Knot diagrams on a punctured sphere as a model of string figures

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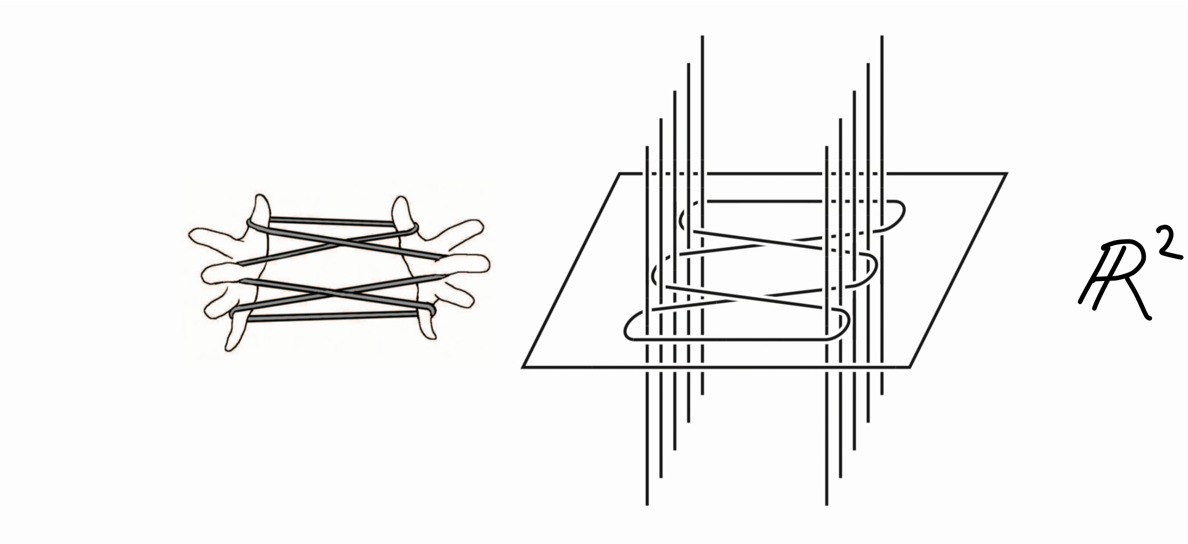
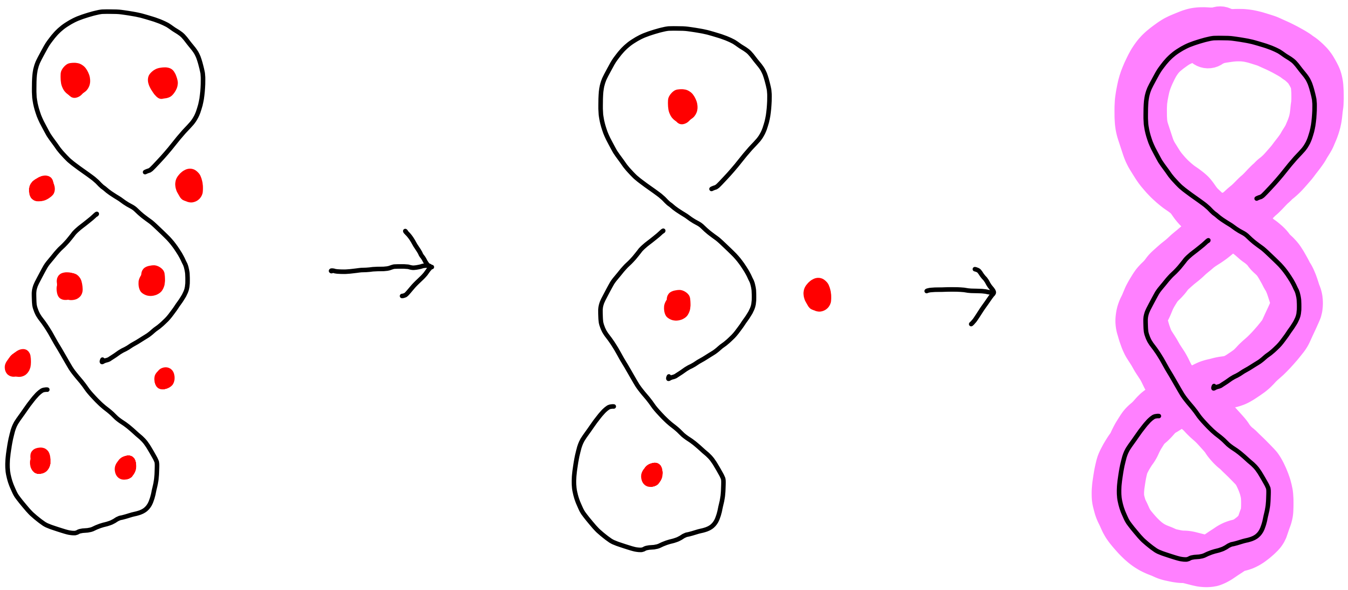
(新井雅章氏 (早大基幹理工修了) との共同研究)



