A MULTI-VARIABLE ALEXANDER POLYNOMIAL FOR A FRAMED TRANSVERSE GRAPH

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Abstract. We propose a definition of the rotation number for a transverse graph diagram. Then we define a multivariable Alexander polynomial for a framed transverse graph (a.k.a ribbon graph). This report is based on a joint work with Zhongtao Wu.

1. The rotation number of a transverse graph diagram

1.1. **A transverse graph.** We consider an oriented connected graph G in S^3 where for each vertex v, there is a disk that separates the incoming and outgoing edges. We call such an orientation a *transverse orientation*. A graph with a transverse orientation is called a *transverse graph*. To our knowledge, the terminology was first used by [2] in their definition of Heegaard Floer homology for graphs.

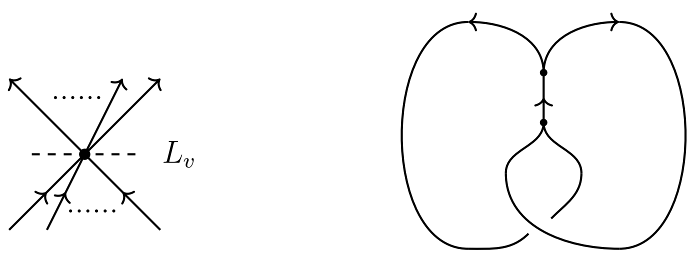


Figure 1. The local picture of a vertex with transverse orientation (left). An oriented trivalent graph without sinks and sources is a transverse graph (right).

Now a diagram D of a transverse graph G on R^2 is a regular projection of G so that

(i) The self-intersections are double points between edges, and at each double point the information of which strand is over and which is under is given.
(ii) Around each vertex v, there is a straight line L_v that separates the edges entering v and the edges leaving v.

If the position of the straight line L_v is clear, it can be omitted in the graph diagram.

Theorem 1.1. *Two diagrams represent the same transverse graph if and only if they can transfer to each other by a finite sequence of moves in Fig. 2.*

For a transverse graph G, let $S^3\backslash G$ denote the complement of G in S^3. **In this report, we consider a transverse graph G for which the meridian of each edge of G represents a non-trivial element in $H_1(S^3\backslash G;\mathbb{Z})$.**

