Unknotting numbers and crossing numbers of spatial embeddings of a planar graph

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Friday Seminar on Knot Theory

L : link $\quad u(L) \leqq \frac{1}{2} c(L)$ (Folklore)

K : nontrivial knot $\quad u(K) \leqq \frac{1}{2}(c(K)-1)$

(Folklore)

G : planar graph

$SE(G) := \{ f : G \to \mathbb{R}^3 : \text{embedding} \}$

$t : G \to \mathbb{R}^3$, $t(G) \subseteq \mathbb{R}^2 \times \{0\}$

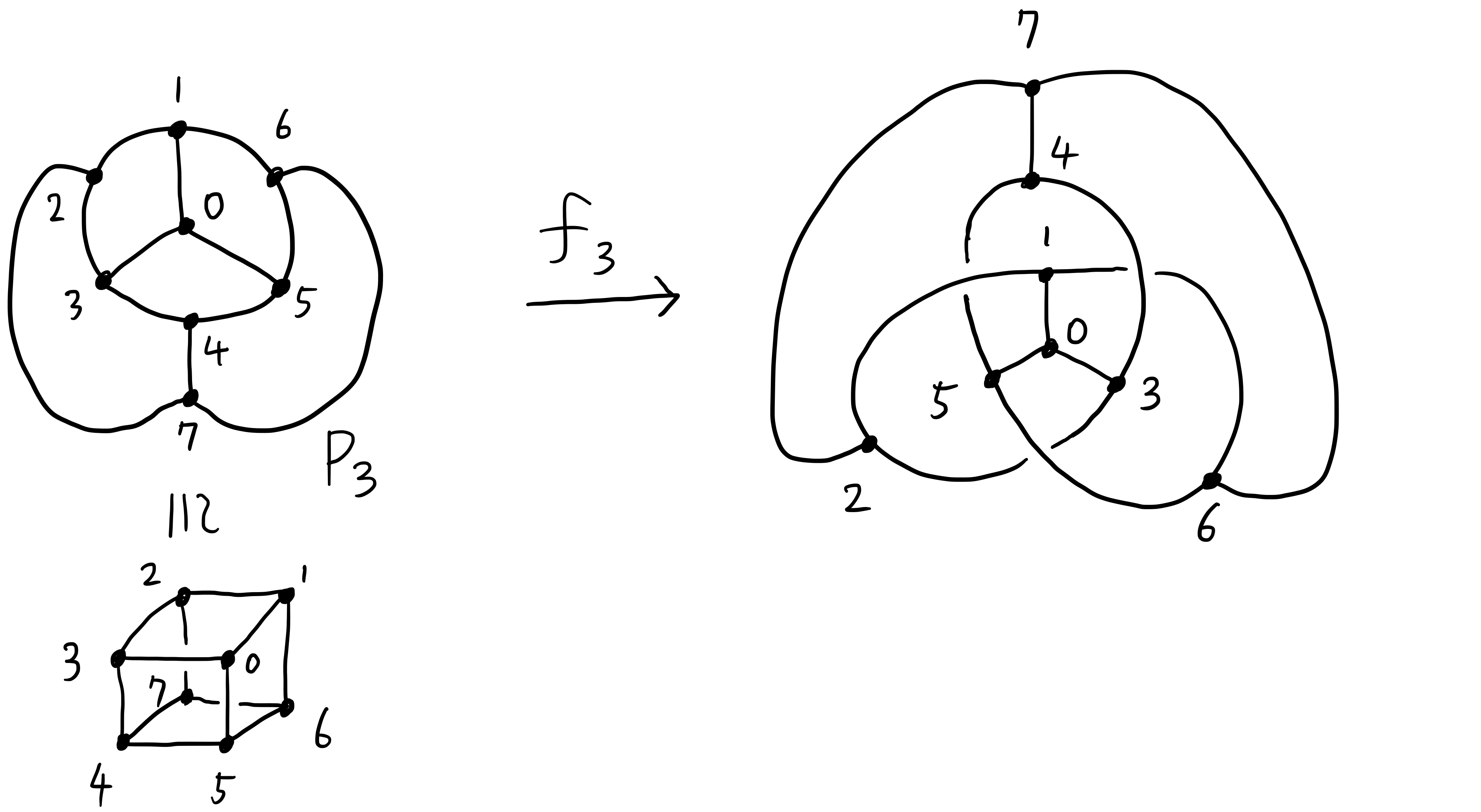
trivial embedding

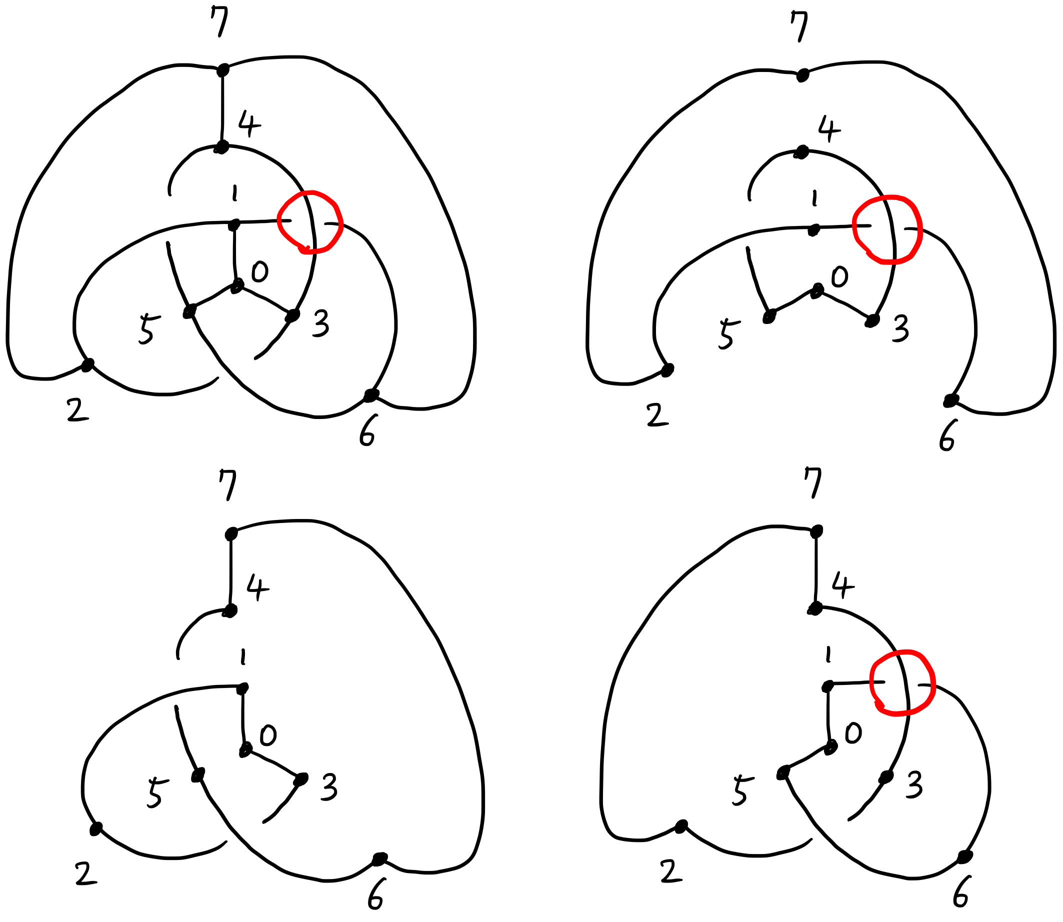
$f \in SE(G)$, $u(f) := d(f, t)$

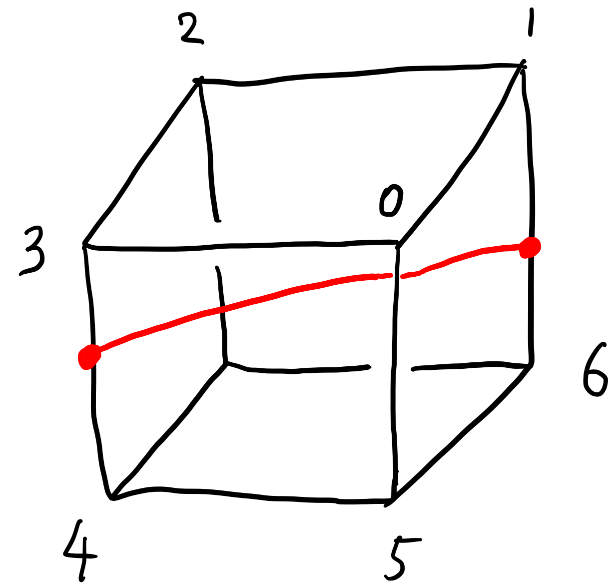
: minimal number of crossing changes from f to t