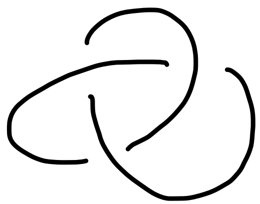
$$\mathbb{R}^2 \to \mathbb{R}^2 \cup \infty = S^2$$



$K \subseteq S^3$: knot $\quad D \subseteq S^2$: diagram of K

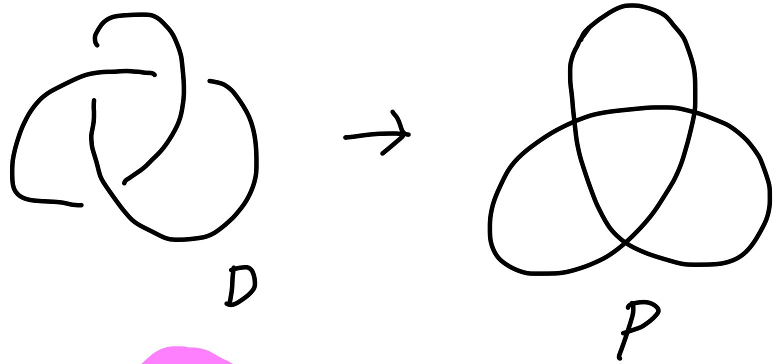
$P = P(D)$: underlying projection of D

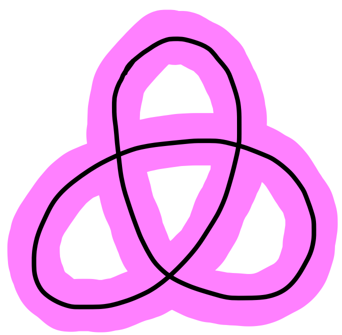
$N(P)$: regular neighbourhood of P in S^2



$K \subseteq S^3 = \mathbb{R}^3 \cup \infty$

$S^2 = \mathbb{R}^2 \cup \infty$





$N(P)$

$$\mathcal{R} = \mathcal{R}(P) := \{ R \mid R \text{ is a component of } S^2 \setminus N(P) \}$$

$C(D)$: the number of crossings of D

$$S \subseteq \mathcal{R}, \quad F(S) := S^2 \setminus \bigcup_{R \in S} R$$

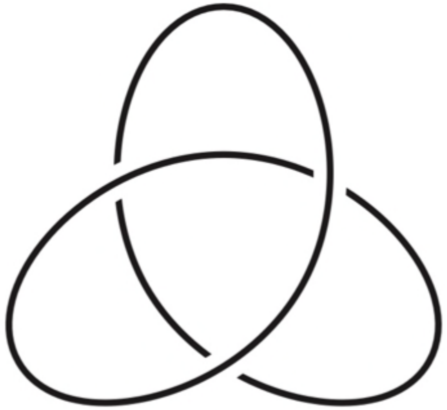
$$E \overset{R1, R2, R3}{\underset{\text{on } F(S)}{\sim}} D$$

