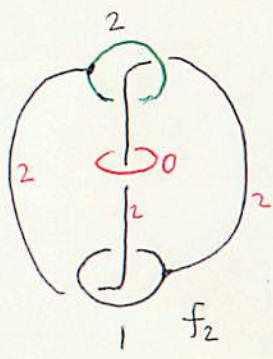
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☆ Habiro's C_n -move



∴ The minimal number of crossing changes between n strings 1 and 2 realizing a C_n -move is 2^{n-1} .

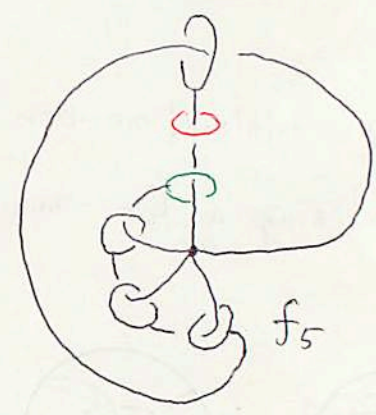
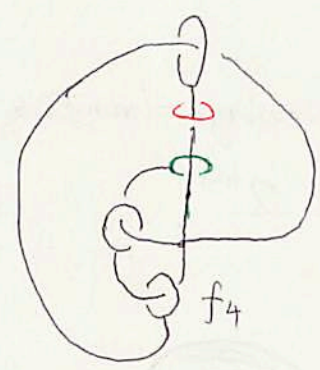
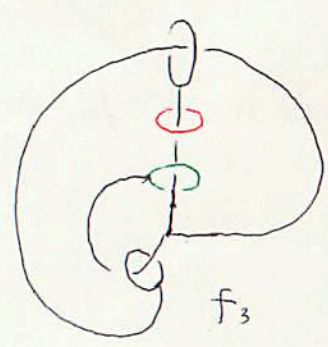




Thm 2 $n \in \mathbb{N}, n \geq 2$.

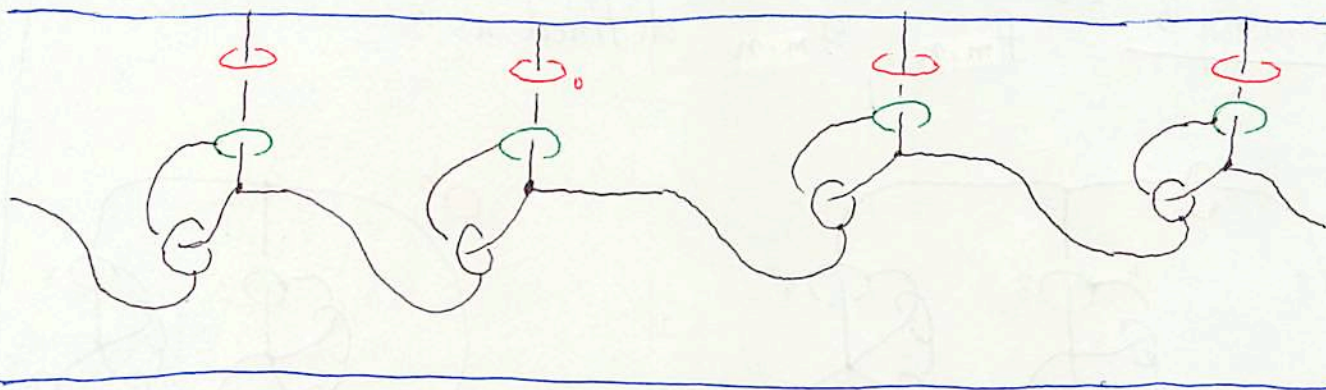
$$\text{don}(f_n, g_n) = 2^n - 2$$

Proof, $\text{don}(f_n, g_n) \leq 2^n - 2$ is as illustrated above (red).



