A multivariable Alexander polynomial for framed trivalent spatial graphs

（Zhongtao Wu氏との共同研究に基づく）

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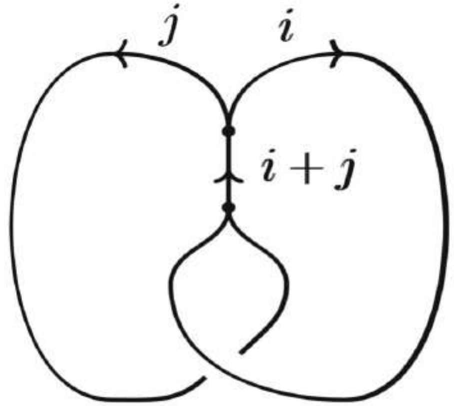
結び目の数理VII＠早稲田大学

§0 Overview

Previous joint work w/ Zhongtao Wu

(D, c) :



D : oriented trivalent graph w/o sources or sinks

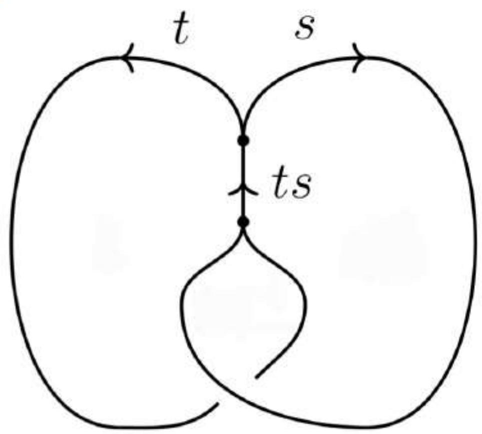
c : {edges} $\to \mathbb{N}$

For (D, c) we call MOY graph

- Defined state sum $\langle D, c \rangle$
- Using rotation number of (D, c), we get normalization $\Delta_{(D,c)}(t) \in \mathbb{Q}(t^{1/2}, t^{-1/2})$, an inv. for framed MOY graphs.

Today :

D :



$t, s \in H_1(S^3 \setminus G; \mathbb{Z})$: homology classes of the oriented meridians of edges

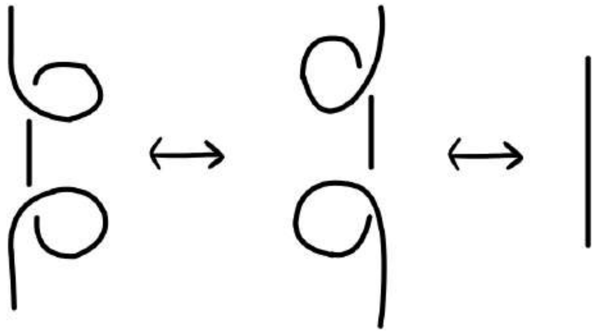
§1 1) Define rotation number of D $\in H_1(S^3 \setminus G; \mathbb{Z})$

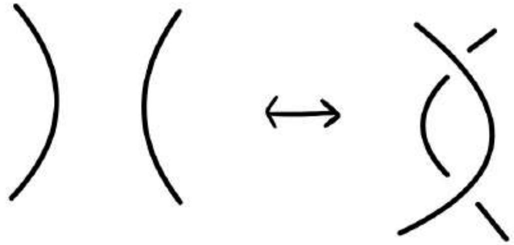
§2 2) Define $\langle D \rangle \in \mathbb{Q}(t_1^{1/2}, \cdots, t_n^{1/2})$

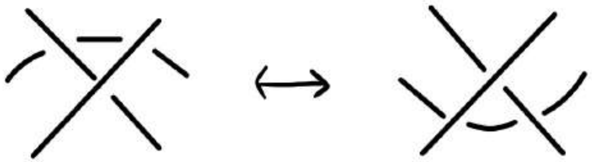
a generating set of $H_1(S^3 \setminus G; \mathbb{Z})$

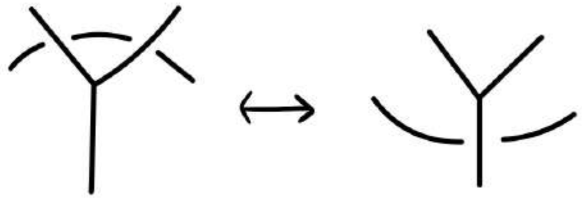
§3 3) Normalization $\Delta_D \in \mathbb{Q}(t_1^{1/2}, \cdots, t_n^{1/2})$

Reidemeister moves for framed trivalent spatial graphs

(I) 

(II) 

(III) 

(IV) 

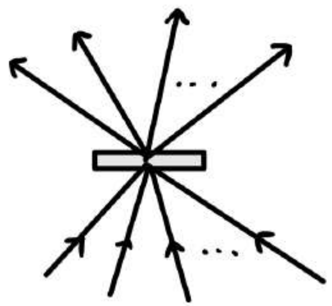
Theorem: Two graph diagrams represents the same framed spatial graph iff they are connected by a finite sequence of moves of (I) ~ (IV).

