グラフゼータおよびねじれアレクサンダー多項式のいくつかの表示

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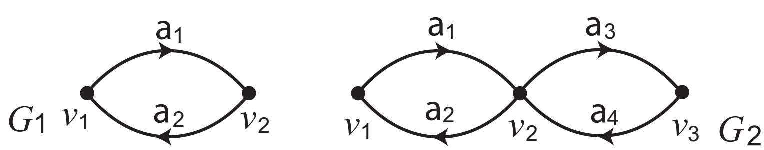
ABSTRACT. 本講演では行列重み付きグラフのゼータ関数に関する結果と，そこから得られる双曲結び目の体積公式について報告しました.

1. GRAPHS

A digraph $G = (V, A)$ is a pair of a set V and a multiset A, where A consists of ordered pairs $a = (u, v)$ of elements $u, v \in V$. An element of A is called a directed edge or an arc of G. If $a = (u, v)$ is an arc of G, then u is called the tail of a, and v the head of a, denoted



by $\mathfrak{t}(a)$ and $\mathfrak{h}(a)$ respectively. If a sequence $c = (a_1 a_2 \cdots a_\ell)$ satisfies $\mathfrak{h}(a_i) = \mathfrak{t}(a_{i+1})$ for all $i = 1, 2, \dots, \ell - 1$, then c is called a path of G, and ℓ is called the length of c, denoted by $|c|$. Further, if $\mathfrak{h}(a_\ell) = \mathfrak{t}(a_1)$, c is said to be closed. Let C be the set of closed paths of G, and C_ℓ the set of closed paths of length ℓ. A prime closed path is a closed path which cannot be written in the form d^k for a shorter $d \in C$. We denote by P the set of prime closed paths of G.

Let $c = (a_1 a_2 \cdots a_\ell) \in C$. The cyclic rearrangement $(a_2 a_3 \cdots a_\ell a_1), (a_3 a_4 \cdots a_1 a_2), \dots,$ $(a_\ell a_1 \cdots a_{\ell-2} a_{\ell-1})$ of c is also a closed path of lenght ℓ. We write $c \sim d$ if d is a cyclic rearrangement of c. This binary relation $\sim$ is an equivalence relation on C. We call it the cyclic equivalence. Let $[C] = C/\sim$ and $[P] = P/\sim$. An element of $[C]$ ($[P]$ resp.) is called a cycle (prime cycle resp.).

2. IHARA ZETA FUNCTION AND ITS EXPRESSIONS

Definition. The following formal power series

$$Z_G(s) = \prod_{\gamma \in [P_G]} \frac{1}{1 - s^{|\gamma|}}$$

is called the Ihara zeta function of G, denoted by $Z_G(s)$. It is called an Euler product expression.

Example. $Z_{G_1}(s) = \dfrac{1}{1 - s^2}$ $\quad (a_1 a_2) \sim (a_2 a_1),\ (a_1 a_2 a_1 a_2) = (a_1 a_2)^2$

For G_2, $(a_2 a_1), (a_2 a_1 a_3 a_4), (a_2 a_1 a_3 a_4 a_3 a_4), (a_2 a_1 (a_3 a_4)^3), \dots, (a_2 a_1 a_3 a_4 a_3 a_4 \cdots a_3 a_4), \dots\dots, \infty.$
Thus it might be difficult to calculate $Z_{G_2}(s)$. However we can calculate the zeta function using a matrix as follows.

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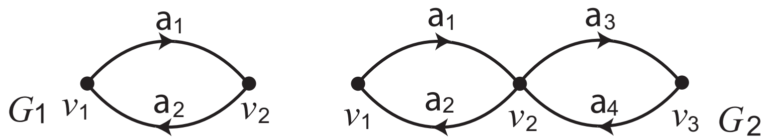


Definition. Let us consider the map

$$\theta : A \times A \longrightarrow \{0,1\} : (a, a') \mapsto \delta_{\mathfrak{h}(a)\mathfrak{t}(a')}.$$

Then we have the edge matrix of G $M_G(\theta) = (\theta(a, a'))_{a,a' \in A}$.

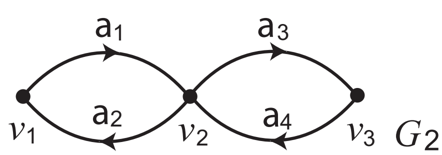
Theorem [Hashimoto].

$$Z_G(s) = \frac{1}{\det(I - sM_G(\theta))}.$$

This expression is called Hashimoto expression.

