Virtualized n-gon moves for virtual knots

守田　夏希　(奈良女子大学)*

概　要

Nakamura-Nakanishi-Satoh-Wada [2] introduced a local deformation called virtualized Δ-move for virtual links, and proved that it is an unknotting operation for virtual knots. In this paper, we introduce virtualized n-gon move as a generalization of virtualized Δ-move. We show that virtualized n-gon move is an unknotting operation for virtual knots when $n \geq 3$, and give a lower bound for the unknotting number, which we call the $v[n]$-unknotting number, in terms of *odd writhes*. This is a joint work with Yeonhee Jang (Nara Women's University).

1. Introduction

Definition 1.1. A *virtualized Δ-move* (or a *$v\Delta$-move*) is a local deformation on a virtual link diagram as shown in Figure 1. We denote it by $v\Delta$ in figures.

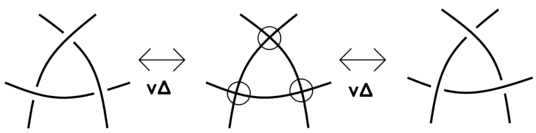

Figure 1:

Two virtual links L and L' are *$v\Delta$-equivalent* to each other if their diagrams are related by a finite sequence of $v\Delta$-moves and generalized Reidemeister moves.

Theorem 1.2. *[2, Theorem 1.3] Any two virtual knots are $v\Delta$-equivalent to each other. In particular, the $v\Delta$-move is an unknotting operation for virtual knots.*

Definition 1.3. Let $n \geq 2$ be an integer. A *virtualized n-gon move* (or a *$v[n]$-move*) is a local deformation on a virtual link diagram as shown in Figure 2. We denote it by $v[n]$ in figures.

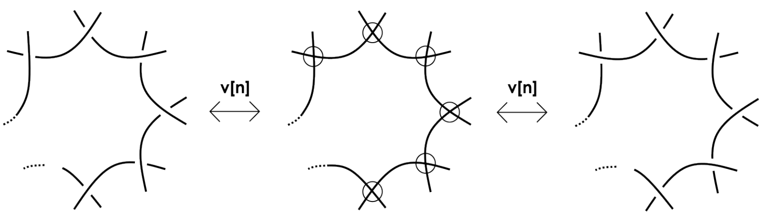

Figure 2:

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*e-mail: `xan_morita@cc.nara-wu.ac.jp`

Two virtual links L and L' are $v[n]$-*equivalent* to each other if their diagrams are related by a finite sequence of $v[n]$-moves and generalized Reidemeister moves.

Proposition 1.4. *Any two virtual knots are $v[n]$-equivalent to each other. In particular, the $v[n]$-move is an unknotting operation for virtual knots.*

Let $u_{v[n]}(K)$ be the minimal number of $v[n]$-moves which is needed to deform a virtual knot K into the trivial knot.

Theorem 1.5. *For any integers $n \geq 3$ and $m \geq 1$, there exists an infinite family $\{K_s\}$ of virtual knots such that $u_{v[n]}(K_s) = m$.*

2. $v[n]$-unknotting number and writhe invariant

Fact 2.1. *A $v[2]$-move and a crossing change are equivalent local deformations, that is, they can be realized by each other (see Figure 3).*

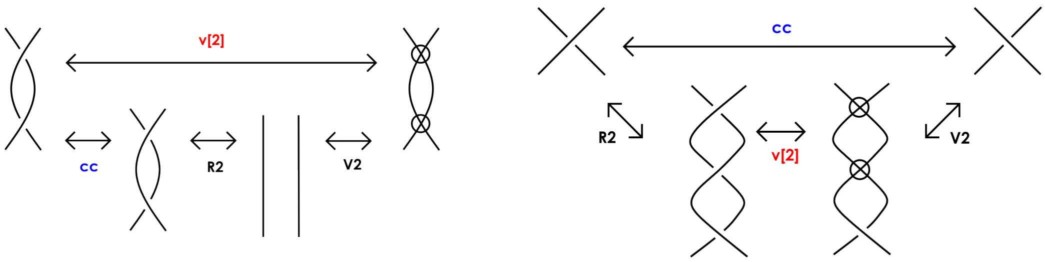

Figure 3:

Fact 2.2. *The $v[3]$-move is the same local deformation as $v\Delta$-move.*

Proposition 2.3. *A $v[n]$-move is realized by a $v[n+1]$-move for any $n \geq 2$.*

Proof. Figure4 illustrates how this can be accomplished. □

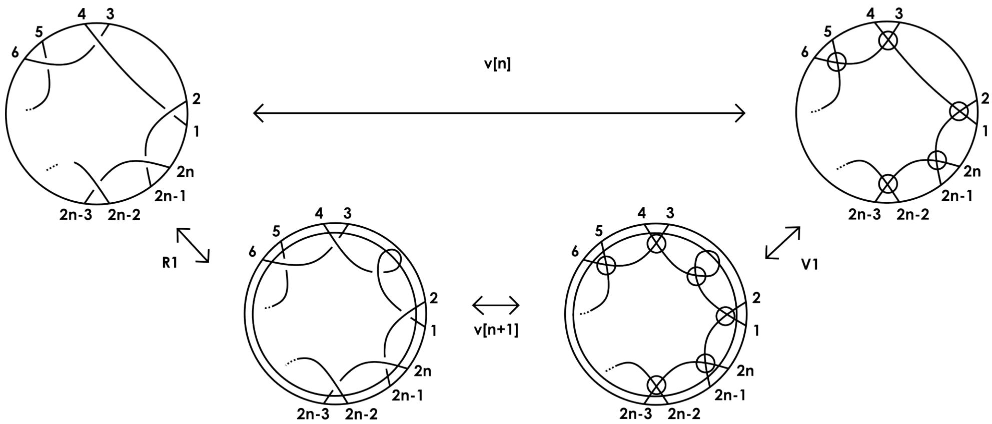

Figure 4:

Proof of Proposition 1.4. It follows from Theorem 1.2 and Proposition 2.3. □

Definition 2.4. For virtual knots K and K', we denote by $d_{v[n]}(K, K')$ the minimal number of $v[n]$-moves needed to deform a diagram of K into one of K'.

By Proposition 2.3, we obtain the following corollary.

Corollary 2.5. *For any virtual knots K and K', we have $d_{v[n]}(K, K') \geq d_{v[n+1]}(K, K')$. In particular, $u_{v[n]}(K) \geq u_{v[n+1]}(K)$.*

Satoh-Taniguchi [4] introduced the *k-writhe* $J_k(D)$ of a virtual knot diagram D for each $k \in \mathbb{Z}$, defined by

$$J_k(D) := \sum_{\mathrm{Ind}(c)=k} \mathrm{sgn}(c).$$

Lemma 2.6. *[4, Lemma 2.3] If D and D' are virtual knot diagrams related by a finite sequence of generalized Reidemeister moves, then $J_k(D) = J_k(D')$ for any $k \neq 0$.*

Hence, $J_k(K)$ is well-defined for a virtual knot K. Also, the *odd writhe* $J(K)$ is

$$J(K) := \sum_{k:\mathrm{odd}} J_k(K).$$

Proposition 2.7. *Let K and K' be virtual knots. Then we have the following.*

(1)

$$d_{v[n]}(K, K') \geq \begin{cases} \dfrac{1}{n}|J(K) - J(K')| & (n : \textit{even}), \\ \dfrac{1}{n-1}|J(K) - J(K')| & (n : \textit{odd}). \end{cases}$$

(2)

$$u_{v[n]}(K) \geq \begin{cases} \dfrac{1}{n}|J(K)| & (n : \textit{even}), \\ \dfrac{1}{n-1}|J(K)| & (n : \textit{odd}). \end{cases}$$