Remark:

- State sum $\langle D \rangle$ comes from Fox derivative, well-defined as an invariant of spatial graphs up to $\pm t_1^{k_1/2} t_2^{k_2/2} \cdots t_n^{k_n/2}$.

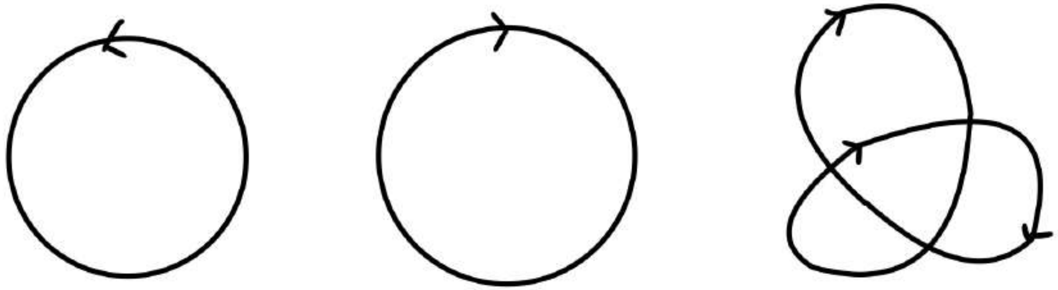
- The normalization Δ_D makes $\langle D \rangle$ stronger by eliminating the ambiguity, and it is possible to consider skein-like relations about Δ_D

- The definition of Δ_D can be extended to any ribbon graph.



§1 Rotation number of a graph

D : oriented plane curve



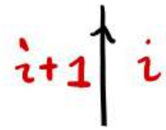
$w(D) = 1$ $\qquad -1 \qquad -2$

rotation number or Whitney index

Theorem (Wesenberg, arXiv: 2010.01422)

$$W(D) = \sum_{r:\text{region}} w(r) - \sum_{c:\text{crossing}} w(c)$$

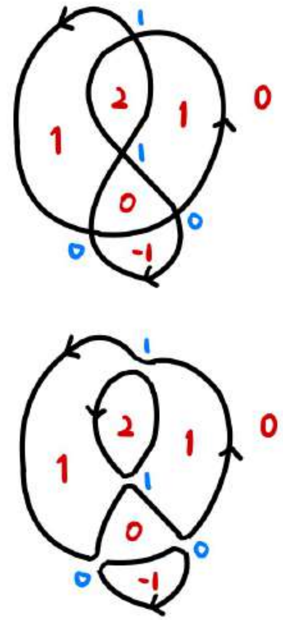
$w(r)$ is defined by

- $w(r) = 0$ if r is unbounded
- 

$w(c)$ is defined by

$w(c) = i$

E.g.



$\sum_r w(r) = 1+2+1+(-1)$
$= 3$

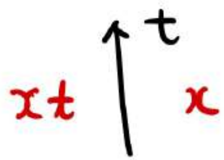
$\sum_c w(c) = 1+1 = 2$

$\therefore\ w(D) = 3-2 = 1$

D : connected graph diagram

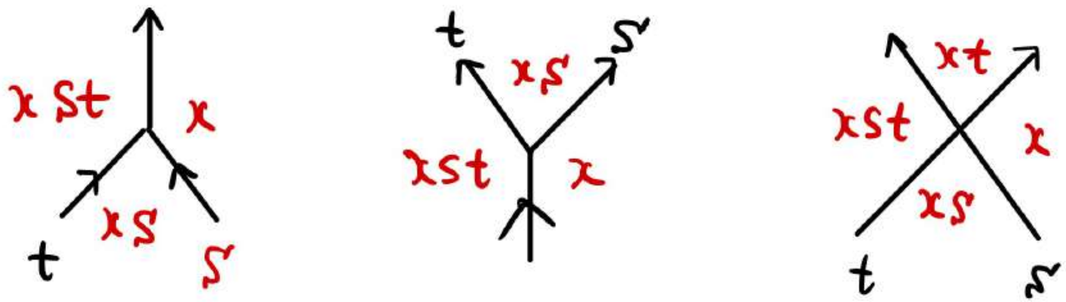
r : a component of $\mathbb{R}^2 \setminus D$