$$c(D) := c(K)$$

$$C(D, S) := \min \{ C(E) \mid E \overset{F(S)}{\approx} D \}$$

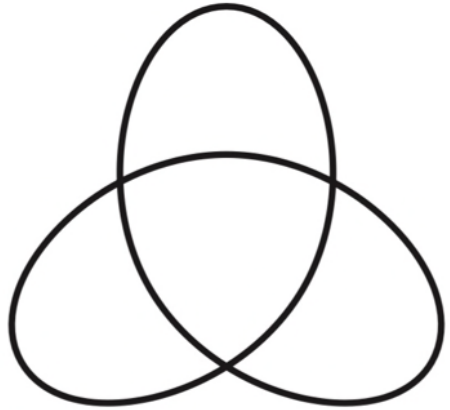
$$c(D) = C(D, \phi) \leq C(D, S) \leq C(D, \mathcal{R})$$

Ex.

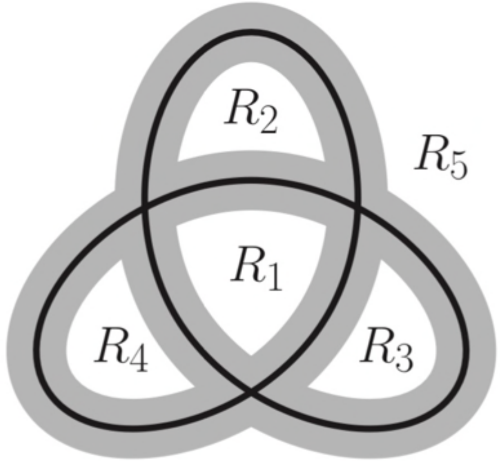


D

$c(D) = 0 \quad C(D) = 3$

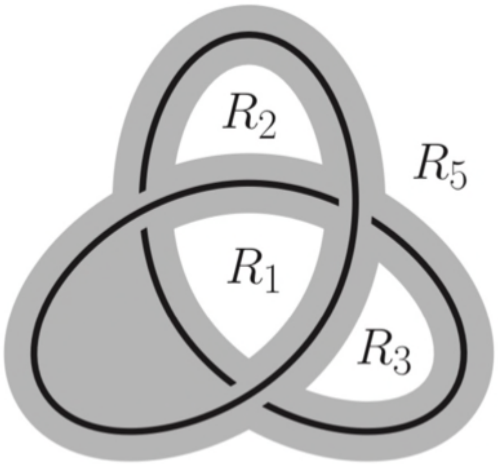


$P = P(D)$

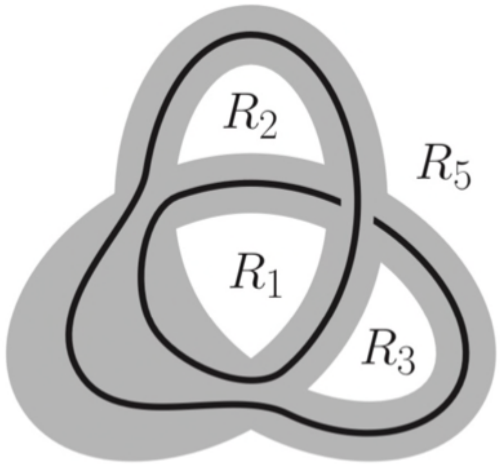


$N(P) = F(\mathcal{R})$

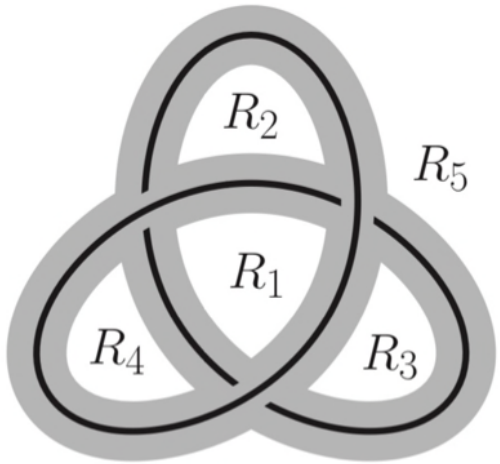
$= F(\{R_1, R_2, R_3, R_4, R_5\})$



$D \subset F(\{R_1, R_2, R_3, R_5\})$



$E \subset F(\{R_1, R_2, R_3, R_5\})$



$D \subset F(\mathcal{R}) = N(P)$

$c(D, \{R_1, R_2, R_3, R_5\}) = c(E, \{R_1, R_2, R_3, R_5\}) = 1$

$c(D, \mathcal{R}) = 3$

Thm 1 $C(D, R) = C(D)$

First proved by Keiji Tagami (National Fisheries University).