Proof. (1) Suppose that $d_{v[n]}(K, K') = m$. Then there exists a sequence of virtual knots

$$K = K_0 \to K_1 \to \cdots \to K_{m-1} \to K_m = K'$$

such that K_i is obtained from K_{i-1} by a single $v[n]$-move for each $i = 1, \ldots, m$. Let $G_0, \ldots, G_m$ be the Gauss diagrams of $K_0, \ldots, K_m$, respectively. We can see that G_i is obtained from G_{i-1} by removing n chords corresponding to n real crossings involved in the $v[n]$-move (see Figure 5). Note that the indices of the other chords are changed by even numbers and the signs do not change. Thus, when n is even, we have

$$|J_*(K_0) - J_*(K_1)| \leq n, \ldots, |J_*(K_{m-1}) - J_*(K_m)| \leq n,$$

and hence

$$|J_*(K) - J_*(K')| \leq mn = d_{v[n]^o}(K, K') \cdot n. \tag{1}$$

When n is odd, note that the number of the chords with odd indices among the n chords is at most $n-1$ since the sum of the indices of the n chords is 0. Thus, we have

$$|J_*(K_0) - J_*(K_1)| \leq n-1, \ldots, |J_*(K_{m-1}) - J_*(K_m)| \leq n-1,$$

and hence

$$|J_*(K) - J_*(K')| \leq m(n-1) = d_{v[n]^o}(K, K') \cdot (n-1). \tag{2}$$

The equalities (1) and (2) imply the desired result.

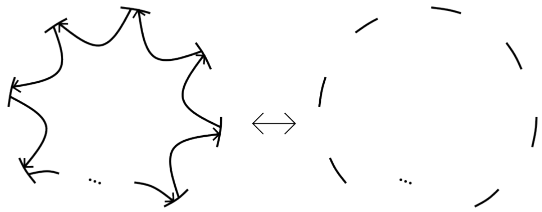

Figure 5:

(2) The inequality follows from (1) together with the fact that $J_*(O) = 0$, where O is the trivial knot. $\square$

3. Proof of Theorem 1.5

In the following, we will construct a family of infinitely many virtual knots K_s with $u_{v[n]}(K_s) = m$.

Let $n \geq 3$ and $m \geq 1$ be integers, and s be a positive odd integer. We consider the virtual knot diagram K_s and its Gauss diagram as shown in Figure 6 and Figure 7, respectively. (The Gauss diagram of K_s is shown in Figure 7 when m=2. In this Gauss diagram, the signs of chords are all $+1$, and the numbers inside the circle indicate indices of chords.) The contents of the box with $(*)$ will have a different shape depending on whether n is odd or even. When n is odd, this diagram have $nm + s + \frac{n-1}{2}$ real crossings $a_{1,1}, \ldots, a_{m,n}$, $b_1, \ldots, b_s$ and $c_1, \ldots, c_{\frac{n-1}{2}}$. When n is even, this diagram have $nm + s + \frac{n-2}{2}$ real crossings $a_{1,1}, \ldots, a_{m,n}$, $b_1, \ldots, b_s$ and $c_1, \ldots, c_{\frac{n-2}{2}}$.

Figure 8 and Figure 9 are examples of K_s in the case of $m = 3, n = 4$ and $m = 3, n = 5$, respectively.

Then we prove that the family $\{K_s\}_{s \in \mathbb{N}}$ satisfies the following.

Claim 1. $u_{v[n]}(K_s) \leq m$.

Claim 2. $u_{v[n]}(K_s) \geq m$.

Claim 3. $K_s \neq K_{s'}$ if and only if $s \neq s'$.