Q. $u(f) \leqq \frac{1}{2} c(f)$?

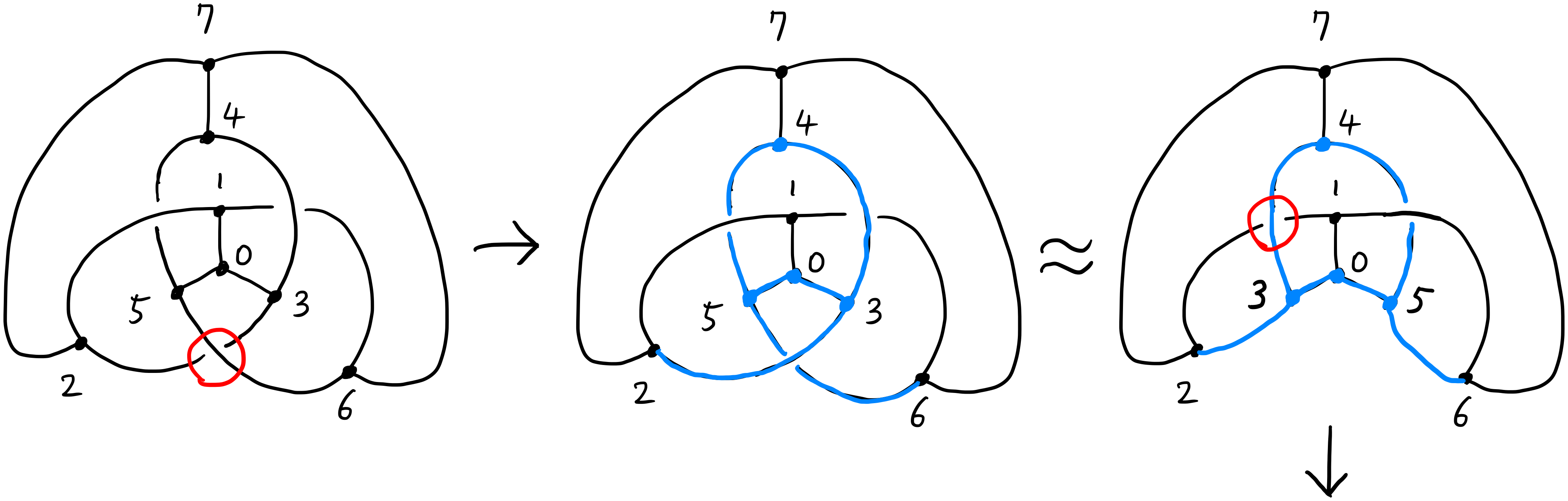
Ans. $^{\exists} f \in SE(^{\exists} G)$ s.t. $u(f) > \frac{1}{2} c(f)$



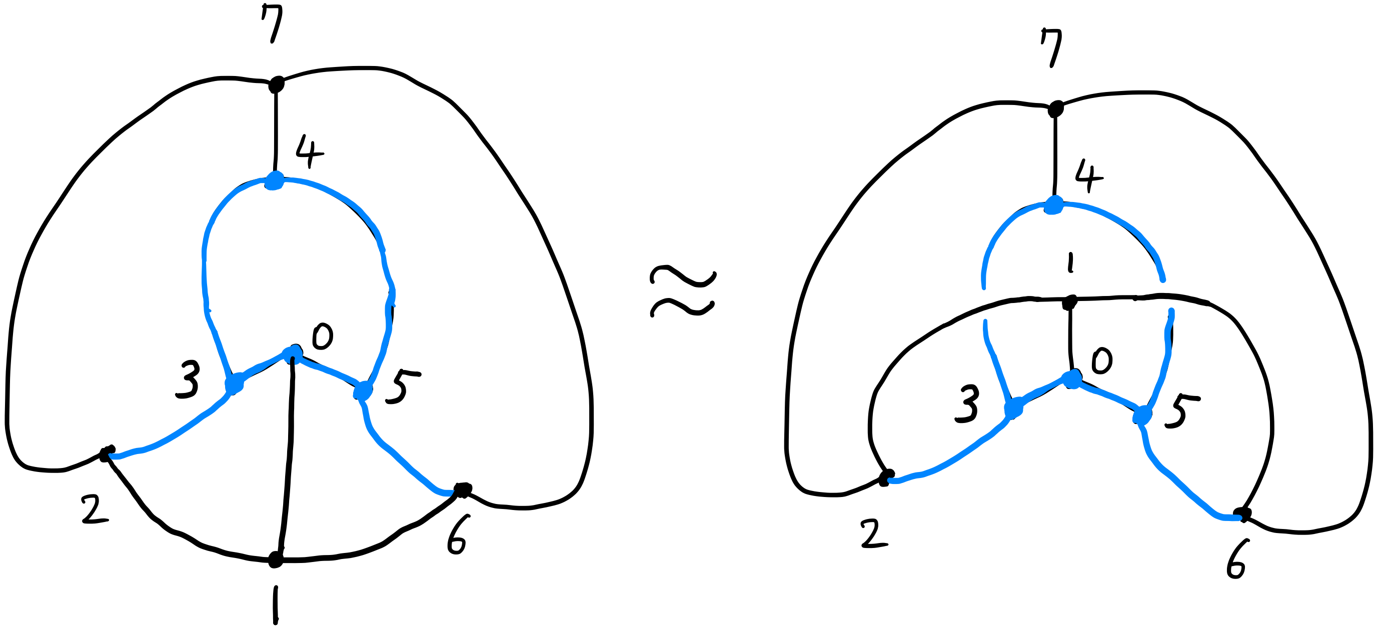




$\therefore\ u(f_3) > 1$



$\therefore\ u(f_3) \leq 2$

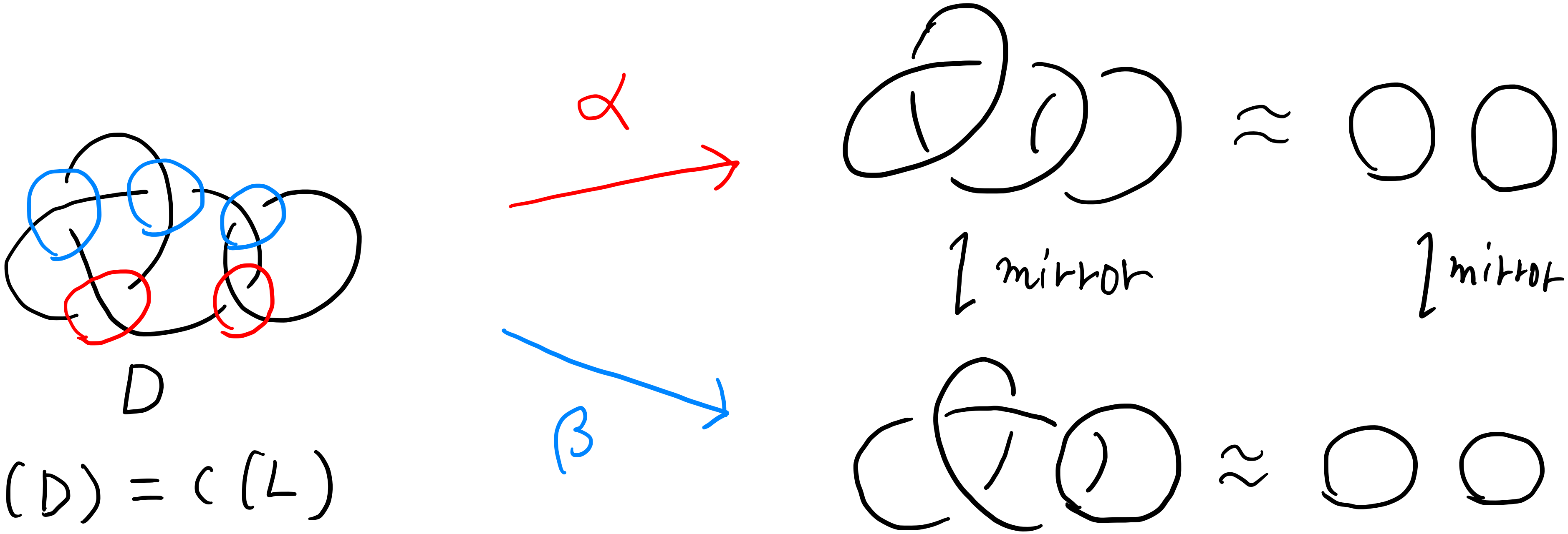


$\therefore\ u(f_3) = 2$

$c(f_3) = 3$

L : link $\quad u(L) \leq \frac{1}{2} c(L)$

(∵)