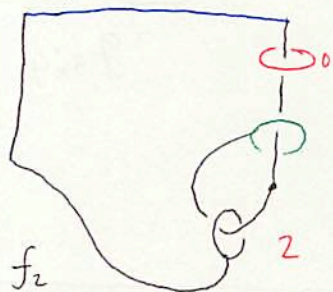
$n=3$

7

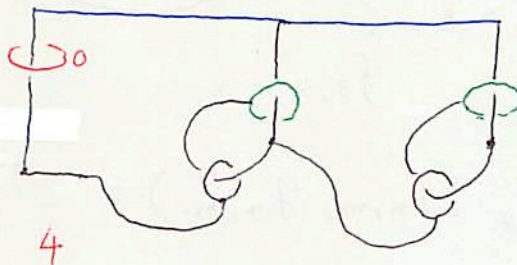


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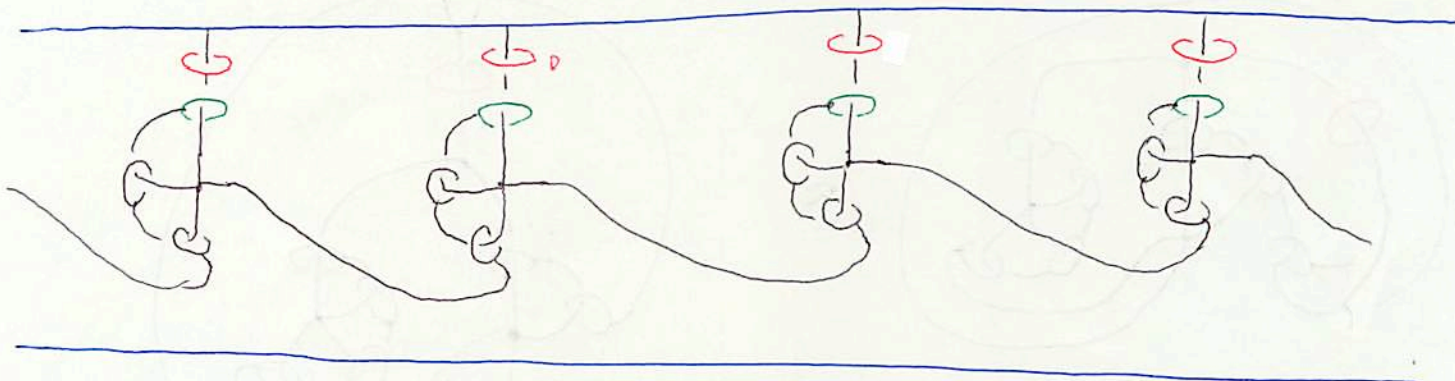
both of