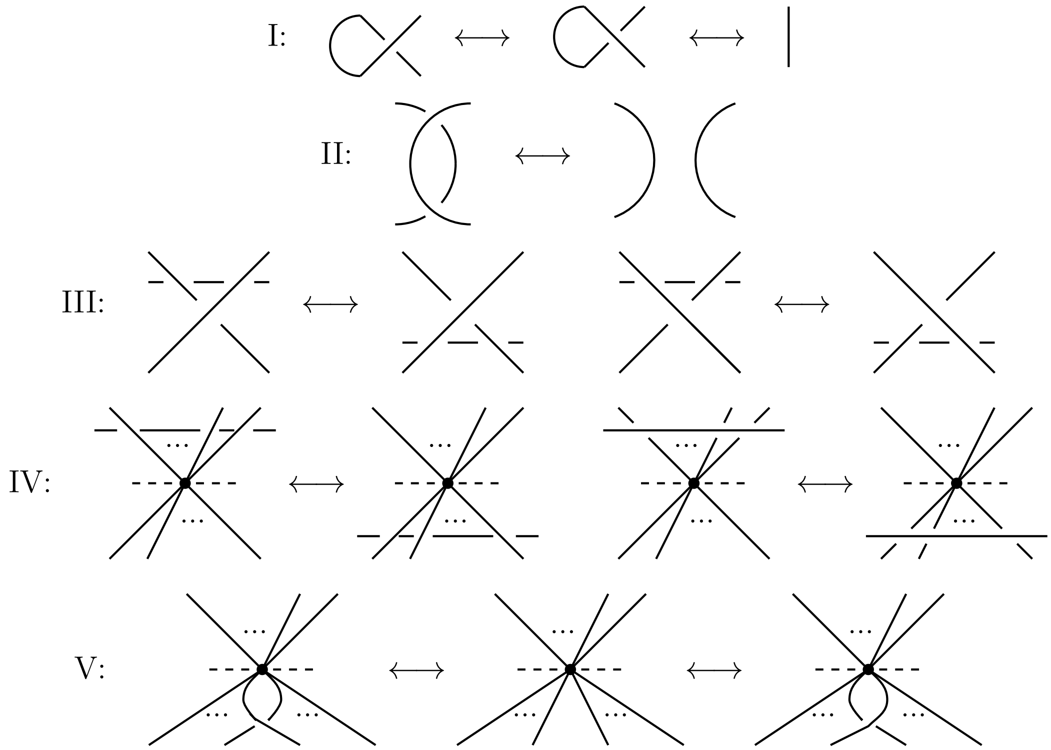


FIGURE 2. Reidemeister moves for transverse graph diagrams. Suppressed orientations of the edges can be added in all compatible ways.

1.2. **Definition of the rotation number for a graph diagram.** For a smooth closed oriented plane curve, the rotation number [10] or Whitney index counts the total number of turns when traveling along the curve. Among many studies, there is a formula which calculates it using information of the regions and the double points of the plane curve. See Theorem 1.3. This formula seems to have been first proposed by Viro [5], and is recently reproved by Wesenberg [9]. Motivated by this formula, we propose a definition of the rotation number for a diagram of a transverse graph.

Let G be a transverse graph in S^3. The first homology group $H_1(S^3\backslash G;\mathbb{Z})$ has a presentation as follows.

(i) *generators* To each edge or loop e of G, we assign a generator, which is the homology class of the oriented meridian of e. We call this element the *color* of e.

(ii) *relators* For any two generators s and t we have a relator $ts = st$. To each vertex v of G, we assign a relator as follows. Suppose the generators corresponding to the incoming (resp. outgoing) edges of v are $s_1, s_2, \cdots, s_k$ (resp. $t_1, t_2, \cdots, t_l$). Then we have a relator $s_1 s_2 \cdots s_k = t_1 t_2 \cdots t_l$.

Consider a connected diagram D of a transverse graph. To each connected component r of $R^2\backslash D$, which we call a *regular region*, we define the color $c(r)$ as follows:

(i) The color of the unique unbounded region is set to be $1 \in H_1(S^3\backslash G;\mathbb{Z})$.

(ii) The colors of the other regions are inductively determined by the rule as exhibited below: when an edge e points upward and its right-hand side region has color $x \in H_1(S^3 \backslash G; \mathbb{Z})$, then the color of its left-hand side region is $x \cdot m_e$, where m_e is the homology class of the meridian of e.

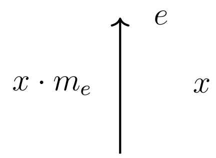


FIGURE 3. The colors of the two regions adjacent to an edge.

For each vertex v of D, we define its color $c(v)$ as follows. Suppose the incoming (resp. outgoing) edges around v are $s_1, s_2, \cdots, s_k$ (resp. $t_1, t_2, \cdots, t_l$), as illustrated in Fig4. Then let x_i (resp. y_j) be the color of the region adjacent to s_i and s_{i+1} (resp. t_j and t_{j+1}) for $1 \le i \le k-1$ (resp. $1 \le j \le l-1$). If $k = 1$ or $l = 1$, there is no x_i or y_j to define. Let

$$c(v) = \prod_{i=1}^{k-1} x_i^{1/2} \prod_{j=1}^{l-1} y_i^{1/2}.$$

Namely we consider all the regions around v except the two which encounter L_v.

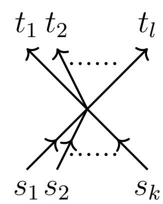


FIGURE 4. A vertex.

For each double point v of D, we regard it as a vertex with indegree 2 and outdegree 2, and define its color as above. Namely

$$c(v) = x^{1/2} y^{1/2},$$

where x is the color of its south corner and y is the color of its north corner.

Lemma 1.2. *Let V be the set of vertices of D, and $X(D)$ be the set of double points of D. After taking the product, $\prod_{v \in V \cup X(D)} c(v)$ becomes an element of $H_1(S^3 \backslash G; \mathbb{Z})$.*

Proof. Omitted. □

Theorem 1.3 (Viro, Wesenberg)**.** *Let D be an oriented plane curve on $\mathbb{R}^2$. Then the rotation number $w(D)$ of D can be calculated as*

$$t^{w(D)} = \prod_r c(r) [\prod_v c(v)]^{-1},$$

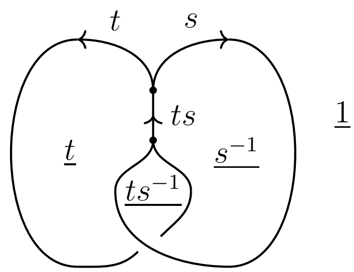


FIGURE 5. A diagram of a trivalent graph.

where r runs through all the component of $R^2 \backslash D$ and v runs through all double points of D.