Define the color $c(r) \in H_1(S^3 \setminus G)$ by

- $c(r) = 1$ for unbounded r.
- 

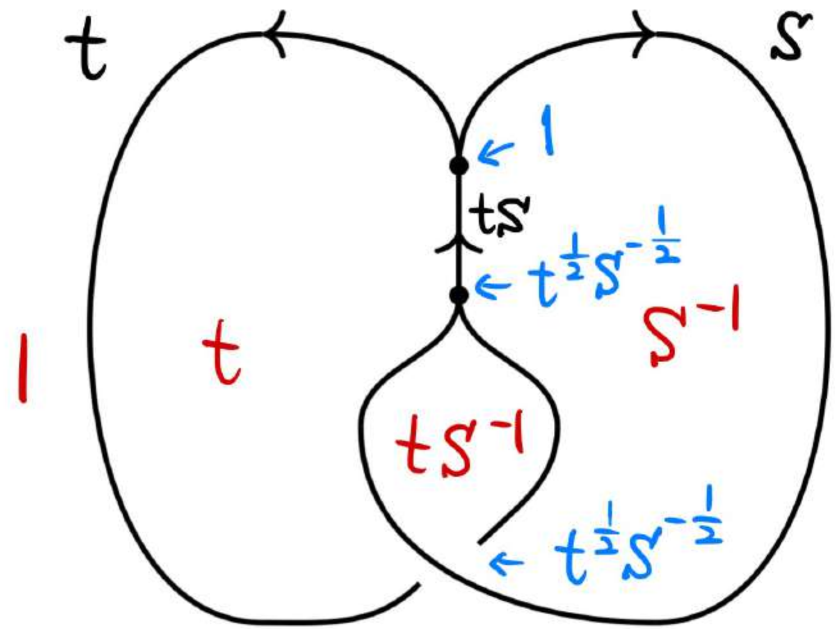
v : a vertex/crossing of D

Define the color $c(v)$ by



$c(v) = x^{\frac{1}{2}} s^{\frac{1}{2}}$ $\qquad$ $x^{\frac{1}{2}} s^{\frac{1}{2}}$ $\qquad$ $x t^{\frac{1}{2}} s^{\frac{1}{2}}$

E.g.



$$\mathrm{Rot}(D) = \frac{ts^{-1} \cdot ts^{-1}}{t^{1/2}s^{-1/2}\, t^{1/2}s^{-1/2}} = ts^{-1}$$

Definition (B.-Wu '24)

$$\mathrm{Rot}(D) = \prod_r c(r) \Big/ \prod_v c(v)$$

is called the rotation number of D $\in H_1(S^3 \setminus G)$

Remark :

- D is a plane curve

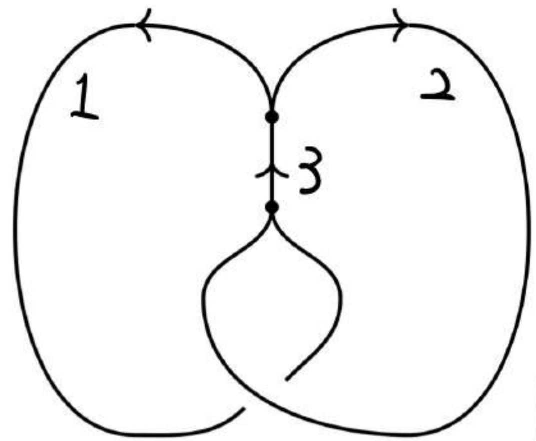
$$\mathrm{Rot}(D) = t^{\omega(D)}$$

- (D, c) is an MOY graph

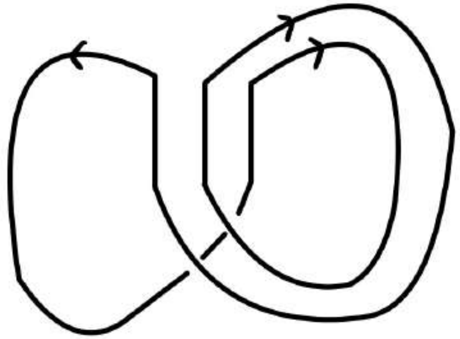
$$\varphi_c : H_1(S^3 \setminus G) \to \mathbb{Z}$$
$$\underbrace{[m_e]}_{\text{meridian of edge } e} \mapsto c(e)$$

Then $\varphi_c(\mathrm{Rot}(G)) = \omega(L(D,c))$

E.g



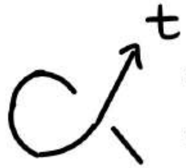
(D, c)