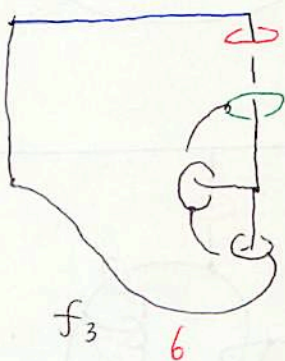
and



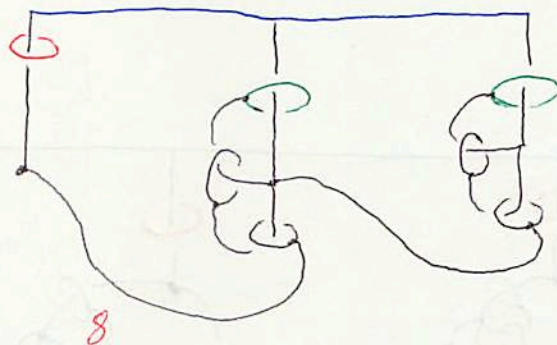
$n=4$



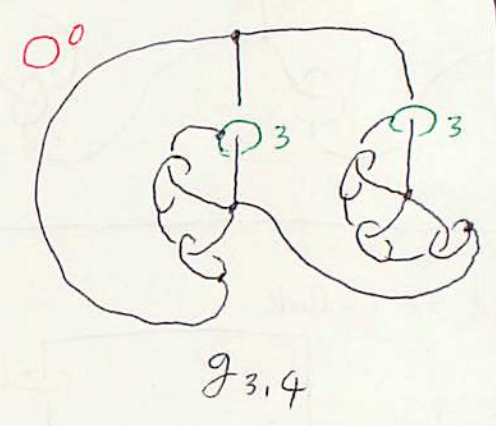