D

$c(D) = c(L)$

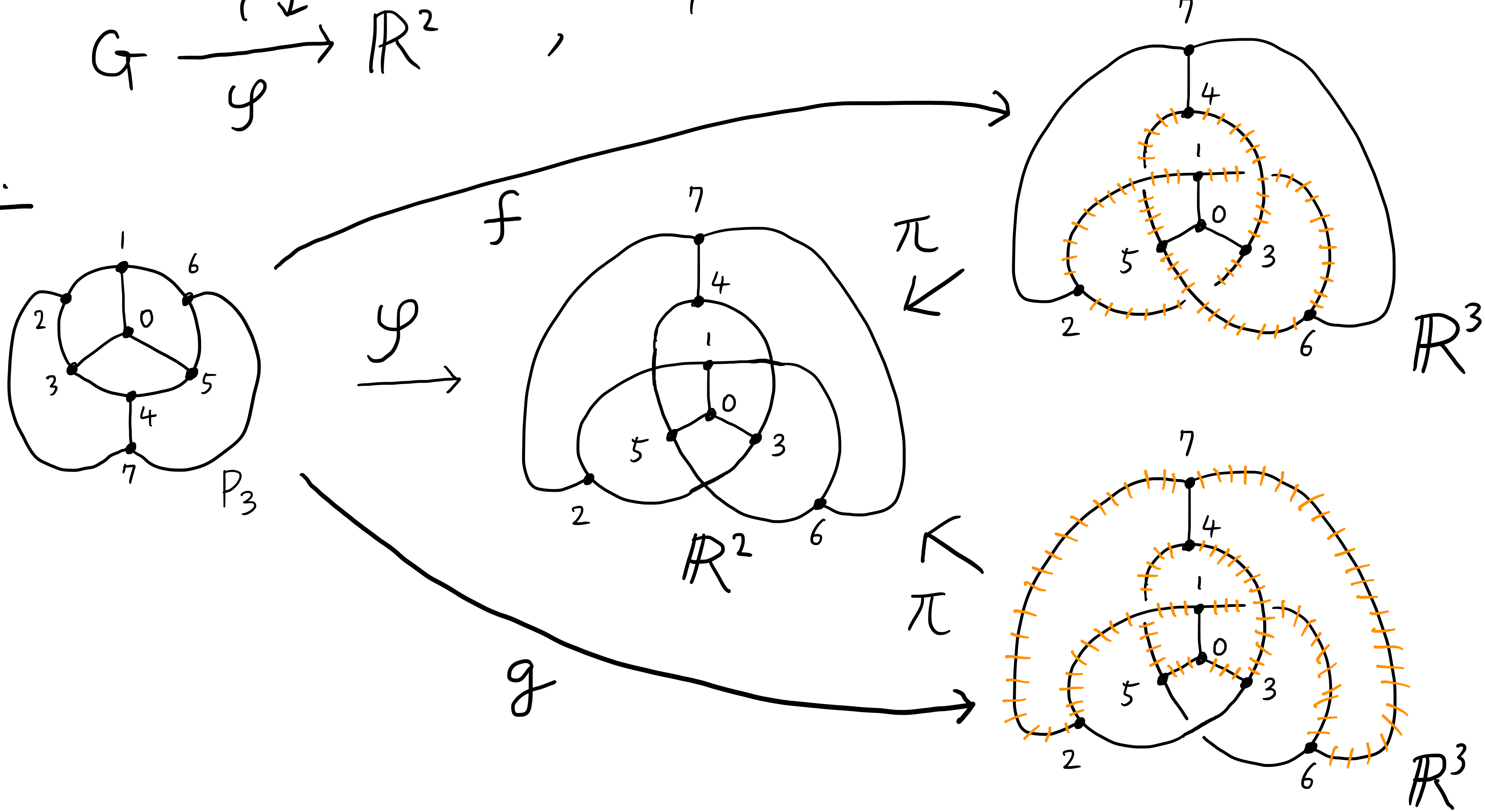
- $\alpha + \beta = c(L)$
- $u(L) \leq \min\{\alpha, \beta\} \leq \frac{1}{2} c(L)$

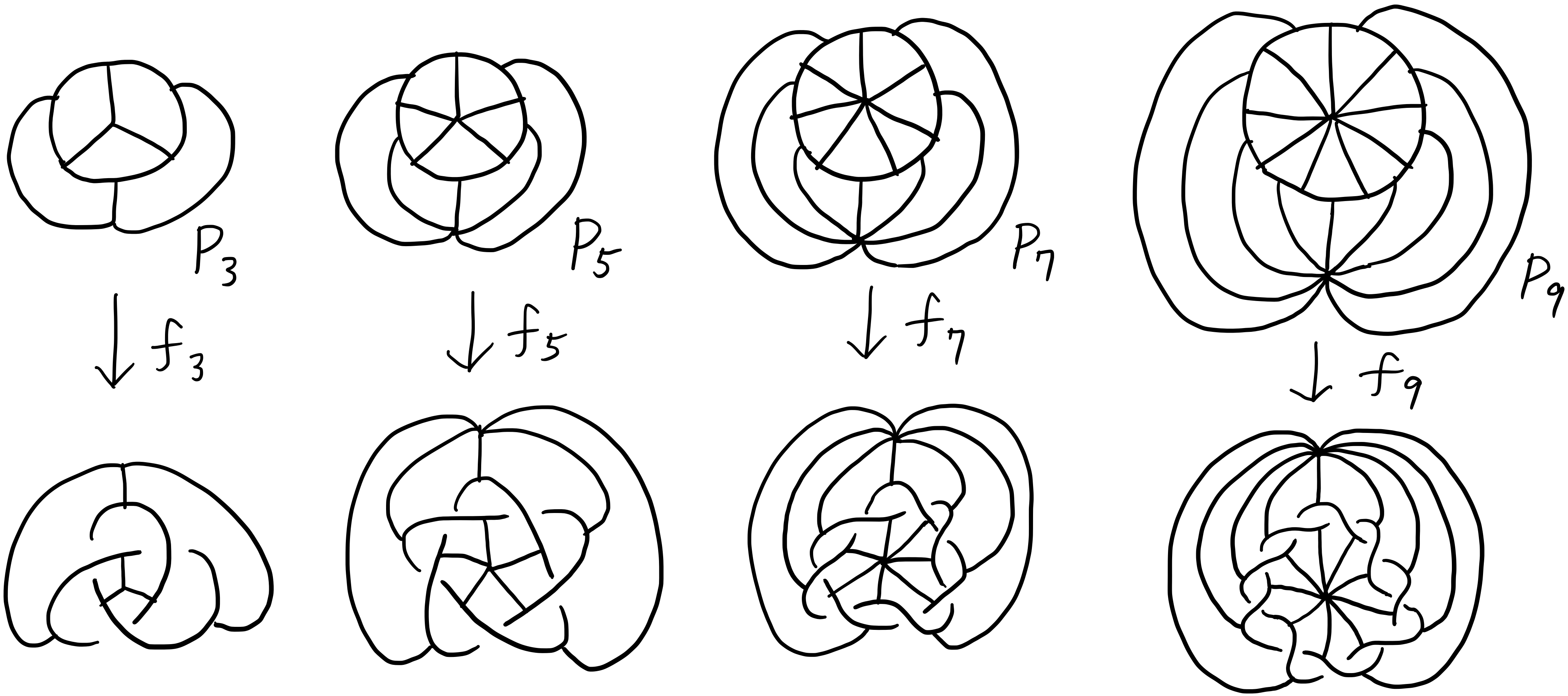
//

generic immersion $\varphi : G \to \mathbb{R}^2$ is a **knotted projection**

$$\overset{\text{def}}{\Longleftrightarrow} \quad \begin{array}{ccc} & {}^{\forall} f \nearrow & \mathbb{R}^3 \\ & \circlearrowright & \downarrow \pi \\ G & \xrightarrow[\varphi]{} & \mathbb{R}^2 \end{array} \quad , \quad f \not\simeq t$$

Ex.