Example.

$$M_{G_2} \quad \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$



$$Z_{G_2}(s) = \frac{1}{1 - 2s^2}.$$

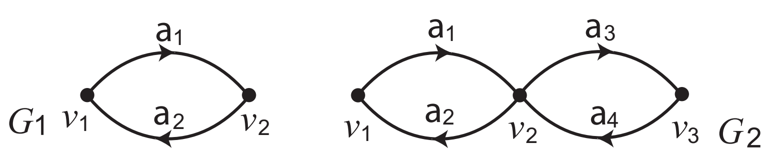
Definition. N_ℓ: the number of closed paths (not necessaily cycles, not necessarily prime) of length ℓ in G.

Theorem.

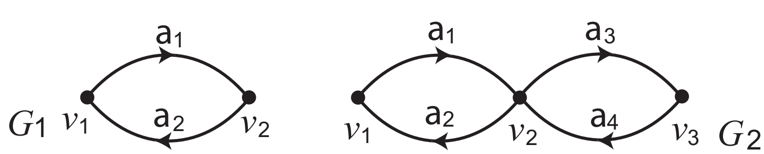
$$Z_G(s) = \exp\left(\sum_{\ell \geq 1} \frac{N_\ell}{\ell} s^\ell\right).$$

This expression is called the exponential expression.

Example.



$$\log Z_{G_1}(s) = \log \frac{1}{1 - s^2} = s^2 + \frac{s^4}{2} + \frac{s^6}{3} + \frac{s^8}{4} + \frac{s^{10}}{5} + \cdots$$
$$N_1 = 0,\ N_2 = 2,\ N_3 = 0,\ N_4 = 2,\ N_6 = 2,\ N_8 = 2,\ N_{10} = 2, \cdots$$
$$N_2 : (a_1a_2), (a_2a_1),\ N_4 : (a_1a_2)^2, (a_2a_1)^2,\ N_4 : (a_1a_2)^3, (a_2a_1)^3.$$



$$\log Z_{G_2}(s) = \log \frac{1}{1-2s^2} = 2s^2 + 2s^4 + \frac{8s^6}{3} + 4s^8 + \frac{32s^{10}}{5} + \cdots$$

$$N_1 = 0,\ N_2 = 4,\ N_3 = 0,\ N_4 = 8,\ N_6 = 16,\ N_8 = 32, \cdots$$

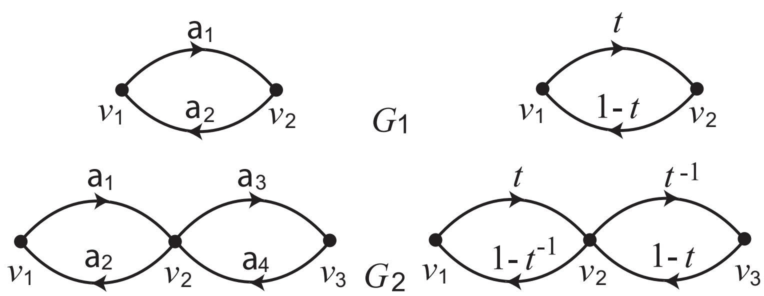
$$N_2 : (a_1a_2), (a_2a_1), (a_3a_4), (a_4a_3),$$

$$N_4 : (a_1a_2)^2, (a_2a_1)^2, (a_3a_4)^2, (a_4a_3)^2, (a_1a_3a_4a_2), (a_3a_4a_2a_1), (a_4a_2a_1a_3), (a_2a_1a_3a_4)$$

3. Weighted zeta functions

Let $G = (V, A)$ be a digraph, and R a commutative $\mathbb{Q}$-algebra. The map $\omega : A \to R$ is called a weight, and (G, ω) is called a weighted digraph.

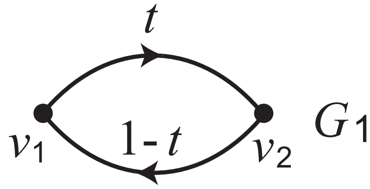
Example. $(G_1, \omega_1) :\ \omega_1(a_1) = t,\ \omega_1(a_2) = 1 - t.$
$(G_2, \omega_2) :\ \omega_2(a_1) = t,\ \omega_2(a_2) = 1 - t^{-1},\ \omega_2(a_3) = t^{-1},\ \omega_2(a_4) = 1 - t.$



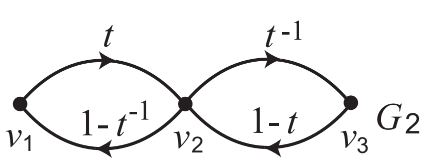
Definition. The matrix A_G is called a weighted adjacency matrix of (G, ω) if $A_G = (a_{uv})_{u,v \in V}$, ($A_{uv}$ is the set of arcs from u to v) , where $a_{uv} = \begin{cases} \sum_{a \in A_{uv}} \omega(a) & \text{if } a = (u,v) \in A_{uv} \\ 0 & \text{if } a = (u,v) \notin A_{uv} \end{cases}$