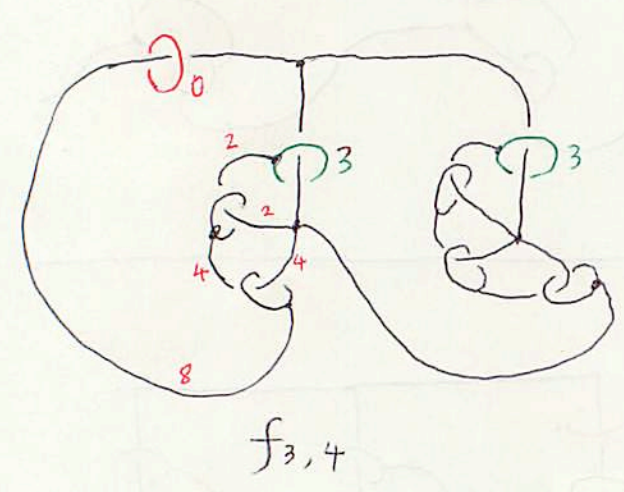
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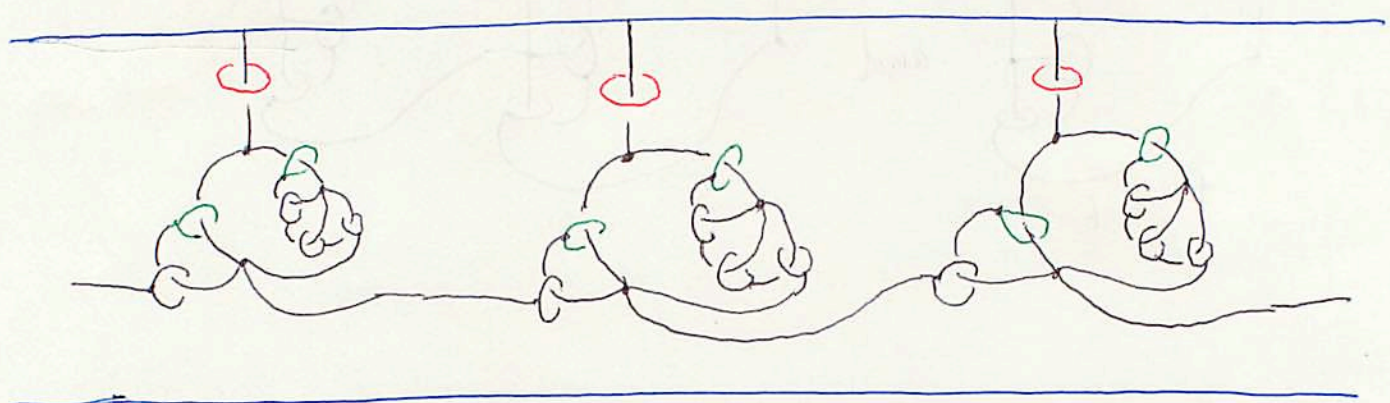
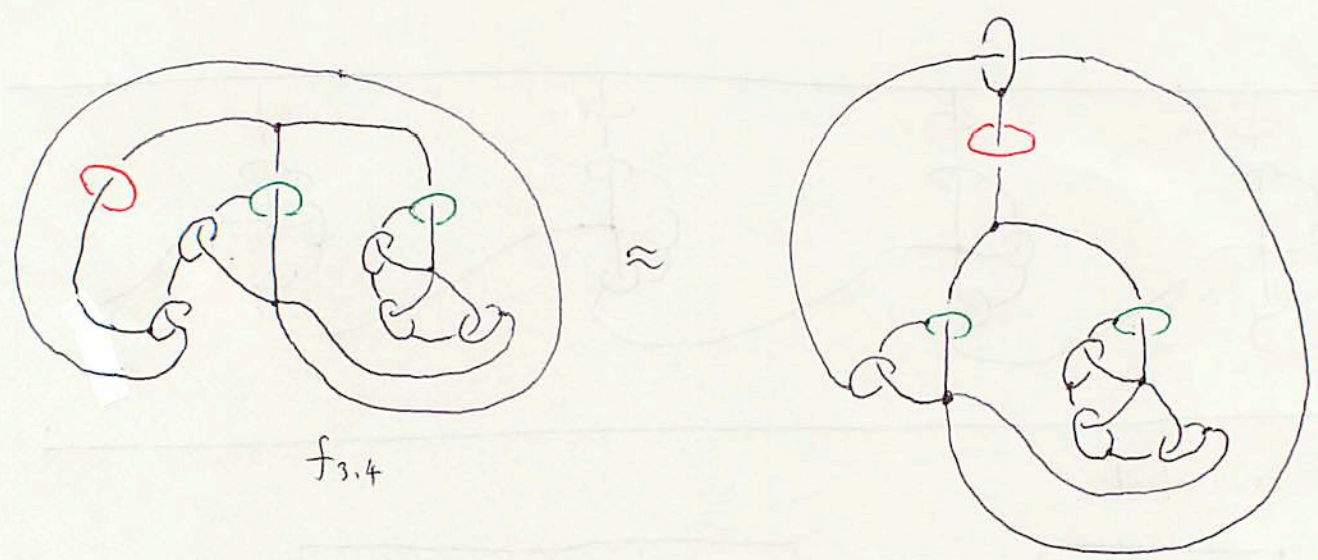