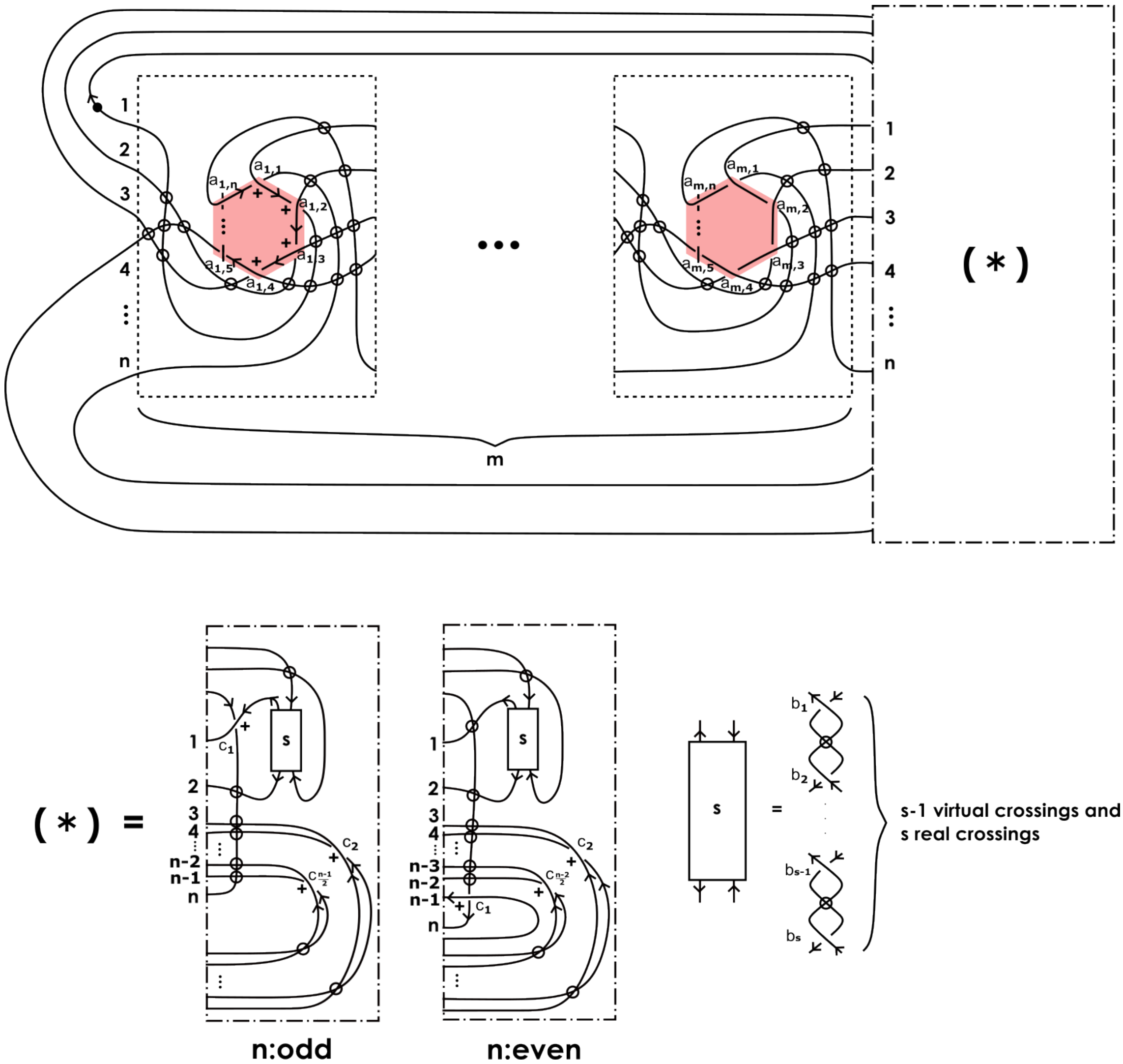

Figure 6:

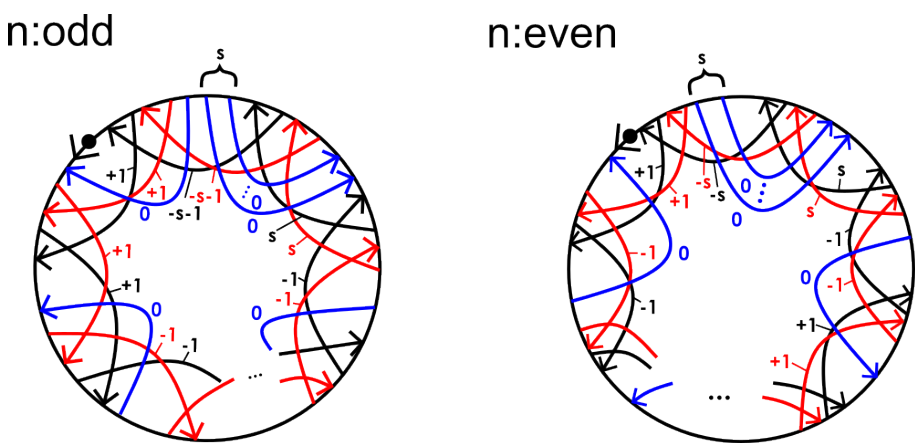

Figure 7: Gauss diagrams of K_s (when m=2)

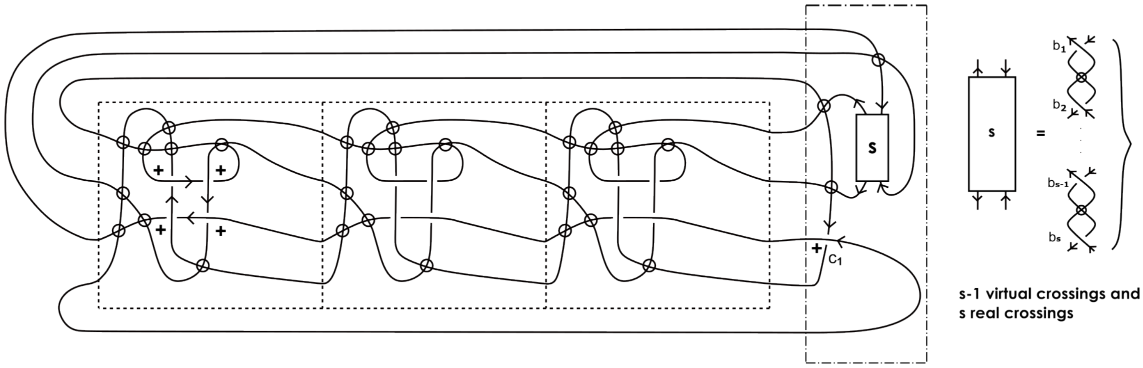

Figure 8:

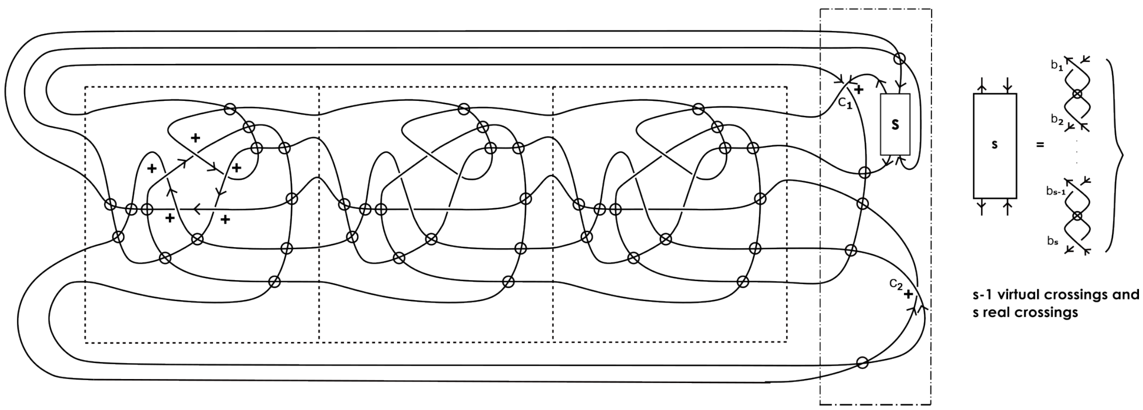

Figure 9:

Proof of Claim 1. The virtual knot K_s can be deformed into the trivial knot after m times of $v[n]$-moves are applied. Figure 10 illustrates how this can be accomplished when $n = 3$. The cases when $n \geq 4$ can be treated similarly, where the last part of Figure 10 is replaced by Figure 11 or Figure 12 according to whether n is odd or even. □

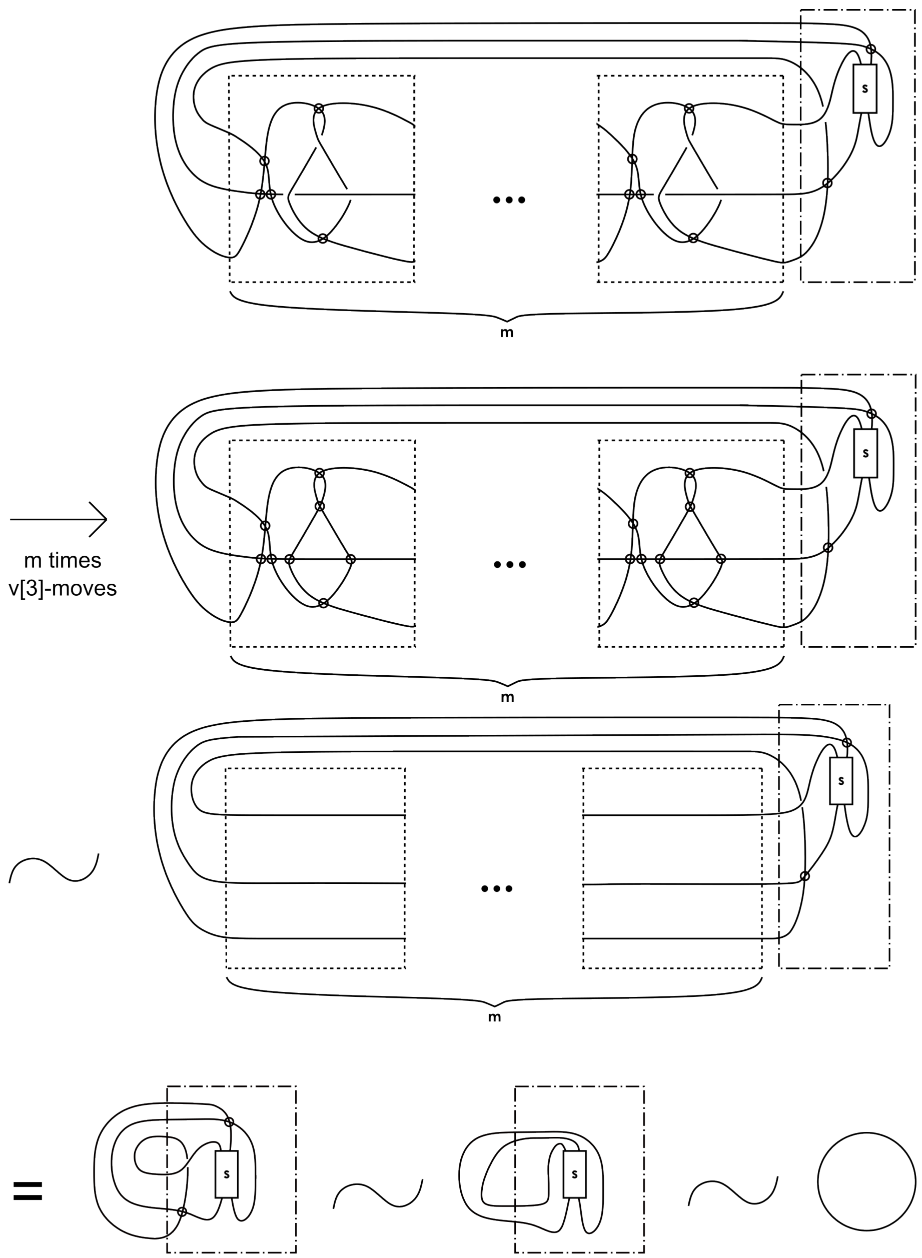

Figure 10:

Proof of Claim 2. When n is odd, we can see that

$$
\begin{aligned}
&J_1(K_s) = \frac{n-1}{2}m, \quad J_{-1}(K_s) = \frac{n-3}{2}m, \quad J_s(K_s) = m, \quad J_{-s-1}(K_s) = m, \\
&J_k(K_s) = 0 \quad \text{if} \quad k \neq 1, -1, s, -s-1, 0,
\end{aligned}
\tag{3}
$$

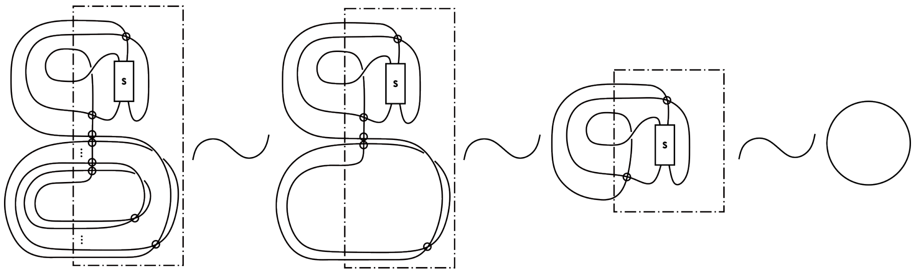

Figure 11:

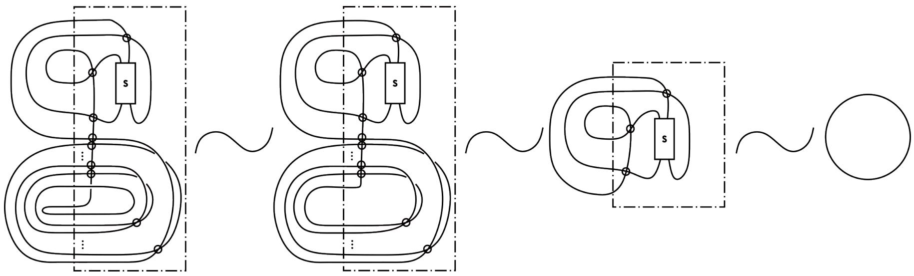

Figure 12:

and hence the odd writhe of K_s is

$$J(K_s) = \frac{n-1}{2}m + \frac{n-3}{2}m + m = (n-1)m.$$

Hence we have $u_{v[n]}(K_s) \geq \frac{1}{n-1} \cdot (n-1)m = m$ by Proposition 2.7.

When n is even, we can see that

$$\begin{aligned} &J_1(K_s) = \frac{n-2}{2}m, \quad J_{-1}(K_s) = \frac{n-2}{2}m, \quad J_s(K_s) = m, \quad J_{-s}(K_s) = m, \\ &J_k(K_s) = 0 \quad \text{if} \quad k \neq 1, -1, s, -s, 0, \end{aligned} \tag{4}$$

and hence the odd writhe of K_s is

$$J(K_s) = \frac{n-2}{2}m + \frac{n-2}{2}m + m + m = nm.$$

Hence we have $u_{v[n]}(K_s) \geq \frac{1}{n} \cdot nm = m$ by Proposition 2.7. □

Proof of Claim 3. By the equalities (3), (4) and Lemma 2.6, we have the conclusion. □

4. Remark

In the talk in the conference, we gave a lower bound for $d_{v[n]}(K, K')$ in terms of "non-zero" writhe. However, we realized that we need to be careful with orientation. Proposition 2.7 in this article is a corrected version.