Now we extend the formula above to a transverse graph diagram.

Definition 1.4. Let D be a connected diagram of a transverse graph. We define the *rotation number* of D to be

$$\mathrm{Rot}(D) = \prod_{r \in F(D)} c(r)[\prod_{v \in V \cup X(D)} c(v)]^{-1},$$

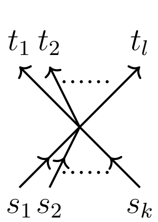
where $F(D)$ is the set of regular regions of D. For a disconnected diagram, its rotation number is defined to be the product of those of its connected components.

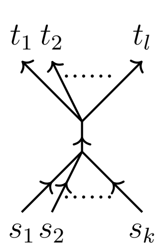
Example 1.5. The rotation number of the diagram D if Fig 5 is calculated as follows. A word with underline indicates the color of the corresponding region. There are four regular regions, two vertices and one crossing. We have $\prod_{r \in F(D)} c(r) = t \cdot s^{-1} \cdot ts^{-1} = t^2 s^{-2}$, and $\prod_{v \in V \cup X(D)} c(v)^2 = (ts^{-1})(ts^{-1}) = t^2 s^{-2}$. As a result,

$$\mathrm{Rot}(D) = t^2 s^{-2} (ts^{-1})^{-1} = ts^{-1}.$$

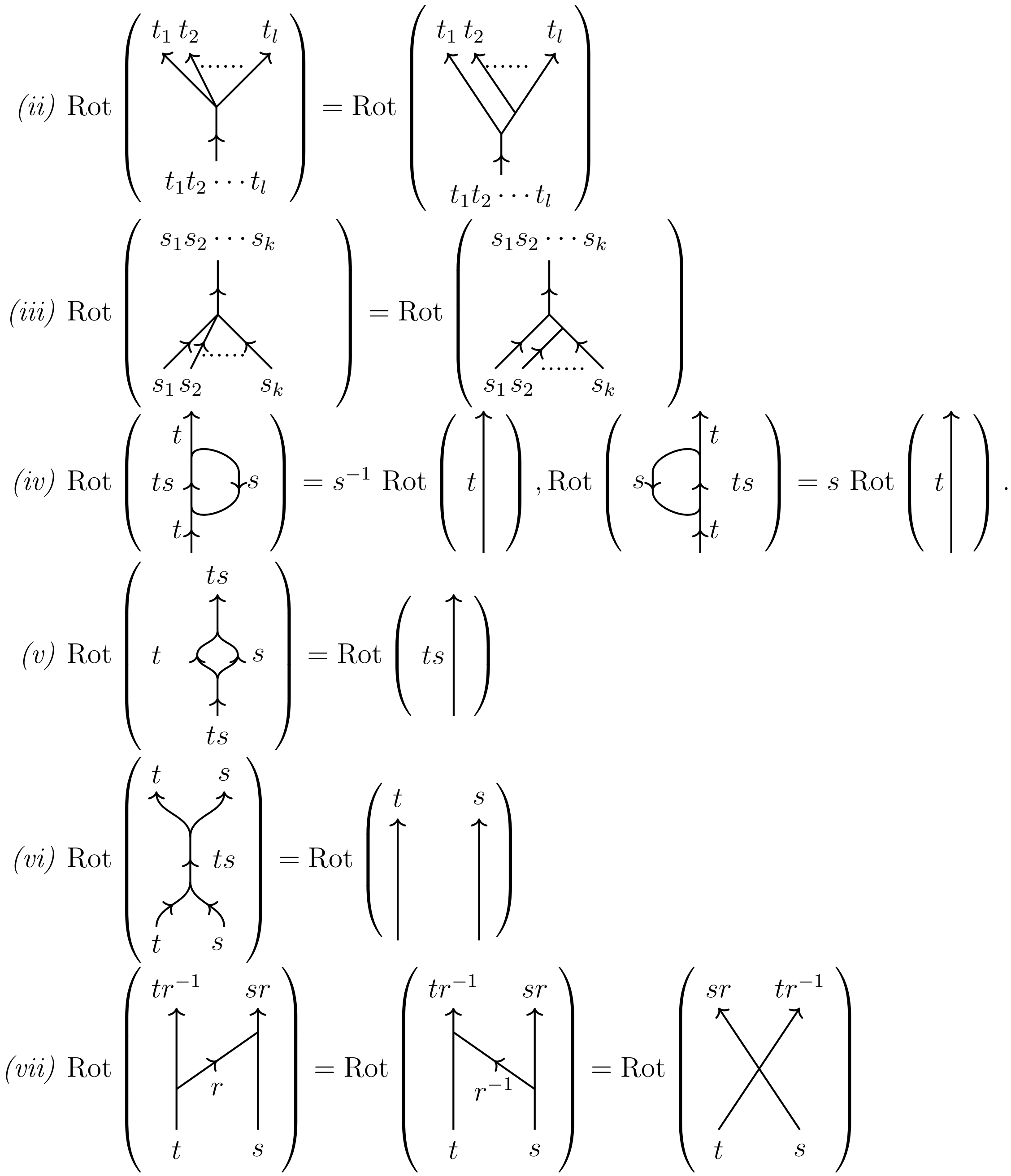
1.3. **Properties.** We discuss some properties of $\mathrm{Rot}(D)$.