Lemma 3 $2 \leq n \leq m, n, m \in \mathbb{N}$
 $f_{n,m}, g_{n,m}$ defined as:



$$do_n(f_{n,m}, g_{n,m}) = 2^n$$

Proof $do_n(f_{n,m}, g_{n,m}) \leq 2^n$ is as illustrated above (red).

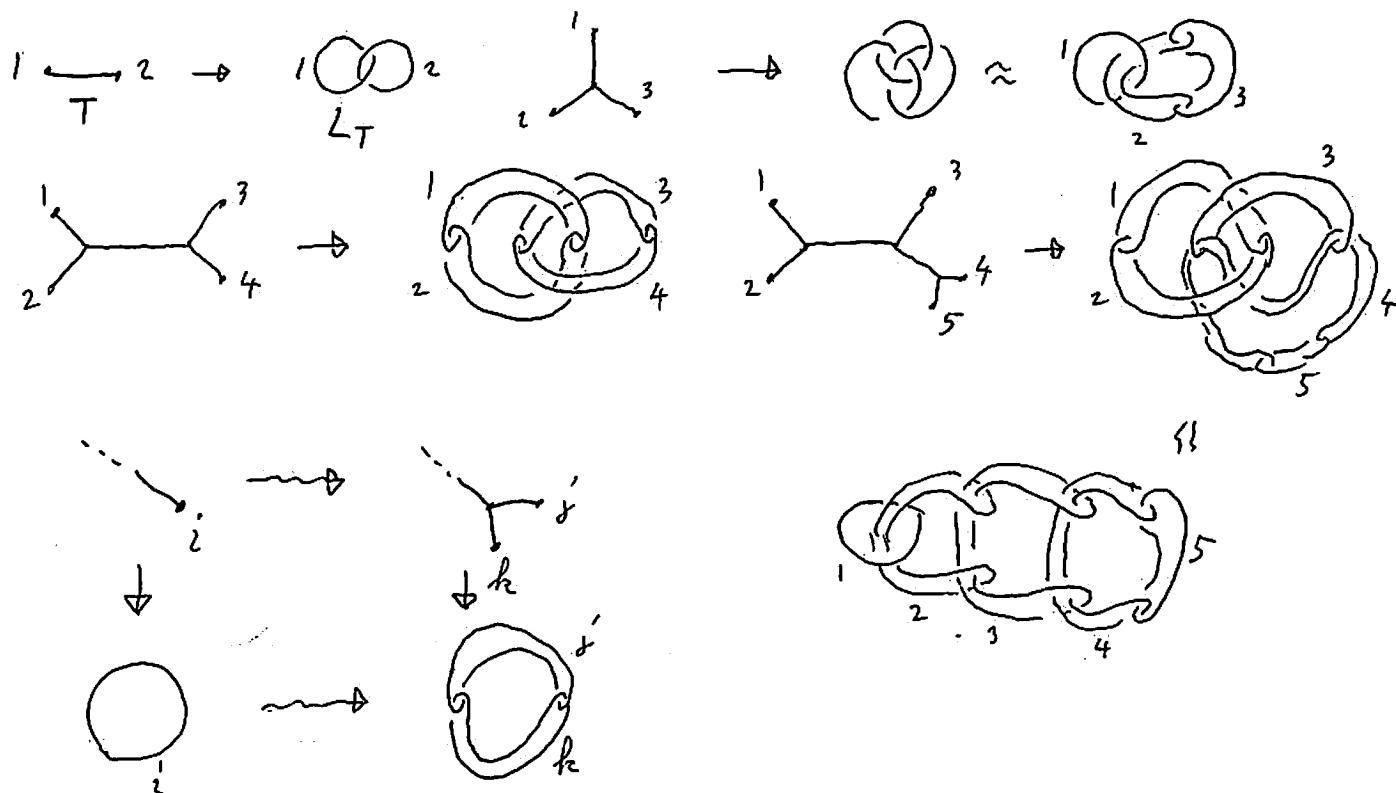


Need to unlink two $f_{2,4}$

★ Component specific unlinking numbers of Milnor type links (9)

T : uni-trivalent tree $V(T) = V_1(T) \sqcup V_3(T)$
(deg=1) (deg=3)

L_T : Milnor type link associated to T



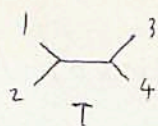
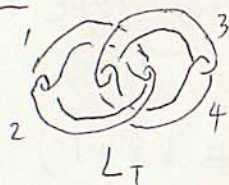
U_n : n -component trivial link.

$u, v \in V_1(T)$: $d_T(u, v) :=$ the number of edges of the path joining u and v
~~we write u for the component of L_T corresponding to $u \in V_1(T)$~~
the component ~~corresponding~~ of L_T corresponding to $u \in V_1(T)$ is denoted also by u .

Thm 4 T : uni-trivalent tree, L_T : Milnor type link associated to T . $u, v \in V_1(T)$.

$$d_{uv}(L_T, U_{|V_1(T)|}) = \begin{cases} \infty & : \text{if } u = v \\ 2d_T(u, v) - 1 & : \text{if } u \neq v \end{cases}$$

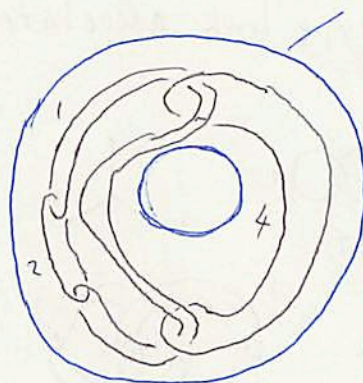
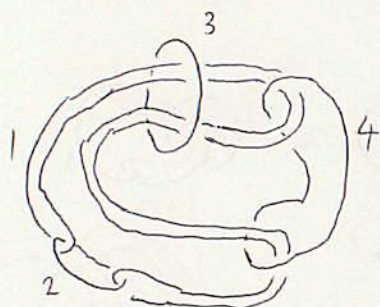
Example