In the following, we introduce an oriented version of virtualized n-gon move, which is related with non-zero writhe. This is called an *oriented virtualized n-gon move* (or a $v[n]^o$-*move*) and is a local deformation on an oriented virtual link diagram as shown in Figure 13. We denote it by $v[n]^o$ in figures. This move is a generalization of $v\Delta^o$-*move* defined in [3].

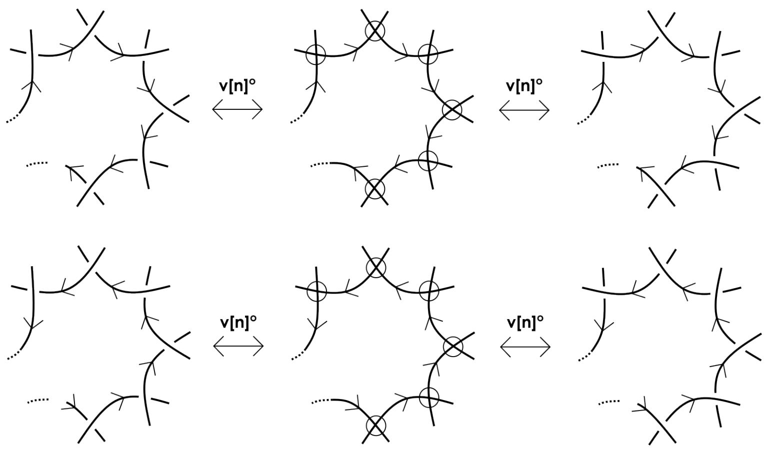

Figure 13:

The *non-zero writhe* $J_*(K)$ is defined by

$$J_*(K) := \sum_{k \neq 0} J_k(K).$$

Proposition 4.1. *Let K and K' be oriented virtual knots. Then we have the following.*

(1) $d_{v[n]^o}(K, K') \geq \dfrac{1}{n}|J_*(K) - J_*(K')|.$

(2) $u_{v[n]^o}(K) \geq \dfrac{1}{n}|J_*(K)|.$

Proof. This can be proved by arguments similar to the proof of Proposition 4.1. Note that it can be easily seen that the indices and signs of any other chords are preserved by the $v[n]^o$-move (see Figure 14, which shows the change of Gauss diagram corresponding to a $v[n]^o$-move). $\square$

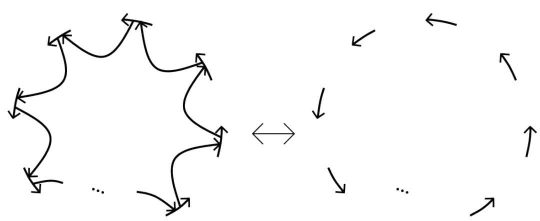

Figure 14:

Then, for the oriented virtual knot K_s constructed in the previous section, we can see that $u_{v[n]^o}(K_s) = m$, where $u_{v[n]^o}(K_s)$ is the minimal number of $v[n]^o$-moves needed to deform K_s into the trivial knot. The proof is similar to that for Proposition 2.7, where we can use Proposition 4.1 to show $u_{v[n]^o}(K_s) \geq m$. We remark that s can also be an even number in the oriented case.

謝辞

研究集会「結び目の数理 VII」において公演の機会を与えてくださり、運営をしてくださった世話人の谷山公規先生、安原晃先生、山口祥司先生、丹下稜斗先生に深く感謝いたします。また、この発表に関して質問やコメントをくださった聴衆の方々にも深く感謝申し上げます。

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