Proposition 1.6. *Under the given colors, we have the following relations.*

(i) Rot $\left(\right.$ 

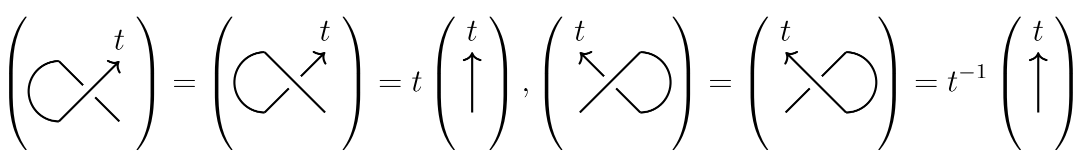
$\left. \right)$ = Rot $\left(\right.$ 

$\left. \right)$



Proof. Obvious from the definition. $\square$

Proposition 1.7. *The rotation number of a diagram does not change under moves II∼V in Fig. 2 and its change under move I is as follows.*



Proof. Omitted. $\square$

Remark 1.8. *Two graph diagrams are called regularly homotopic if they are connected by a finite sequence of moves of II, III and IV. Prop. 1.7 tells us that the rotation number* $\mathrm{Rot}(D)$ *is a regular homotopy invariant for a transverse graph. Nikkuni* [3] *showed that the Wu invariant* [8, 7] *is a complete regular homotopy invariant for graphs. We expect to clarify the relationship between Wu invariant and* $\mathrm{Rot}(D)$ *in a near future.*

2. A multi-variable Alexander polynomial

2.1. **Kauffman state.** We recall the definition of Kauffman state that we introduced in [1, Definition 2.4]. Suppose D is a connected diagram of a transverse graph. We can obtain a *decorated diagram* (D, δ) by putting a base point δ on an edge of D and drawing a circle around each vertex of D. Then we define

(i) $\mathrm{Cr}(D)$: denotes the set of crossings, including the types and which are the double points of the diagram and the type which are the intersection points around each vertex between the incoming edges with the circle.

(ii) $\mathrm{Re}(D)$: denotes the set of regions, including the *regular regions* of R^2 separated by D and the *circle regions* around the vertices. *Marked regions* are the regions adjacent to the base point δ, and the others are called *unmarked regions*.

(iii) Corners: For a crossing of type or , there are four corners around it, and we call them the *north*, *south*, *west*, and *east corners* of the crossing. Around a crossing of type there are three corners, and we call the one inside the circle region the *north* corner, the one on the left of the crossing the *west* corner and the one on the right the *east* corner. Note also that every corner belongs to a unique region in $\mathrm{Re}(D)$.