Example.

$$A_{G_1} \quad \begin{matrix} 0 & t \\ 1\text{-}t & 0 \end{matrix}$$



$$A_{G_2} \quad \begin{matrix} 0 & t & 0 \\ 1\text{-}t^{-1} & 0 & t^{-1} \\ 0 & 1\text{-}t & 0 \end{matrix}$$

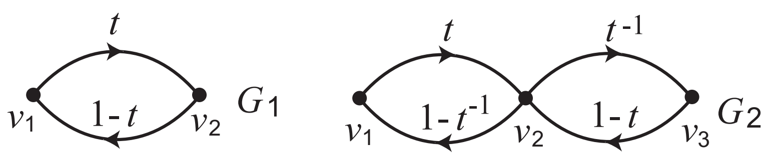


The same thing as in §2 applies to weighted case ! i.e., We may consider the weighted zeta function $Z_G(s; \omega)$.

$$Z_G(s;\omega) = \frac{1}{\det(I - sA_G(\omega))} = \frac{1}{\det(I - sM_G(\omega))}$$

$$= \exp\left(\sum_{\ell \geq 1} \frac{N_\ell(\omega)}{\ell} s^\ell\right).$$

Example. $Z_{G_1}(s;\omega_1) = \dfrac{1}{\det(I - sA_{G_1}(\omega_1))} = \dfrac{1}{1 + (-t + t^2)\, s^2}$,



$$Z_{G_2}(s;\omega_2) = \frac{1}{\det(I - sA_{G_2}(\omega_2))} = \frac{1}{1 + \left(2 - t - \frac{1}{t}\right) s^2}$$

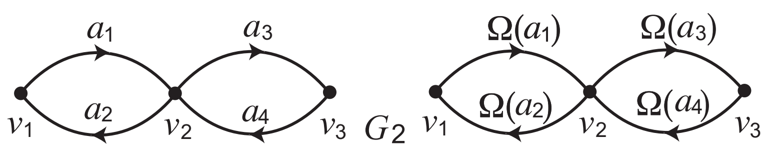
Moreover, we can have the same result as in the 'matrix weight' case. Let :

$$\Theta : A \times A \to \mathrm{Mat}(R) : (a, a') \mapsto \delta_{\mathfrak{h}(a)\mathfrak{t}(a')} \Omega(a').$$

The edge matrix $M_G(\Omega)$ whose (a, a')-entry is the block matrix $\Omega(a')$ for $a' = (v_i', v_j')$, and the adjacency matrix is :

$$A_G(\Omega) = \left(\sum_{a \in A_{uv}} \Omega(a)\right)_{u,v \in V}$$

Example.



$$M_{G_2}(\Omega) = \begin{pmatrix} O & \Omega(a_2) & \Omega(a_3) & O \\ \Omega(a_1) & O & O & O \\ O & O & O & \Omega(a_4) \\ O & \Omega(a_2) & \Omega(a_3) & O \end{pmatrix}, \quad A_{G_2}(\Omega) = \begin{pmatrix} O & \Omega(a_1) & O \\ \Omega(a_2) & O & \Omega(a_3) \\ O & \Omega(a_4) & O \end{pmatrix}.$$

Definition. Set :

$$N_\ell(\Omega) = \sum_{c \in C_\ell} \operatorname{tr} \Omega(c).$$

Theorem ([1], [4]).

$$Z_G(s;\Omega) = \prod_{\gamma \in [P_G]} \frac{1}{\det(I - s^{|\gamma|}\Omega(\gamma))} = \frac{1}{\det(I - sM_G(\Omega))}$$

$$= \frac{1}{\det(I - sA_G(\Omega))} = \exp\left(\sum_{\ell \geq 1} \frac{N_\ell(\Omega)}{\ell} s^\ell\right)$$

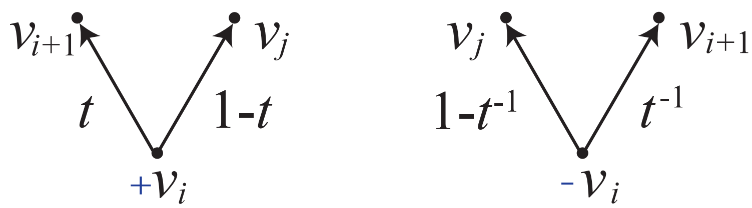
4. The knot graph and the volume of a hyperbolic knot

Let $G_K = (V, A)$ be a knot graph of a knot K. For the detail of the knot graph, see [2]. We define the Alexander weight as follows:

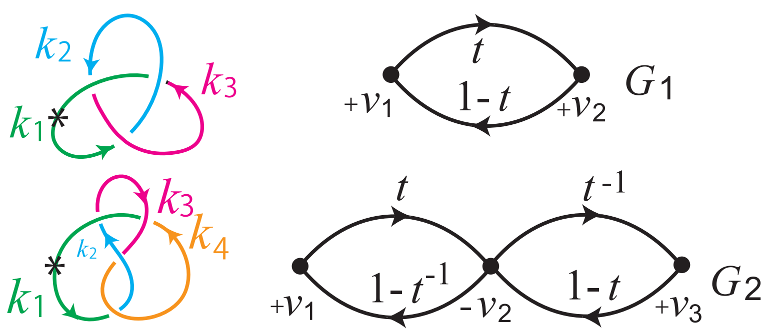
Definition. Let $\omega : A \to R = \mathbb{Q}[t^{\pm 1}]$ be

$$\omega(a) = \omega(v_i, v_j) = \begin{cases} t^{\text{sign}(v_i)} & \text{if } a = (v_i, v_{i+1}) \\ 1 - t^{\text{sign}(v_i)} & \text{if } a = (v_i, v_j). \end{cases}$$

Then we call the map ω the Alexander weight. See the next figure.



Example. The knot graph for the trefoil is G_1 and it for the figure eight knot is G_2. Suppose ω is the Alexander weight.



Then we have :

$$Z_{G_1}(s;\omega) = \frac{1}{\det(I - sA_{G_1}(\omega_1))} = \frac{1}{1 + (-t + t^2)\, s^2}, \quad Z_{G_1}(1;\omega) = \frac{1}{1 - t + t^2}.$$

$$Z_{G_2}(s;\omega) = \frac{1}{\det(I - sA_{G_2}(\omega_2))} = \frac{1}{1 + \left(2 - t - \frac{1}{t}\right) s^2}, \quad Z_{G_2}(1;\omega) = \frac{1}{-t + 3 - \frac{1}{t}}.$$

Similarly, we have the twisted Alexander polynomial using the matrix weighted zeta function !.

Definition.

$$\Delta_{K,\rho}(t) = \frac{\det\left(\Phi\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1 \le i,j \le m-1}}{\det \Phi(x_m - 1)} \left(=: \frac{\Delta^1_{K,\rho}(t)}{\det \Phi(x_m - 1)}\right).$$

Theorem ([4]). Let G_K be the knot graph for a knot K, and suppose that Ω is the twisted Alexander weight for G_K. Then

$$\frac{1}{\Delta^1_{K,\rho}(t)} = Z_{G_K}(1;\Omega) = \prod_{\gamma \in [P_{G_K}]} \frac{1}{\det(I - \Omega(\gamma))} = \frac{1}{\det(I - M_{G_K}(\Omega))}$$

$$= \frac{1}{\det(I - A_{G_K}(\Omega))} = \exp\left(\sum_{\ell \geq 1} \frac{N_\ell(\Omega)}{\ell}\right)$$

Let Ω_n be the twisted Alexander weight for the knot graph G_K of a hyperbolic knot K, which corresponds to the lift of the holonomy representation $\rho_n : \pi_1(E_K) \to \mathrm{SL}(n;\mathbb{C})$. Further, we set $q_k(t) = \mathrm{tr}\,(A_{G_K}(\Omega_n))^k \in \mathbb{C}[t^{\pm 1}]$ $(k = 1, 2, \ldots, d)$ and

$$\vec{q}(t) = \big(-q_1(t), -1!q_2(t), -2!q_3(t), \ldots, -(d-1)!q_d(t)\big).$$

By using above expressions, we have:

Theorem ([4]). Let K be a hyperbolic knot, and let $\zeta(\neq 1) \in S^1$.

$$\mathrm{Vol}(S^3 \setminus K) = 4\pi \lim_{n\to\infty} \frac{1}{n^2} \log \left| \sum_{k=0}^{mn} \frac{B_k(\vec{q}(\zeta))}{k!} \right|,$$

where B_k is the Bell polynomial and m is the number of vertices in the knot graph G_K.

Remark. In the previous work [3], we have obtained a similar formula for a fibered hyperbolic knot.

References

[1] T. Endo, T. Komatsu, N. Konno, H. Mitsuhashi and I. Sato, *The second matrix-weighted zeta function of a graph (in Japanese)*, MSJ Autumn Meeting 2024, Division of Applied Mathematics, Abstract 71–74.

[2] H. Goda, *Twisted Alexander polynomial and Matrix-weighted zeta function*, Kyushu J. Math. 74 (2020), 211–221.

[3] H. Goda and T. Morifuji, *A volume presentation of a fibered knot*, Tohoku Math. J. **76** (2024), 423–443.

[4] H. Goda and T. Morifuji, *Some expressions of graph zetas and the twisted Alexander polynomials, and the knot volume*, preprint.

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