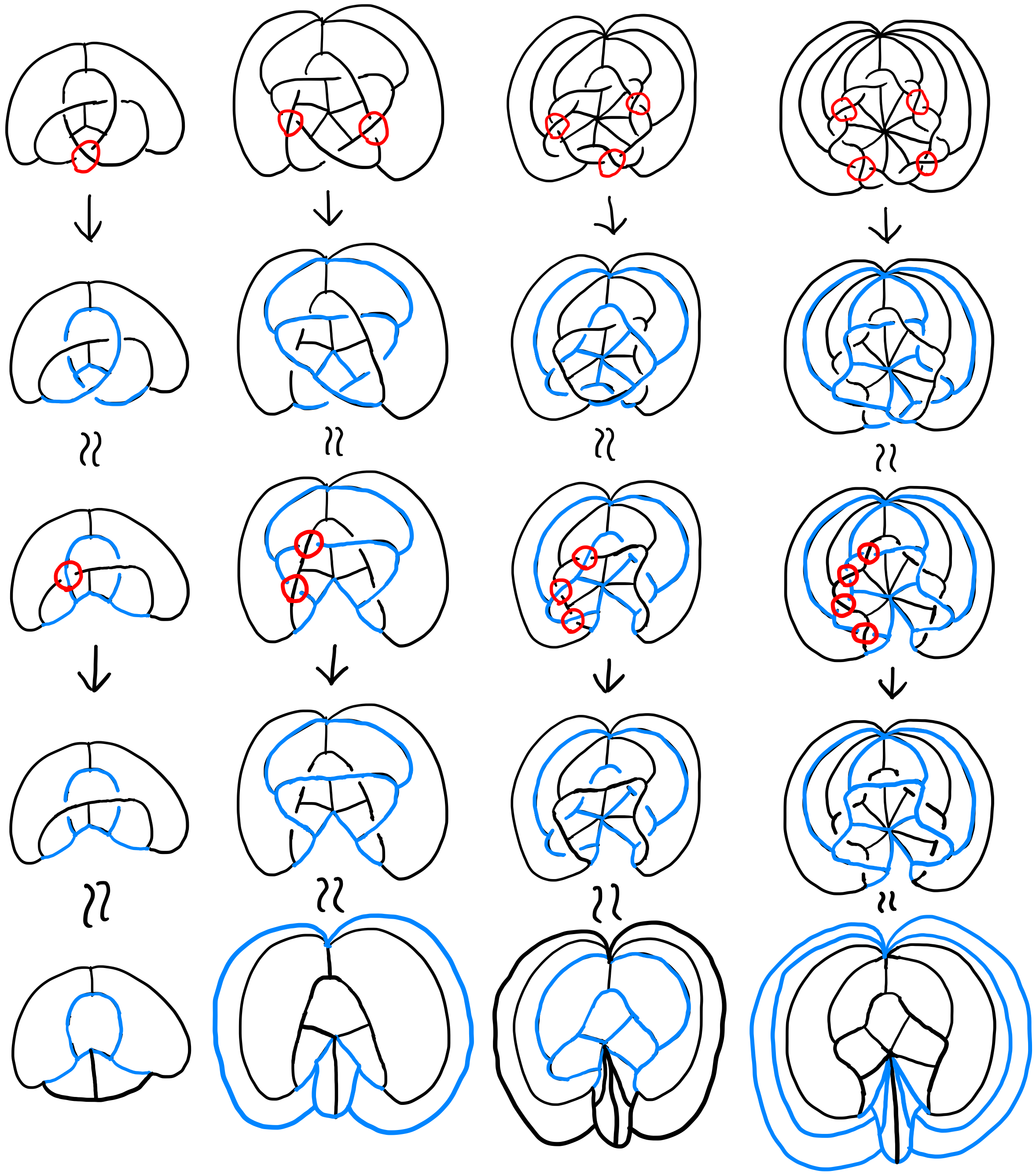


Thm

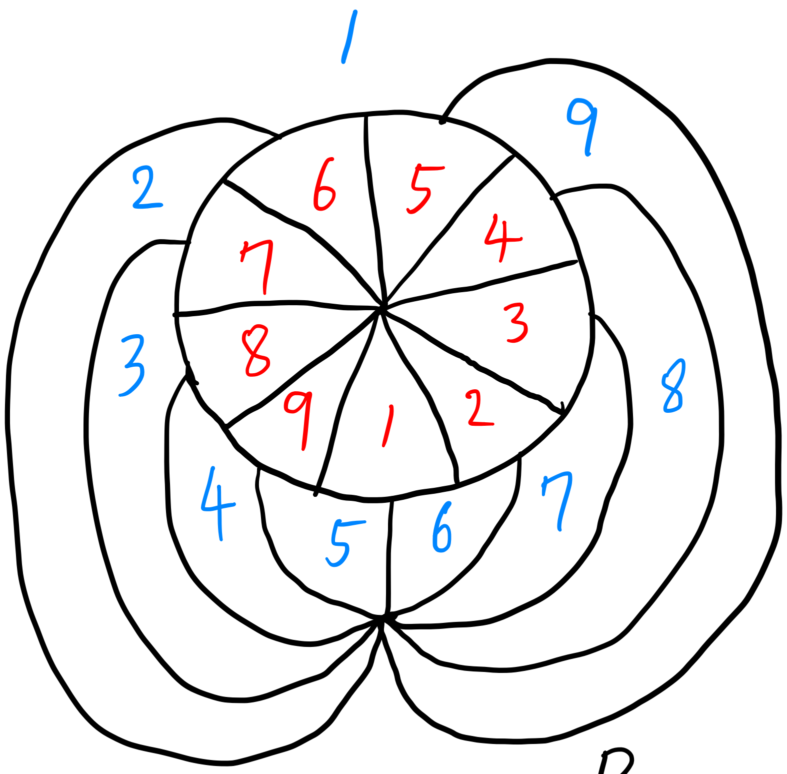
$u(f_{2n+1}) = 2n.$

Sketch proof

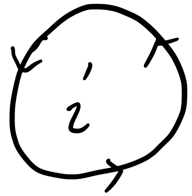
∘ $u(f_{2n+1}) \leqq 2n$



∘ $u(f_{2n+1}) \geqq 2n$ (Case $n = 4$)



P_9



region cycle

$i \in \mathbb{Z}/9\mathbb{Z}$

$f \in SE(P_9)$

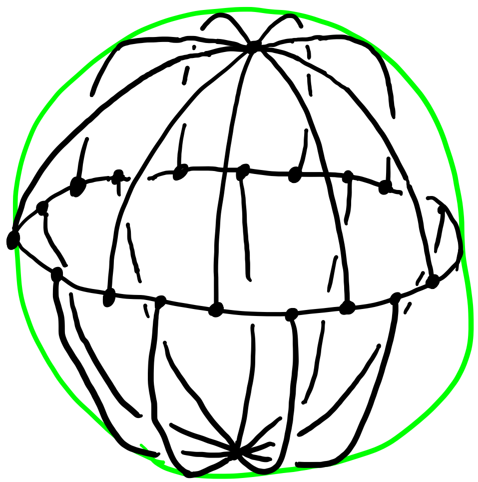
$$\mathcal{L}(f) := (l_1(f), l_2(f), l_3(f), l_4(f)) \in \mathbb{Z}^4$$

$$l_1(f) := \sum_{i=1}^{9} lk(f(i), f(i))$$

$$l_2(f) := \sum_{i=1}^{9} lk(f(i), f(i+1)) + \sum_{i=1}^{9} lk(f(i), f(i-1))$$

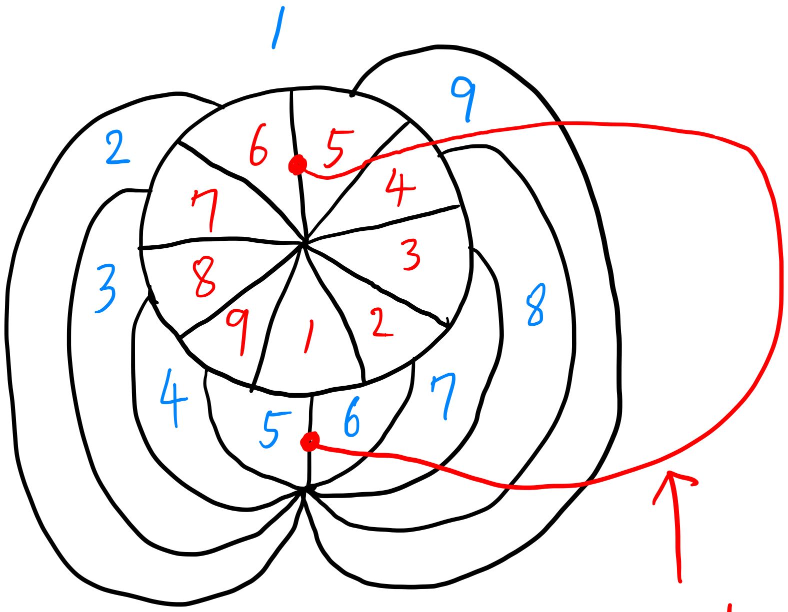
$$l_3(f) := \sum_{i=1}^{9} lk(f(i), f(i+2)) + \sum_{i=1}^{9} lk(f(i), f(i-2))$$

$$l_4(f) := \sum_{i=1}^{9} lk(f(i), f(i+3)) + \sum_{i=1}^{9} lk(f(i), f(i-3))$$



$P_9 \subseteq S^2$

$$\mathcal{L}(f_9) = (9, 0, 0, 0)$$



$lk(f(5), f(5)) : \pm 1$
$lk(f(6), f(6)) : \pm 1$ } $l_1(f) : \pm 2$

$lk(f(5), f(6)) : \mp 1$
$lk(f(6), f(5)) : \mp 1$ } $l_2(f) : \mp 2$

$\therefore \mathcal{L}(f) : \pm(2, -2, 0, 0)$

crossing change between these 2 edges

$$B_9 = \{ (2,0,0,0), (0,2,0,0), (0,0,2,0), (0,0,0,2), (1,1,0,0), (0,1,1,0), (0,0,1,1), (1,-1,0,0), (0,1,-1,0), (0,0,1,-1), (2,-2,0,0), (1,-2,1,0), (0,1,-2,1), (0,0,1,-2), (0,0,0,1) \}$$