$$d_{uv}(L_T, U_{|V_1(T)|}) = u_{uv}(L_T) \quad (10)$$

$$u_{1,4}(L_T) = 4$$

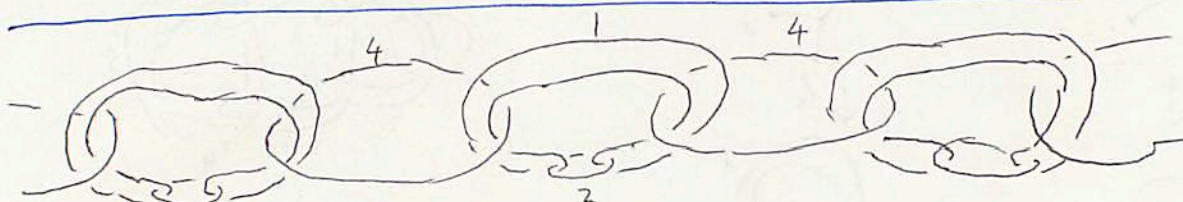
SS



$$\mathbb{S}^3 - \overset{\circ}{N}(3)$$

$$\cong \mathbb{S}^1 \times \mathbb{D}^2$$

↑ universal cover.

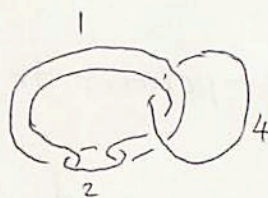


Need to unlink both of

$$\cong \mathbb{D}^2 \times \mathbb{R}$$



and



2

+

2

= 4

Lemma 5

T : tree , $u, v, w \in V_1(T)$,

P : path in T with $\partial P = \{u, v\}$, $x \in V_3(T)$ s.t. $wx \in E(T)$
(If $u=v$, $P=\{u\}$)

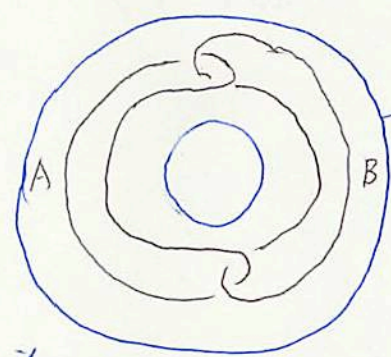
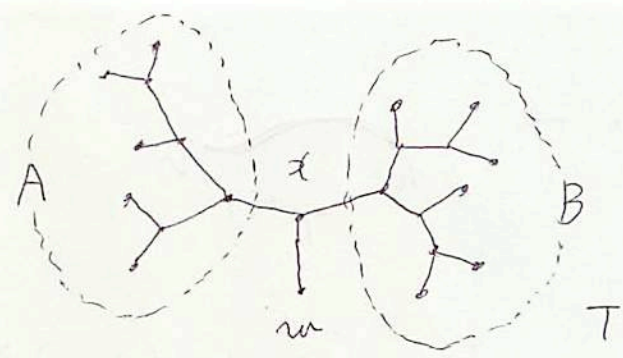
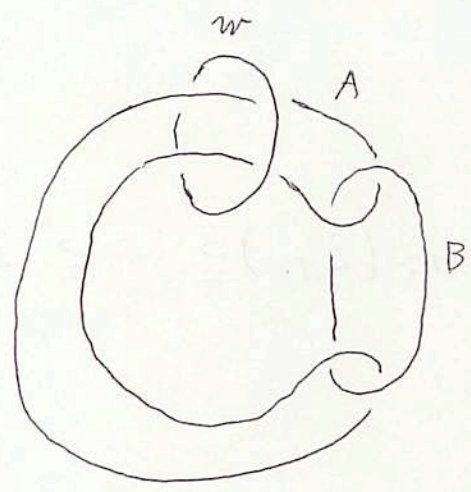
$T - \{w\}$: uni-trivalent tree obtained from T by deleting w and ignoring w ,

(1) If $w \notin P$, then $u_{uv}(L_T) \cong u_{uv}(L_{T-\{w\}})$.