Thm 2 $\varphi : S^1 \to S^2$: generic immersion

$P = \varphi(S^1)$, $N = N(P) \subseteq S^2$: regular neighbourhood

$\forall \psi : S^1 \to N$: generic immersion with $\psi \underset{\text{homotopic}}{\simeq} \varphi$ on N,

$c(\psi) \geqq c(\varphi)$

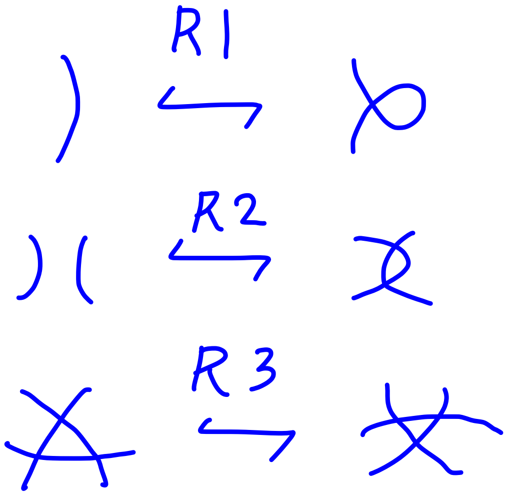
@ Turaev cobracket (arranged for our purpose)

F : oriented surface

$$\mathcal{H} := \{\varphi : S^1 \to F \mid \varphi : \text{generic immersion}, \underbrace{\varphi \not\simeq 0}_{\varphi \text{ is not null-homotopic on } F}\}$$

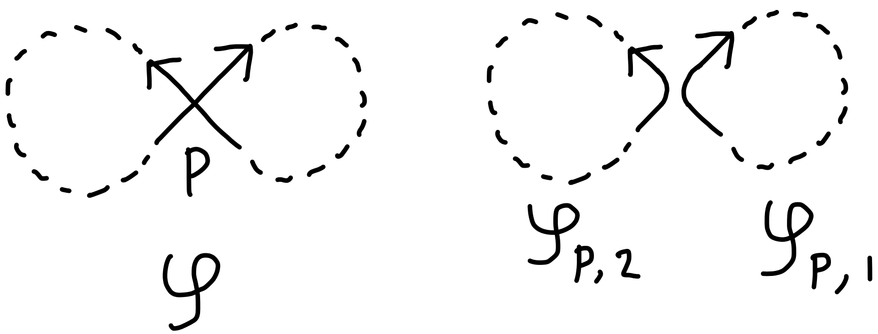
$$H := \mathcal{H} / \simeq, \quad [\varphi] := \{\psi \in \mathcal{H} \mid \psi \simeq \varphi\}$$

$$\underset{\text{homotopic on } F}{\varphi \simeq \psi} \iff \varphi \xrightarrow[\text{on } F]{R1,\, R2,\, R3} \psi$$