By a purely combinatorial argument, we can show :

$$\forall \varepsilon_i \in \{-1, 0, 1\}, \ \forall a_i \in B_9 \ , \ i = 1, 2, \cdots, 7,$$

$$\sum_{i=1}^{7} \varepsilon_i a_i \neq (9, 0, 0, 0)$$

$$\therefore \ u(f_9) > 7 \ /\!/$$

$$(9, 0, 0, 0) = 4(2, 0, 0, 0) + (1, 1, 0, 0) - (0, 1, 1, 0) + (0, 0, 1, 1) - (0, 0, 0, 1)$$

Problem Find relations between $c(f)$ and $u(f)$!

planar graph G is trivializable $\overset{\text{def.}}{\Longleftrightarrow}$ G has no knotted projections

Prop. G : trivializable

$${}^{\forall} f \in SE(G), \quad u(f) \leqq \frac{1}{2} c(f)$$

- $$\lim_{n\to\infty} \frac{u(f_{2n+1})}{c(f_{2n+1})} = \lim_{n\to\infty} \frac{2n}{2n+1} = 1$$

Thm. ${}^{\forall} G$: planar graph, ${}^{\exists} A, B \in \mathbb{R}$ s.t.

$${}^{\forall} f \in SE(G), \quad u(f) \leqq A \cdot c(f) + B$$

Find relations between $c(f)$ and $u(f)$! What does it mean?

○ $\mathcal{K}$: the set of all knots, X : set

$f : \mathcal{K} \to X$: knot invariant

Problem! Decide $f(\mathcal{K}) \subseteq X$

Ex. $\Delta : \mathcal{K} \to \mathbb{Z}[t^{\pm 1}]$

$\quad\quad K \longmapsto \Delta_K(t)$: Alexander Polynomial

$$\Delta(\mathcal{K}) = \{ P(t) \in \mathbb{Z}[t^{\pm 1}] \mid P(t^{-1}) = P(t),\ P(1) = 1 \}$$

(Alexander)

X, Y : sets , $f : \mathcal{K} \to X$, $g : \mathcal{K} \to Y$: knot invariants

$$(f, g) : \mathcal{K} \to X \times Y$$
$$K \mapsto (f(K), g(K))$$

Problem 2 Decide $(f, g)(\mathcal{K}) \subseteq X \times Y$

"relation between f and g"

Thm [T. Sakai], [H. Kondo]

$$\Delta(\mathcal{K}) = \Delta(\{K \in \mathcal{K} \mid u(K) = 1\})$$

$$\Longleftrightarrow \quad (u, \Delta)(\mathcal{K}) \cap \{1\} \times \mathbb{Z}[t^{\pm 1}] = \{1\} \times \Delta(\mathcal{K})$$

- $c : \mathcal{K} \to \mathbb{Z}_{\geqq 0}$
  $$\begin{array}{ccc} \mathcal{K} & \to & \mathbb{Z}_{\geqq 0} \\ \Psi & & \Psi \\ K & \mapsto & c(K) \end{array}$$
  $c(K)$: crossing number of K

- $u : \mathcal{K} \to \mathbb{Z}_{\geqq 0}$
  $$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathbb{Z}_{\geqq 0} \\ \Psi & & \Psi \\ K & \longmapsto & u(K) \end{array}$$
  $u(K)$: unknotting number of K

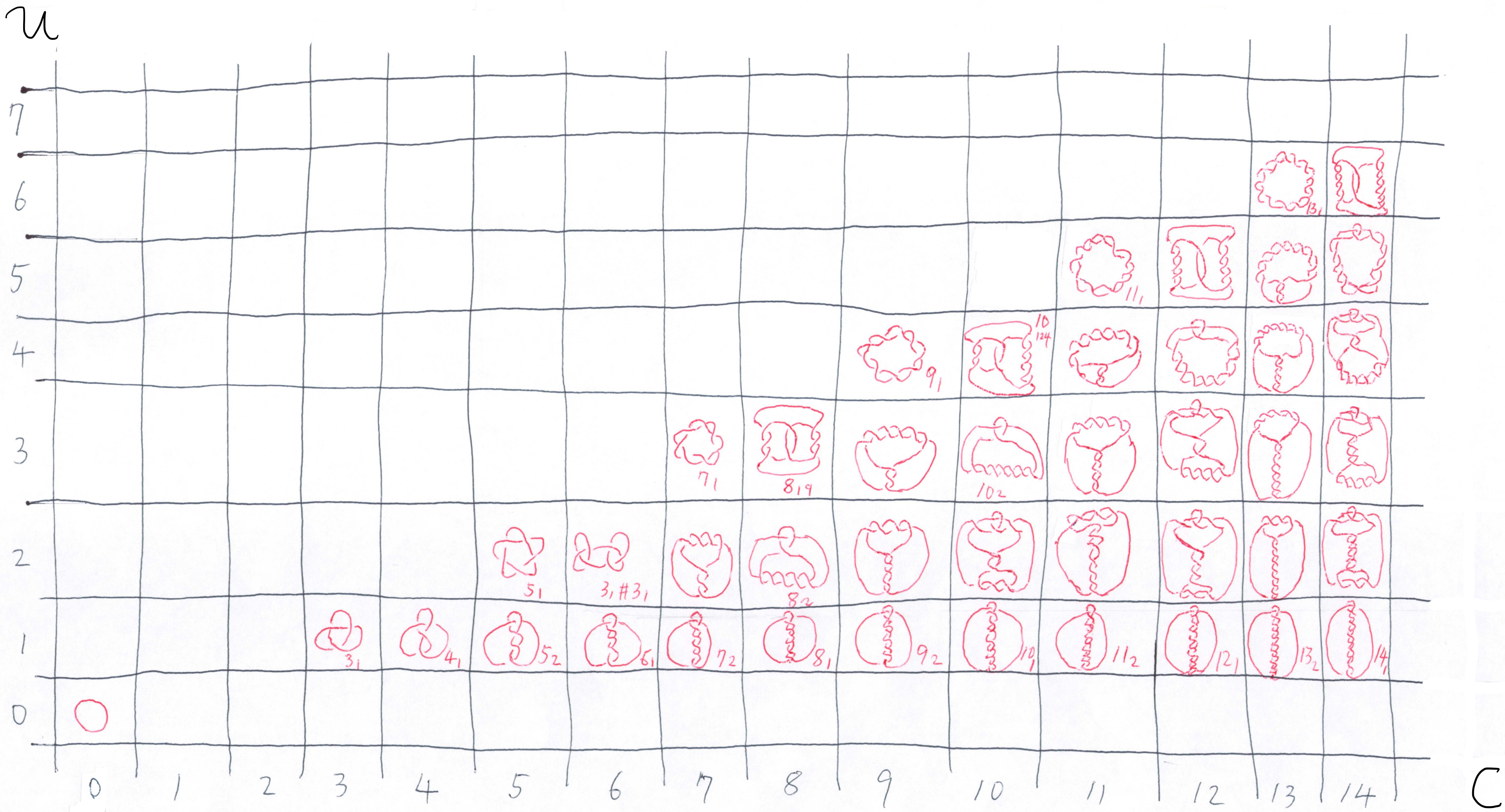
- $\text{braid}-1 : \mathcal{K} \to \mathbb{Z}_{\geqq 0}$
  $$\begin{array}{ccc} \mathcal{K} & \to & \mathbb{Z}_{\geqq 0} \\ \Psi & & \Psi \\ K & \longmapsto & \text{braid}(K)-1 \end{array}$$

- $\text{bridge}-1 : \mathcal{K} \to \mathbb{Z}_{\geqq 0}$
  $$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathbb{Z}_{\geqq 0} \\ \Psi & & \Psi \\ K & \longmapsto & \text{bridge}(K)-1 \end{array}$$

o(c, u)

Thm (Almost Folklore)