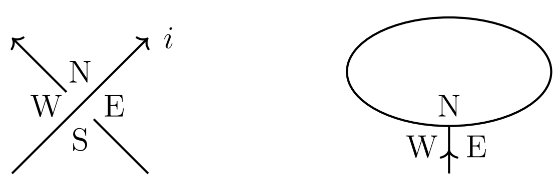


Fig 6 is an example of decorated diagram. Note that each meridian is nontrivial, there are exactly two regions R_u and R_v adjacent to δ. Under the assumption that D is connected, we have $|\,\mathrm{Re}(D)| = |\,\mathrm{Cr}(D)| + 2$. A *Kauffman state*, or simply, a *state* for a decorated diagram (D, δ) is a bijective map

$$s: \mathrm{Cr}(D) \to \mathrm{Re}(D) \backslash \{R_u, R_v\},$$

which sends a crossing in $\mathrm{Cr}(D)$ to one of its corners. Let $S(D, \delta)$ denote the set of all states.

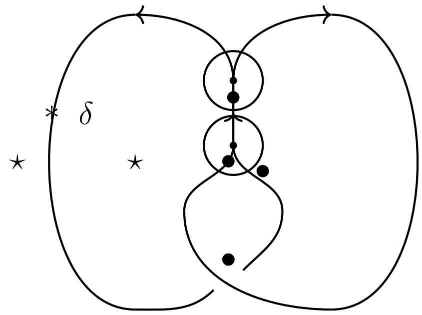


FIGURE 6. There are four regular regions, two circle regions, one crossing of type , three crossings of type . A Kauffman state associated to (D, δ) is marked out by $\bullet$'s .

2.2. **Kauffman state sum.** The aim here is to define a value $\langle D \rangle$, which is the multi-variable version of the Kauffman state sum that we defined in [1].

Definition 2.1. Suppose δ is on an edge with color t, and the colors of the marked regions are x and xt respectively. Define

$$|\delta| = x - xt.$$

Note that all edges have non-trivial colors, we have $|\delta| \neq 0 \in \mathbb{Z}H_1(S^3 \backslash \mathrm{G}; \mathbb{Z})$.

Definition 2.2. Choose a base point δ on an edge. Suppose (D, δ) is a connected decorated diagram with N crossings $C_1, C_2, \cdots, C_N$ in $\mathrm{Cr}(D)$ and $N+2$ regions $R_1, R_2, \cdots, R_{N+2}$ in $\mathrm{Re}(D)$.

(i) Define the local contributions $M_{C_p}^{\triangle}$ and $A_{C_p}^{\triangle}$ associated to each corner $\triangle$ around the crossing C_p as in Fig. 7.

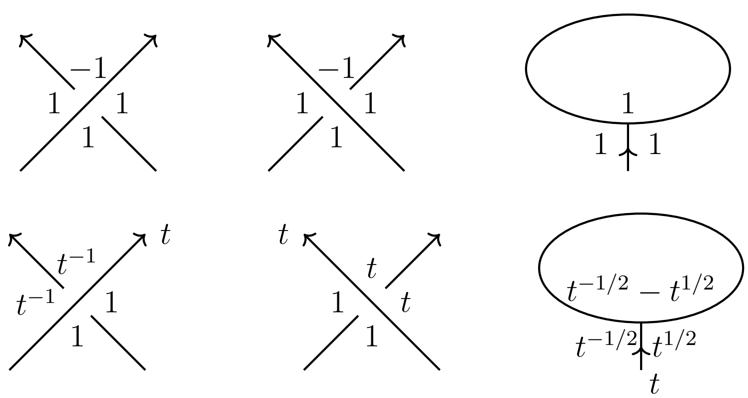


FIGURE 7. The local contributions $M_{C_p}^{\triangle}$ (top) and $A_{C_p}^{\triangle}$ (bottom). Here t indicates the color of the nearby edge.

(ii) For each state $s \in S(D, \delta)$, let

$$M(s) := \prod_{p=1}^{N} M_{C_p}^{s(C_p)},$$

$$A(s) := \prod_{p=1}^{N} A_{C_p}^{s(C_p)}.$$

(iii) The *state sum* is defined as

$$\langle D \rangle := |\delta|^{-1} \sum_{s \in S(D,\delta)} M(s) \cdot A(s). \tag{1}$$

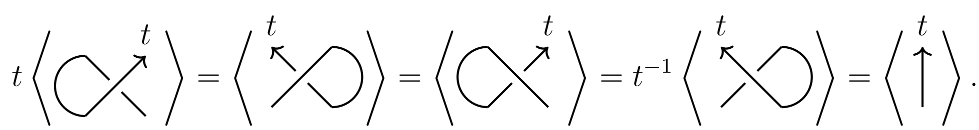
For all the cases that $S(D,\delta) = \emptyset$ or D is disconnected, we let $\langle D \rangle = 0$.