Turaev cobracket

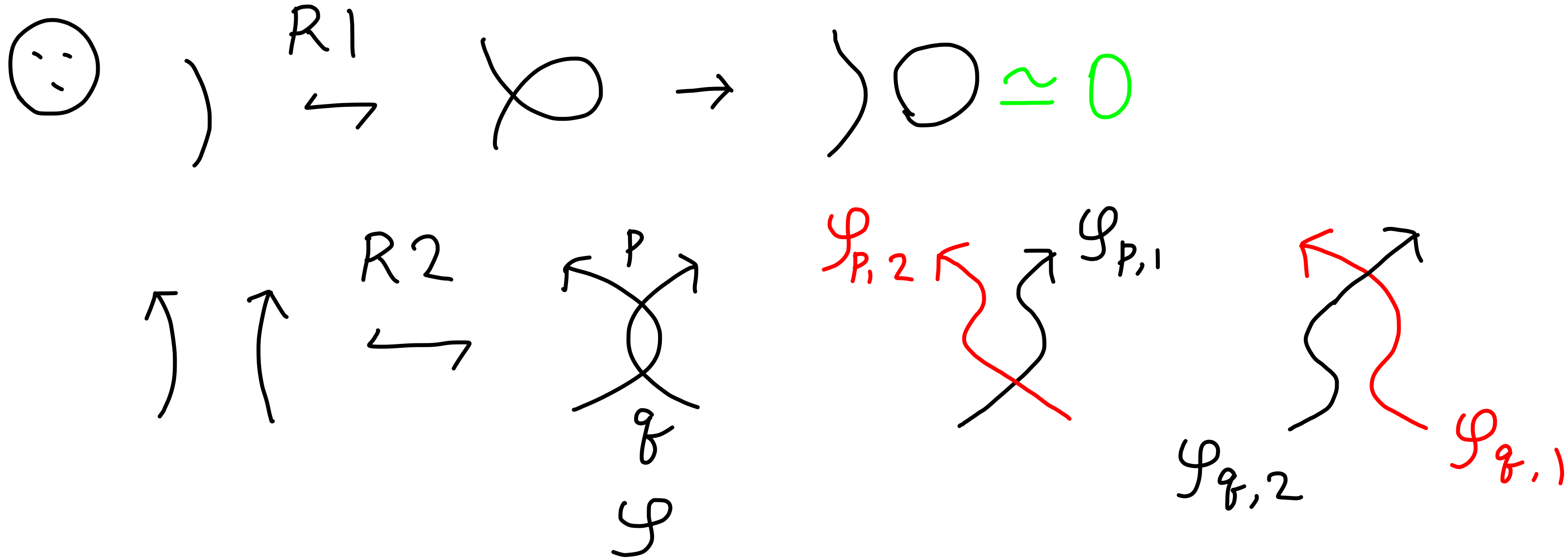
$$\begin{array}{rcl} \tau : H & \longrightarrow & \mathbb{Z}(H \times H) \\ \rotatebox{90}{$\in$} & & \\ [\varphi] & \longmapsto & \displaystyle\sum_{p \in D(\varphi)} \Big(([\varphi_{p,1}], [\varphi_{p,2}]) - ([\varphi_{p,2}], [\varphi_{p,1}]) \Big) \end{array}$$