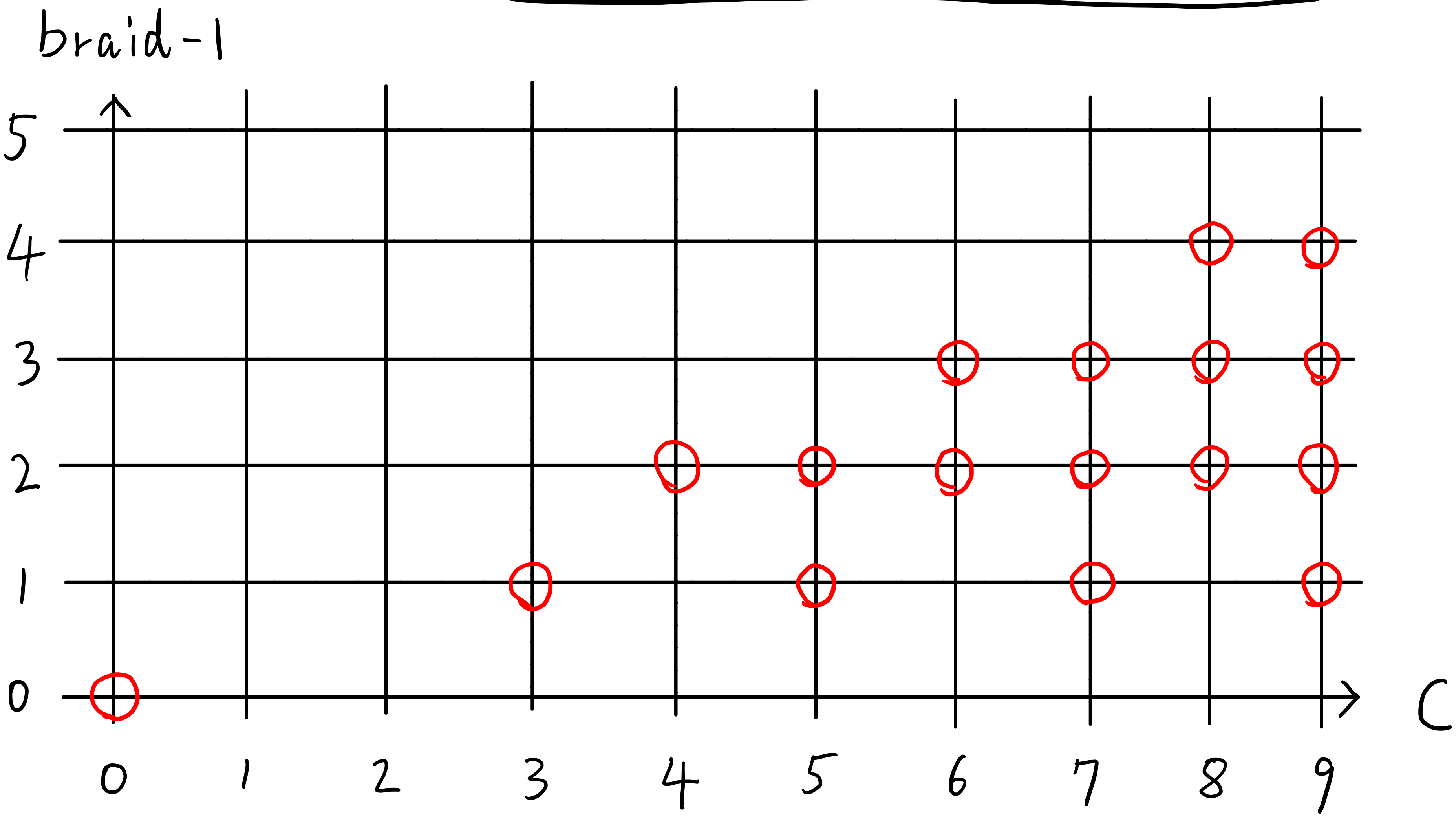
$$(c,u)(K) = \{(0,0)\} \cup \{(x,y) \in \mathbb{Z}^2_{>0} \mid y \leq \tfrac{1}{2}(x-1)\}$$



∘ (c, braid-1)

Thm [Ohyama] $^\forall K \in \mathcal{K}$

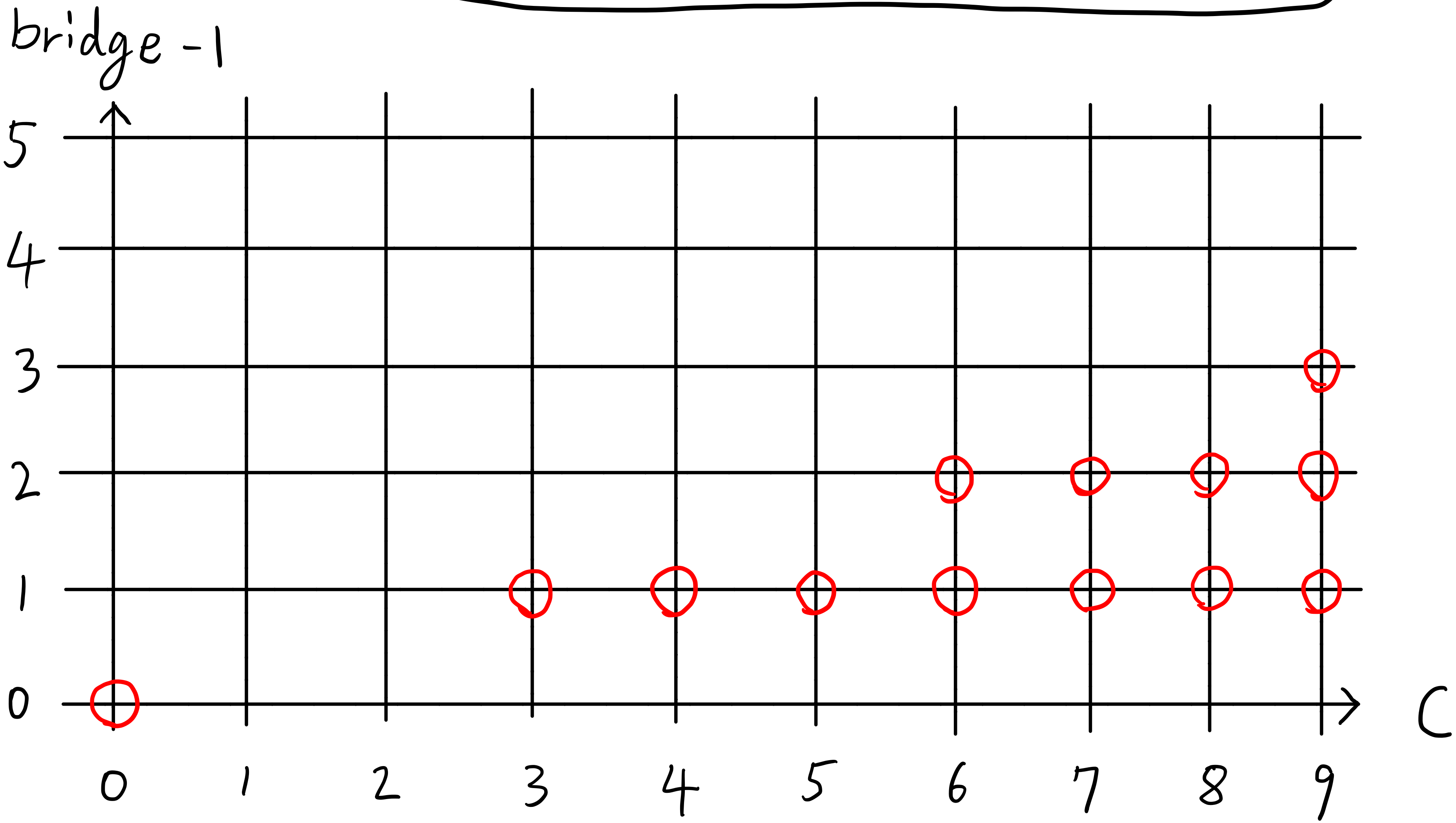
$$\text{braid-1}(K) \leqq \frac{1}{2} c(K)$$



∘ (c, bridge −1)