(2) If $w \in P$, then $u_{uv}(L_T) \cong 2 u_{uv}(L_{T-\{w\}})$.

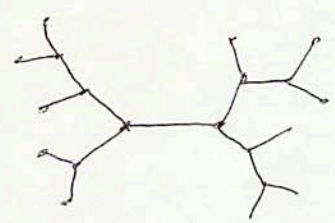
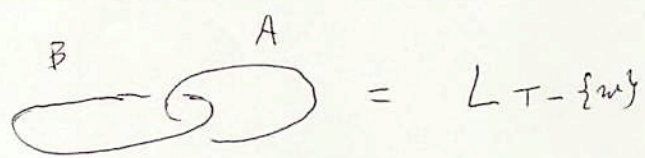
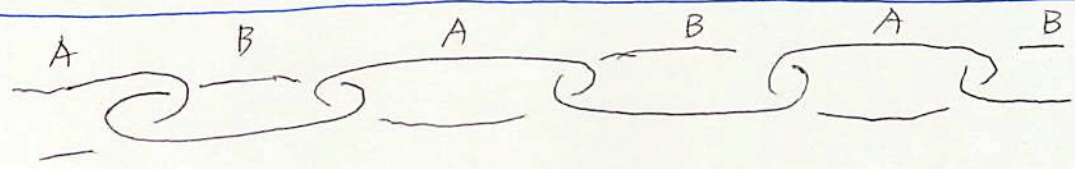
Proof,

$L_T \approx$



$S^3 - N(w)$
 $\cong S^1 \times D^2$

unlu, cover



In the case (2), we need to unlink

both of $\begin{matrix} B & A \\ \text{---} & \text{---} \end{matrix}$ and $\begin{matrix} A & B \\ \text{---} & \text{---} \end{matrix}$

$T - \{w\}$

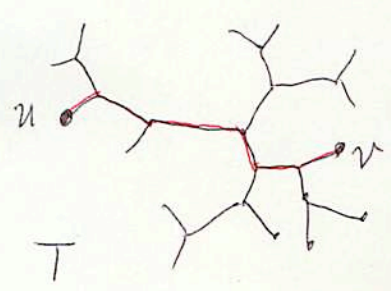
Proof of Thm 4

By an induction using Lemma 5 we have

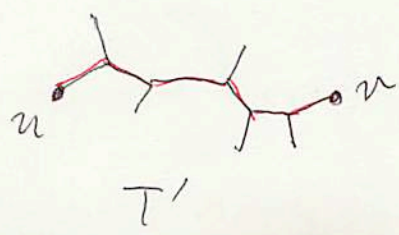
$$\text{unl}(L_T) \geq \begin{cases} \infty & \text{if } u=v \\ 2d_T(u,v)-1 & \text{if } u \neq v \end{cases}$$

When

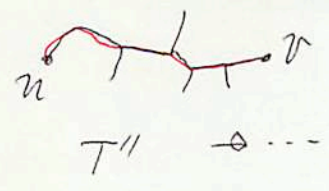
$u \neq v$, $\text{unl}(L_T) \leq 2d_T(u,v)-1$ is easily shown.



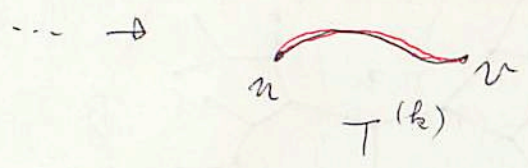
$\rightarrow$



$\rightarrow$



$\rightarrow \dots$



$$\mathcal{U}_{uv}(L_T) \geq \mathcal{U}_{uv}(L_{T'}) \geq 2 \mathcal{U}_{uv}(L_{T''}) \geq \dots \geq 2^{k-1} \mathcal{U}_{uv}(L_{T^{(k)}})$$

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