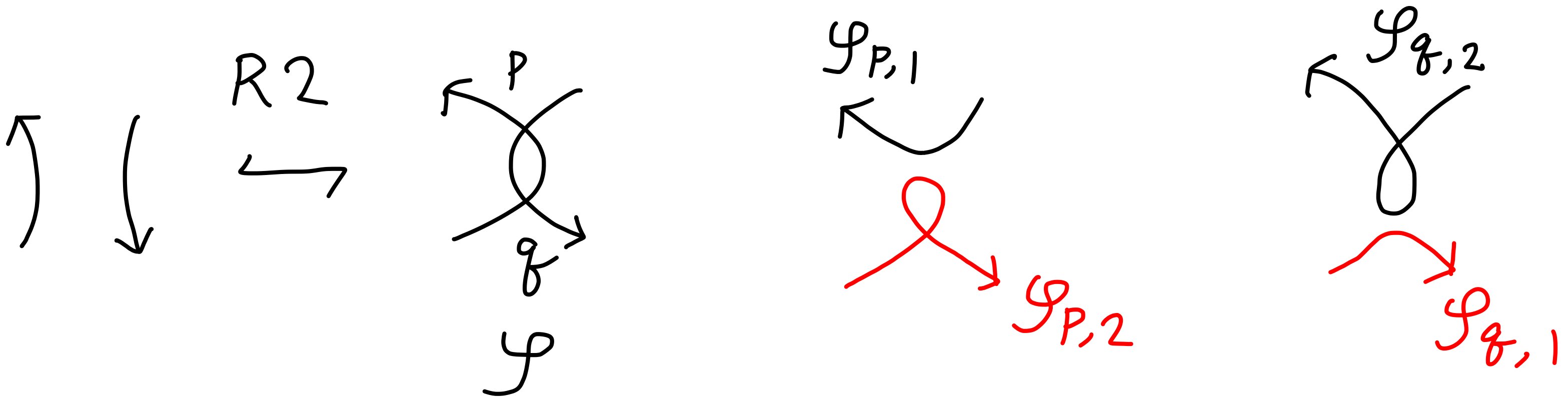
$\mathcal{C}(\varphi) :=$ the set of crossings of φ

$$D(\varphi) := \{ p \in \mathcal{C}(\varphi) \mid \varphi_{p,1} \not\simeq 0,\ \varphi_{p,2} \not\simeq 0 \}$$

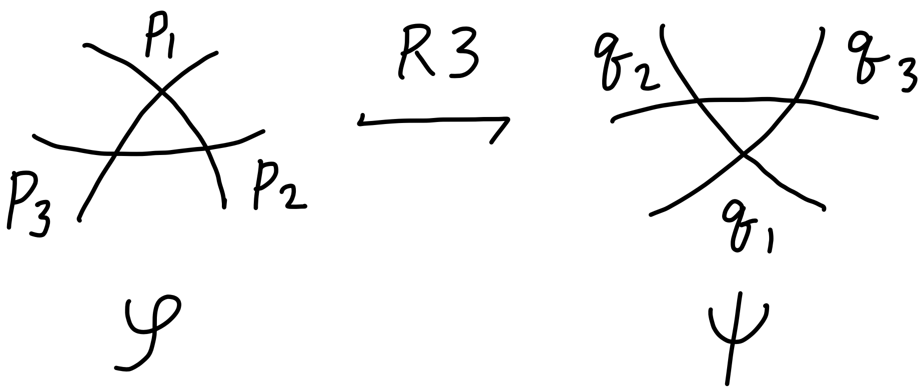
Prop.3 τ is a well-defined homotopy invariant.



$$\left([\varphi_{p,1}],[\varphi_{p,2}]\right) - \left([\varphi_{p,2}],[\varphi_{p,1}]\right)$$
$$+ \left([\varphi_{q,1}],[\varphi_{q,2}]\right) - \left([\varphi_{q,2}],[\varphi_{q,1}]\right) = 0$$

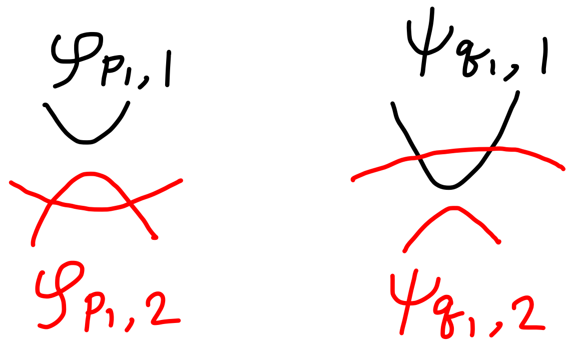


$$
\left([\varphi_{p,1}],\,[\varphi_{p,2}]\right)-\left([\varphi_{p,2}],\,[\varphi_{p,1}]\right)
$$
$$
+\left([\varphi_{q,1}],\,[\varphi_{q,2}]\right)-\left([\varphi_{q,2}],\,[\varphi_{q,1}]\right)=0
$$



$$
[\varphi_{p_i,j}]=[\psi_{q_i,j}]
$$
$$
i=1,2,3,\quad j=1,2
$$

Ex.