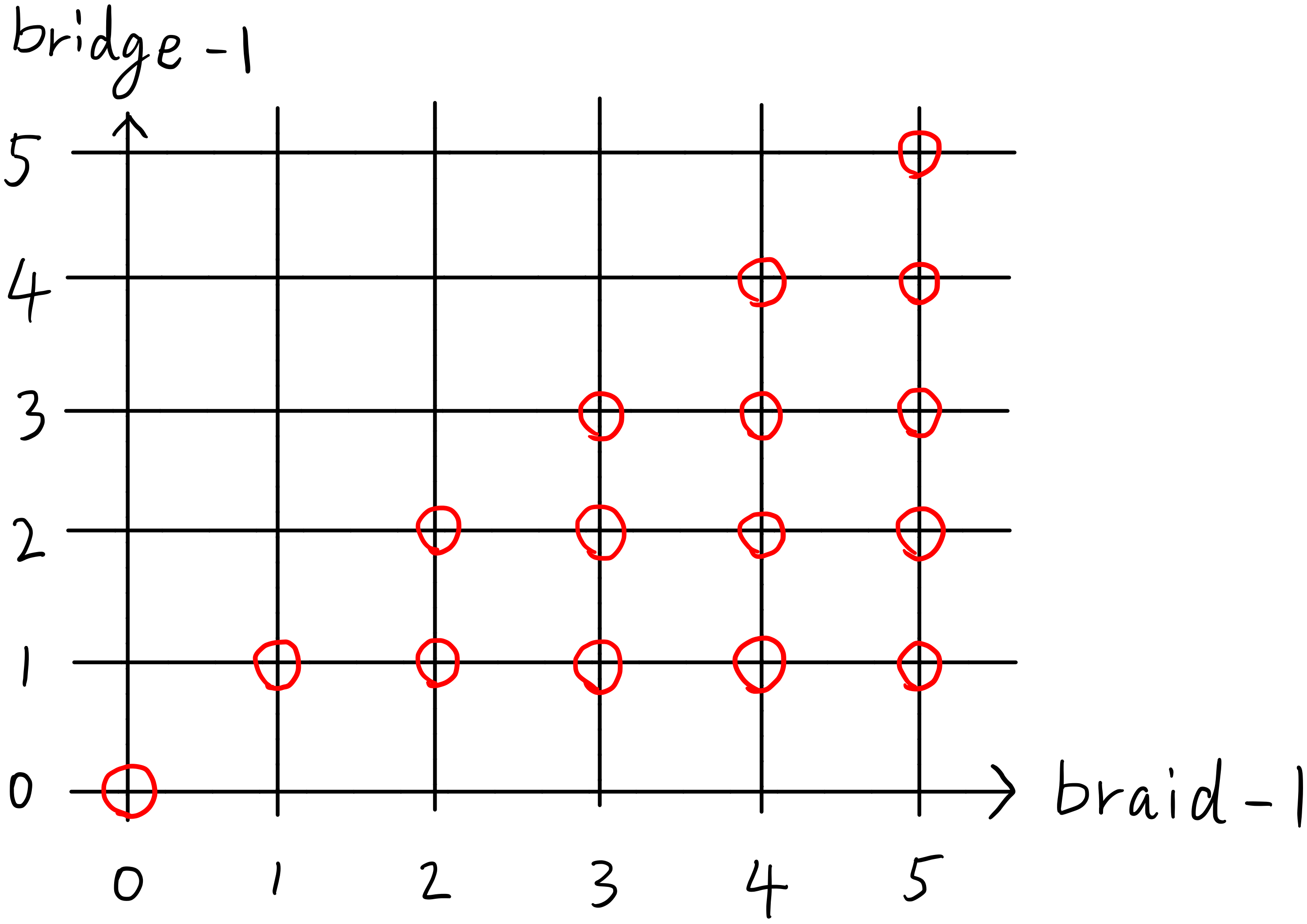
Fox conjecture $\forall K \in \mathcal{K}$

$$\text{bridge}-1\,(K) \leqq \frac{1}{3}\, c\,(K)$$

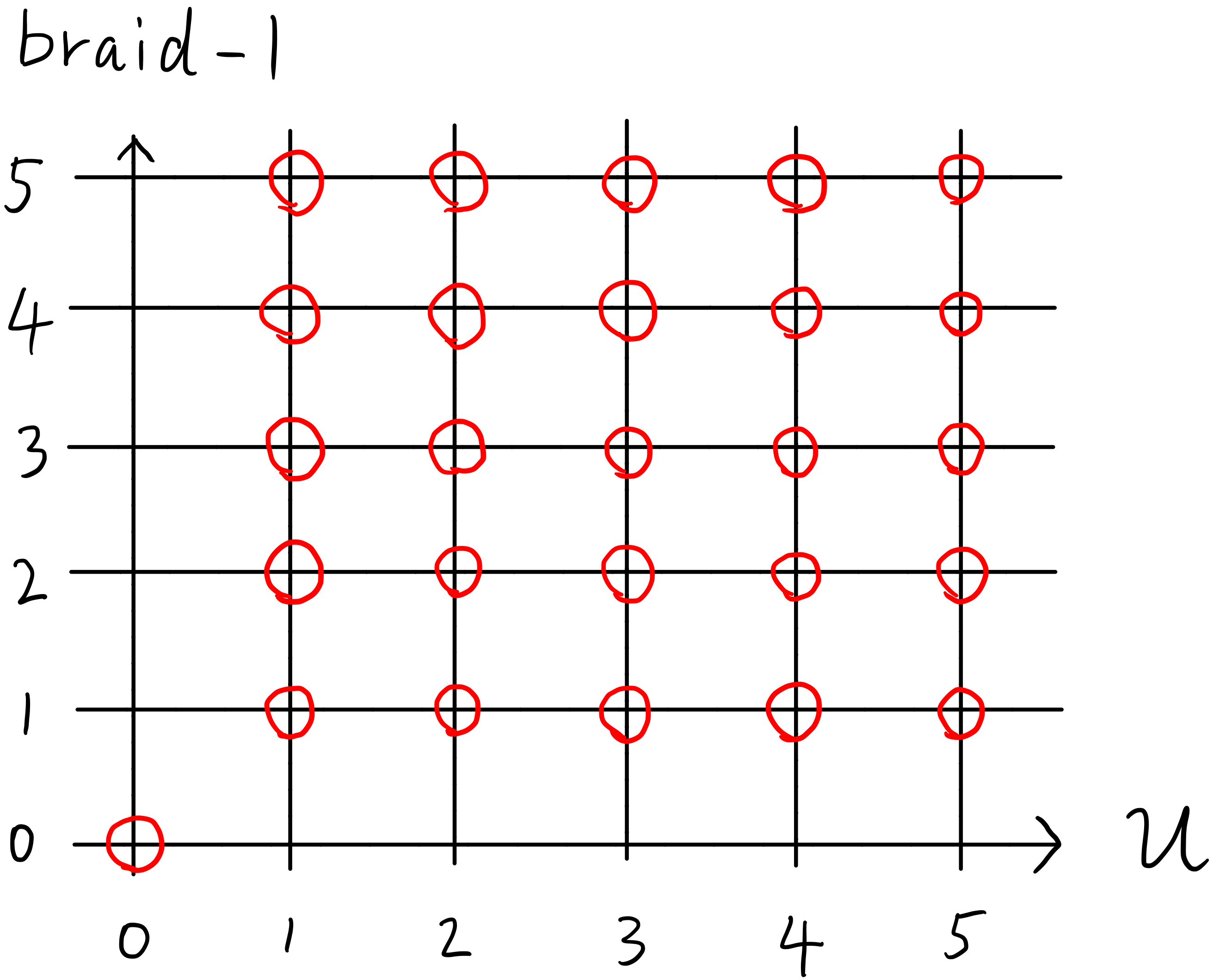


• (braid-1, bridge-1)

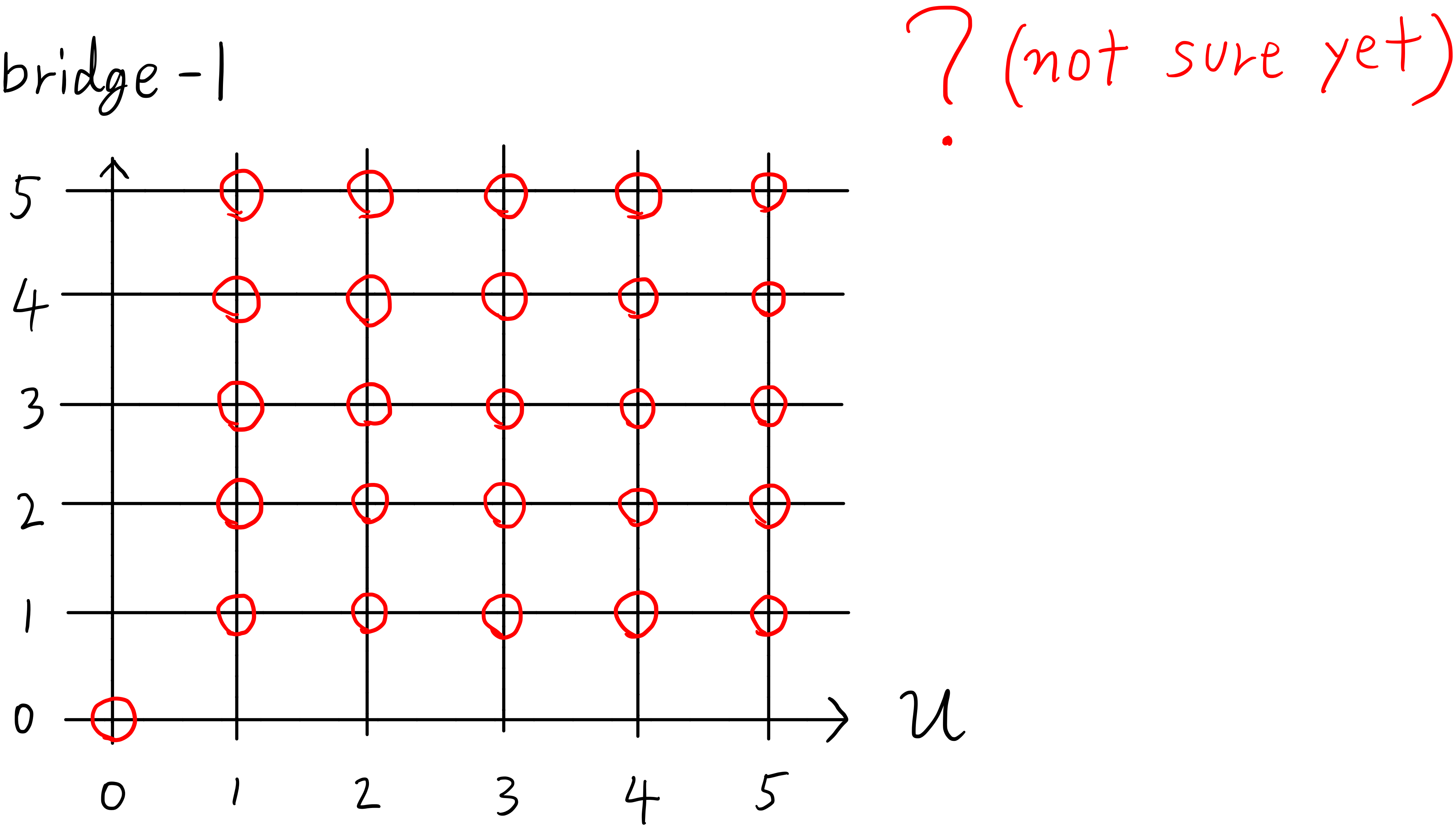
$$\text{bridge-1}(K) \leq \text{braid-1}(K)$$