For an MOY graph (G, c), there is a homormorphism $\phi_c : H_1(S^3 \backslash G) \to \mathbb{Z}$ that assigns the oriented meridian of an edge e to $c(e)$. The state sum $\langle D, c \rangle$ defined in [1, Definition 2.11] can be obtained from $\langle D \rangle$ by replacing each element in $H_1(S^3 \backslash G)$ with its image under ϕ_c. In particular $\langle D, c \rangle \neq 0$ implies that $\langle D \rangle \neq 0$.

Proposition 2.3. *The state sum $\langle D \rangle$ does not depend on the choice of the base point δ.*

Proof. Omitted. $\square$

Proposition 2.4. *The state sum $\langle D \rangle$ is invariant under the Reidemeister moves (II) – (V) in Fig. 2, and its variations under Reidemeister move (I) are given as below.*



Proof. Omitted. $\square$

2.3. **A multi-variable Alexander polynomial.** In this part, we present a way of normalizing the state sum so that it becomes an invariant for framed transverse graphs. This work is a generalization of [1, Section 3.2].

Let G be a transverse graph. Recall that a *framing* of G is an embedded compact surface $F \subset S^3$ in which G is sitting as a deformation retract. A *framed graph* is a graph equipped with a framing. More precisely, each vertex of G is replaced by a disk in F where the vertex is the center of the disk, and each edge of G is replaced by a strip $[0,1] \times [0,1]$ where $[0,1] \times \{0,1\}$ is attached to the boundaries of its adjacent vertex disks and the edge is $\{\frac{1}{2}\} \times [0,1]$. It is obvious to see that a framed transverse graph is equivalent with a ribbon graph defined in [4], where a vertex is replaced by a coupon while an edge is replaced by an annulus.

Each graph diagram of G in R^2 has a *blackboard framing*, whose projection in R^2 is the tubular neighborhood of the graph diagram in R^2. Hereafter, a framed transverse graph will be represented by graph diagrams with blackboard framing. For framed transverse graphs, we have the following result.

Lemma 2.5. *Any two graph diagrams for a framed transverse graph can be connected by a sequence of Reidemeister moves in Fig. 8.*

Starting from a framed graph diagram, we hope to construct an appropriate factor that cancels the change of the state sum coming from Reidemeister move (I) in Proposition 2.4 while keeping invariant under the other types of Reidemeister moves.