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$$n : \mathbb{Z}(H \times H) \to \mathbb{Z}_{\geqq 0}$$

$$n\left(a_1(x_1, y_1) + \cdots + a_k(x_k, y_k)\right) := |a_1| + \cdots + |a_k|$$

$\boxed{\begin{array}{l} \underline{\text{Prop. 4}} \quad c([\varphi]) = \min\{c(\psi) \mid \psi \simeq \varphi\} \\ c([\varphi]) \geqq \frac{1}{2} n(\tau([\varphi])) \end{array}}$

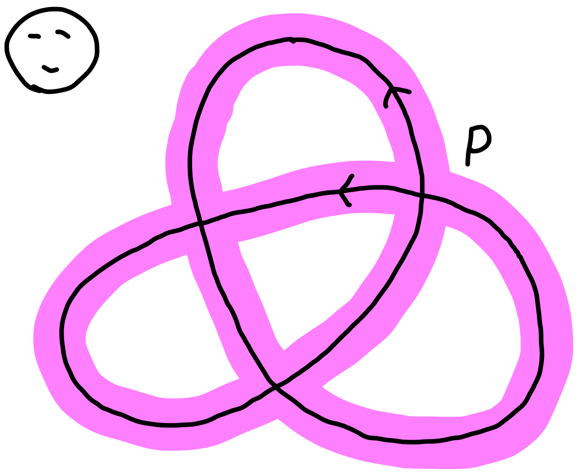
Prop. 5 $\varphi : S^1 \to S^2$: generic immersion,

$P = \varphi(S^1)$, $F = N(P)$: regular neighbourhood of P

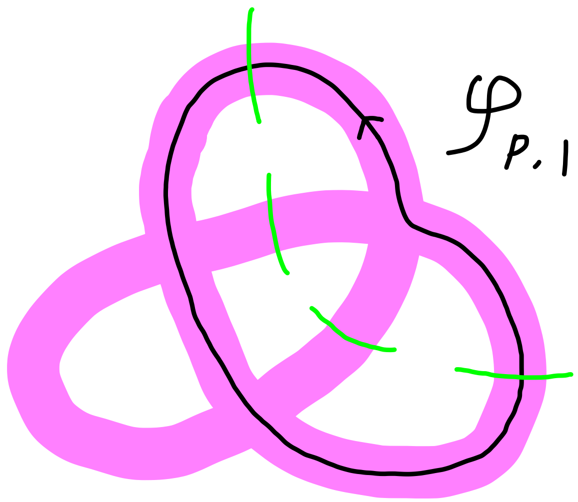
$\varphi : S^1 \to F$: generic immersion

Thm 2 $\Updownarrow$

$\Rightarrow n(\tau([\varphi])) = 2c(\varphi)$, hence $c([\varphi]) = c(\varphi)$



$\varphi(S^1)$ $F = N(P)$



characterize $\varphi_{P,1}$

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Zhiyun Cheng informed us that

Thm 2 is an immediate consequence of :

[Hass - Scott] Intersections of curves on surfaces
Israel J. Math. 1985

THEOREM 4.2. *Let f be a general position loop on an orientable surface F. If f has excess self-intersection, then f has a singular 1-gon or 2-gon.*

$\mathcal{R} = \mathcal{R}(P) := \{ R \mid R \text{ is a component of } \mathbb{S}^2 \setminus N(P) \}$

$$\emptyset \subseteqq S_1 \subseteqq \cdots \subseteqq S_k \subseteqq \mathcal{R}$$