∘ (u , braid-1)



• (u , bridge - 1)



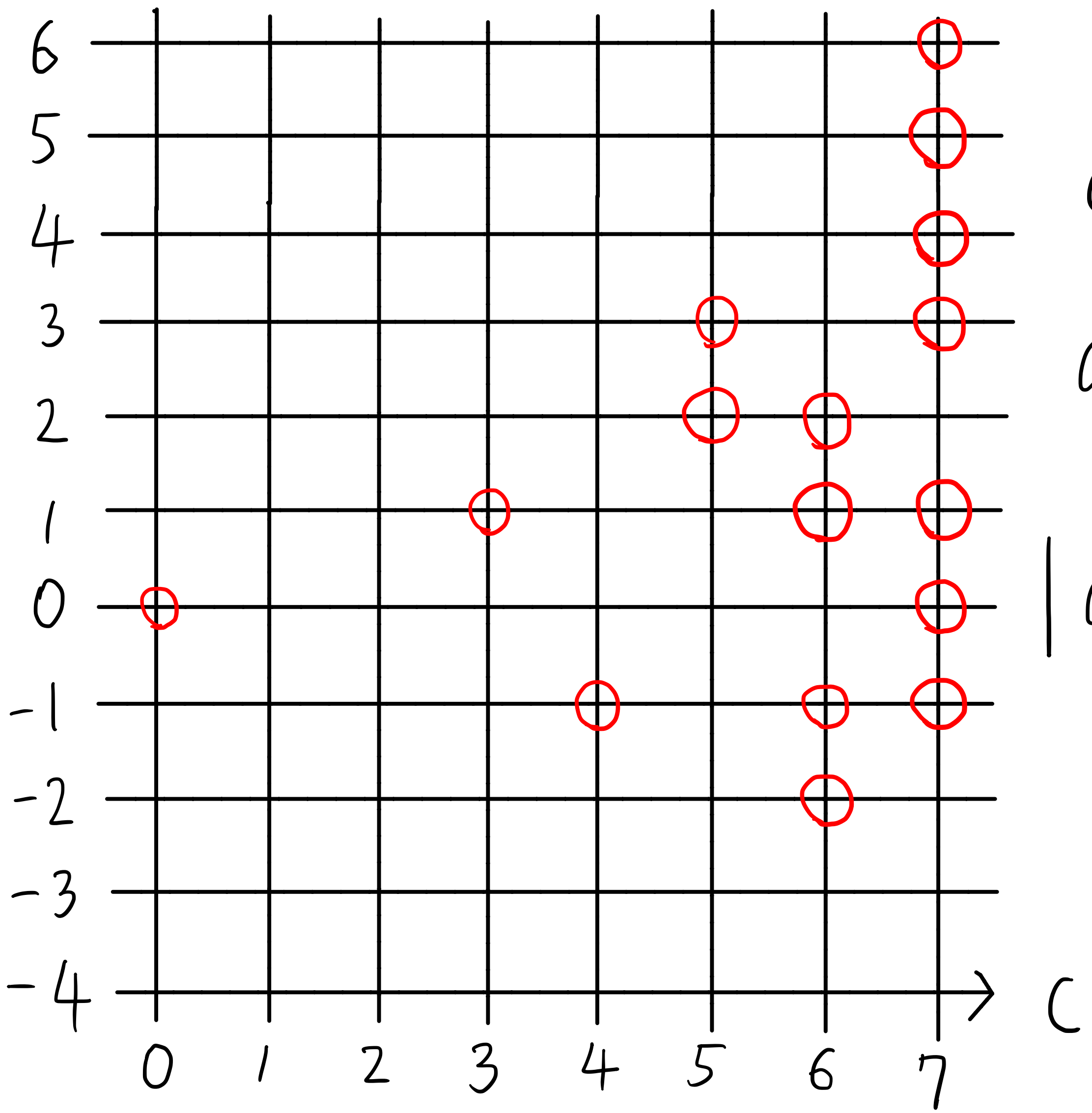
Thm. ${}^\forall f : \mathcal{K} \to \mathbb{R}$: knot invariant,

${}^\exists \varphi : \mathbb{Z}_{\geqq 0} \to \mathbb{R}$: map s.t.

${}^\forall K \in \mathcal{K}$, $f(K) \leqq \varphi(c(K))$

(∵) $\varphi(m) := \max\{f(K) \mid c(K) = m\}$ //

$$(c, a_2)(K) \subseteq \mathbb{Z}_{\geqq 0} \times \mathbb{Z}$$



$$c\big((2, 2n+1)\text{-torus knot}\big) = 2n+1$$

$$a_2\big((2, 2n+1)\text{-torus knot}\big) = \frac{n(n+1)}{2}$$

$$|a_2(K)| \overset{?}{\leqq} \frac{c(K)^2 - 1}{8}$$

$c : SE(G) \to \mathbb{Z}_{\geqq 0}$, $u : SE(G) \to \mathbb{Z}_{\geqq 0}$

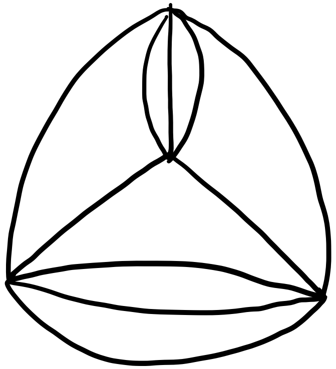
$(c, u) : SE(G) \to \mathbb{Z}^2_{\geqq 0}$

Prop. G : planar graph with a cycle

$\Rightarrow$ $(c, u)(SE(G)) \geqq (c, u)(\mathcal{K}) = (c, u)(SE(\bigcirc))$

Ex. $^{\exists}G$: planar graph $G \geqq \bigcirc\,\bigcirc$ s.t.

$(c, u)(SE(G)) \not\geqq (c, u)(SE(\bigcirc\bigcirc))$

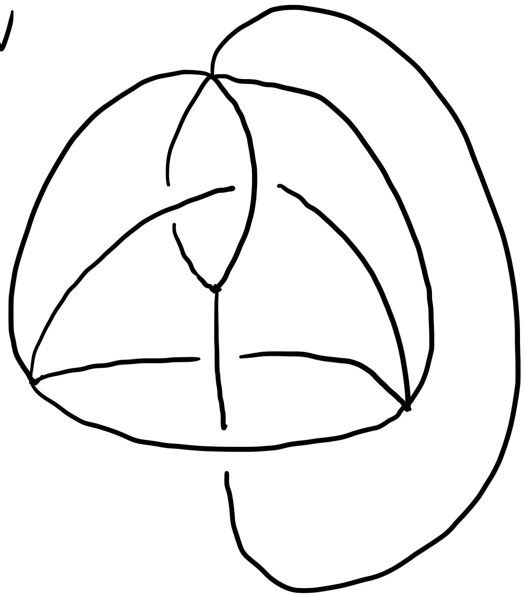


G

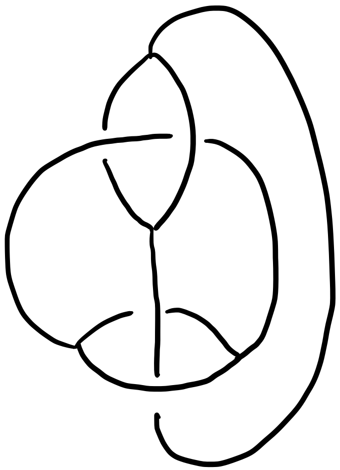
$^{\forall} f \in SE(G)$, $c(f) \neq 2$

$\therefore (c,u)(SE(G)) \not\ni (2,1) \in (c,u)(SE(\bigcirc\bigcirc))$

$(c(f), u(f)) = (2,1)$



$\cong$

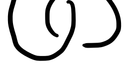


$c(g) = 4$

$u(g) = 2$

$c = 4$

$u = 2$

$\because$ contains 6  s

$\therefore u = 2$

$u \leqq \frac{1}{2} c$

$\therefore c = 4$

Thank you very much

for your listening.