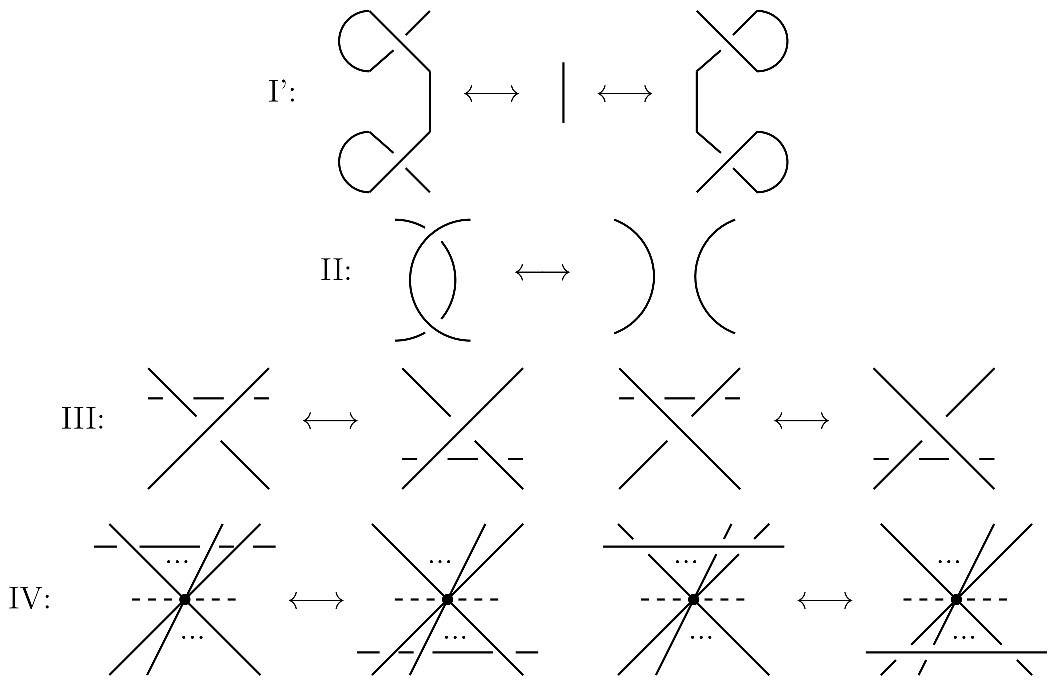


FIGURE 8. Reidemeister moves for framed transverse graph diagrams (ribbon graphs).

From now on, we use the blackboard-bold letters $\mathbb{G}$ and $\mathbb{D}$ to denote a framed trivalent graph and diagram, respectively.

Definition 2.6. For a framed transverse graph diagram $\mathbb{D}$, define the normalized Alexander polynomial by

$$\Delta_{\mathbb{D}} := \mathrm{Rot}(D)^{1/2} \cdot \langle D \rangle. \tag{2}$$

Theorem 2.7. *$\Delta_{\mathbb{D}}$ is a topological invariant of the framed transverse graph $\mathbb{G}$.*

Proof. Since both $\langle D \rangle$ and $\mathrm{Rot}(D)$ are invariant under Reidemeister moves (II, III, IV), it is enough to study their behavior under move (I'). Suppose the graph on the left hand side of (I') is $\mathbb{D}_1$ and the right one is $\mathbb{D}_2$. Then by proposition 2.4 $\langle D_1 \rangle = t^{-1} \langle D_2 \rangle$, where t is the color of the corresponding edge. By Proposition 1.7 we have $\mathrm{Rot}(D_1) = t^2 \mathrm{Rot}(D_2)$. Therefore $\Delta_{\mathbb{D}_1} = \Delta_{\mathbb{D}_2}$ under move (I'). □

3. FUTURE STUDIES

There are several questions whose answers remain open to us.

(i) For a graph diagram D of a transverse graph, what is the relation of $\mathrm{Rot}(D)$ and Wu invariant?
(ii) Is $\mathrm{Rot}(D)$ a complete regular homotopy invariant for transverse graphs?
(iii) For a framed trivalent graph without sink or source, what is the relation of $\Delta_{\mathbb{D}}$ and Viro's $gl(1|1)$-Alexander polynomial defined in [6]?
(iv) How to extend the definition of $\mathrm{Rot}(D)$ or even $\Delta_{\mathbb{D}}$ to a graph with sources/sinks?

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References

[1] Y. Bao and Z. Wu, *An Alexander polynomial for MOY graphs*, Selecta Math. (N.S.), 26 (2020), pp. Paper No. 32, 44.
[2] S. Harvey and D. O'Donnol, *Heegaard Floer homology of spatial graphs*, Algebr. Geom. Topol., 17 (2017), pp. 1445–1525.
[3] R. Nikkuni, *On the Wu invariants for immersions of a graph into the plane*, Homology Homotopy Appl., 12 (2010), pp. 45–60.
[4] N. Y. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys., 127 (1990), pp. 1–26.
[5] O. Viro, *First degree invariants of generic curves on surfaces*, U.U.D.M. report, ISSN 1101-3591, (1994).
[6] O. Y. Viro, *Quantum relatives of the Alexander polynomial*, Algebra i Analiz, 18 (2006), pp. 63–157.
[7] W. Wen-tsün, *On the isotopy of a complex in a Euclidean space. I*, Sci. Sinica, 9 (1960), pp. 21–46.
[8] ——, *A theory of imbedding, immersion, and isotopy of polytopes in a euclidean space*, Science Press, Peking, 1965.
[9] D. Wesenberg, *A new formula for rotation number*, arXiv:2010.01422, (2020).
[10] H. Whitney, *On regular closed curves in the plane*, Compositio Math., 4 (1937), pp. 276–284.

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