Thm 1

$$\Rightarrow c(K) = C(D, \emptyset) \leqq C(D, S_1) \leqq \cdots \leqq C(D, S_k) \leqq C(D, \mathcal{R}) = C(D)$$

$$C(D) = C(P) = m \quad \Rightarrow \quad |\mathcal{R}| = m + 2$$

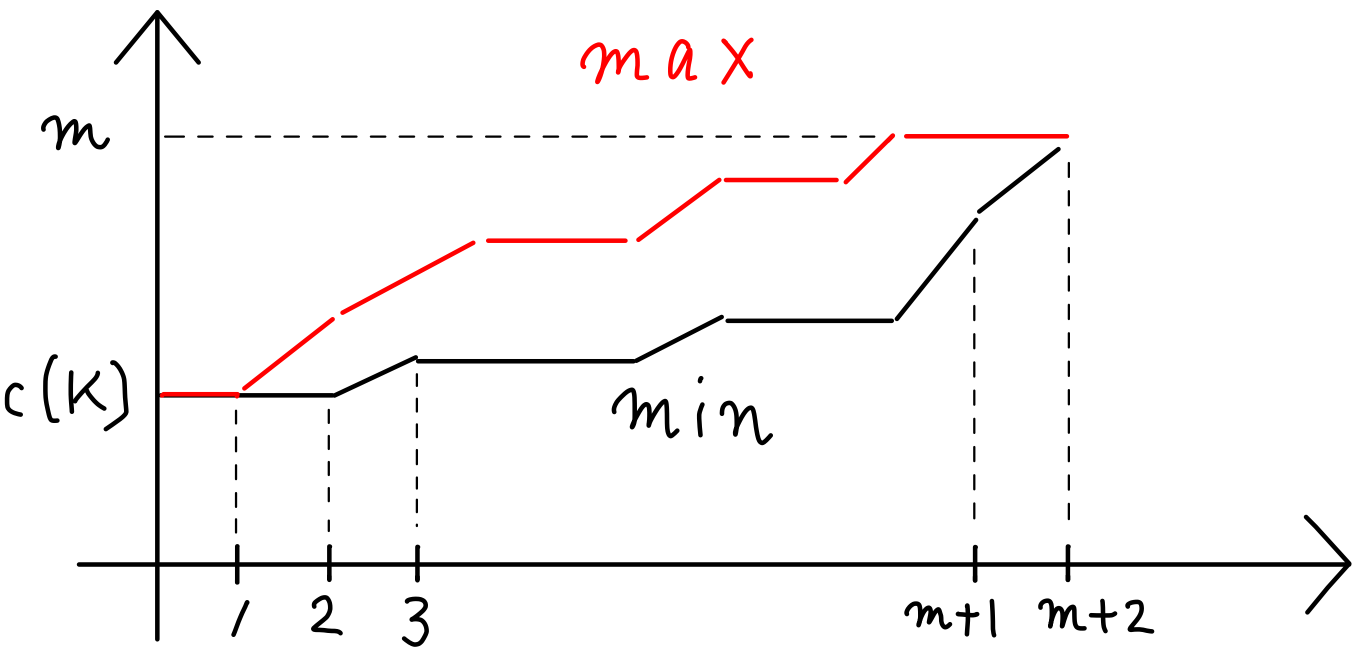
$0 \leqq n \leqq m+2$

$$C_{\max}(D, n) := \max\{ C(D, S) \mid |S| = n \}$$

$$C_{\min}(D, n) := \min\{ C(D, S) \mid |S| = n \}$$

Proposition 6 *Let D be a knot diagram on $\mathbb{S}^2$ with $C(D) = m$. Then we have the following inequalities.*

$$\begin{array}{ccccccccccc}
c_{\max}(D,0) & = & c_{\max}(D,1) & \leq & c_{\max}(D,2) & \leq \cdots \leq & c_{\max}(D,m+1) & \leq & c_{\max}(D,m+2) \\
\| & & \| & & \vee\!| & \cdots & \vee\!| & & \| \\
c_{\min}(D,0) & = & c_{\min}(D,1) & = & c_{\min}(D,2) & \leq \cdots \leq & c_{\min}(D,m+1) & \leq & c_{\min}(D,m+2).
\end{array}$$



Thank you

for your listening!