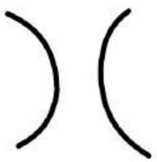
$L(D, c)$

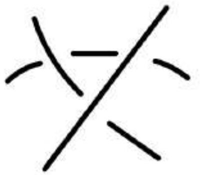
$$\varphi_c(\mathrm{Rot}(G)) = \varphi_c(t s^{-1})$$
$$= 1 - 2 = -1$$

Properties of Rot(D)

1) Rot() = Rot() = t Rot($\uparrow^{t}$)

Rot() = Rot() = t^{-1} Rot($\uparrow^{t}$)

2) Rot() = Rot()

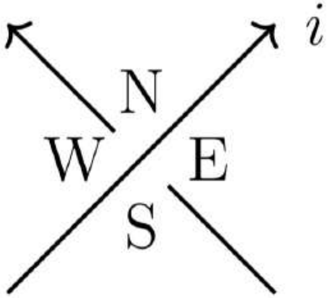
3) Rot() = Rot()

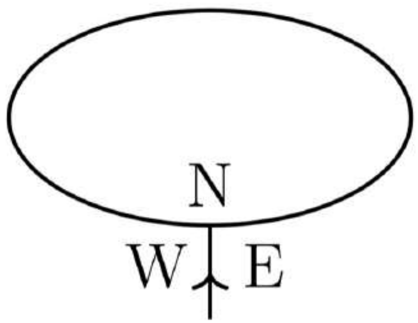
4) Rot() = Rot()

§2 State sum $\langle D \rangle$

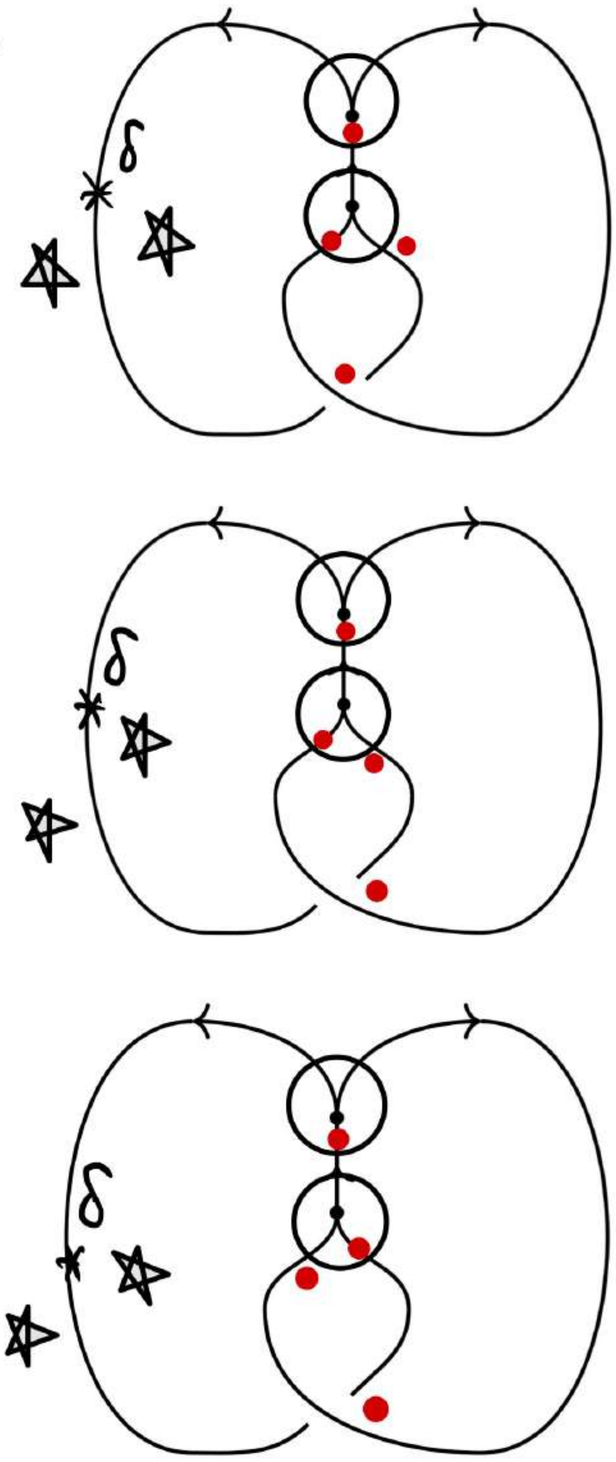
- choose a basepoint δ and mark the adjacent regions by $\star$
- Introduce a circle to each vertex

<u>A state S</u> is an assignment of an adjacent unmarked region or circle to a vertex / crossing

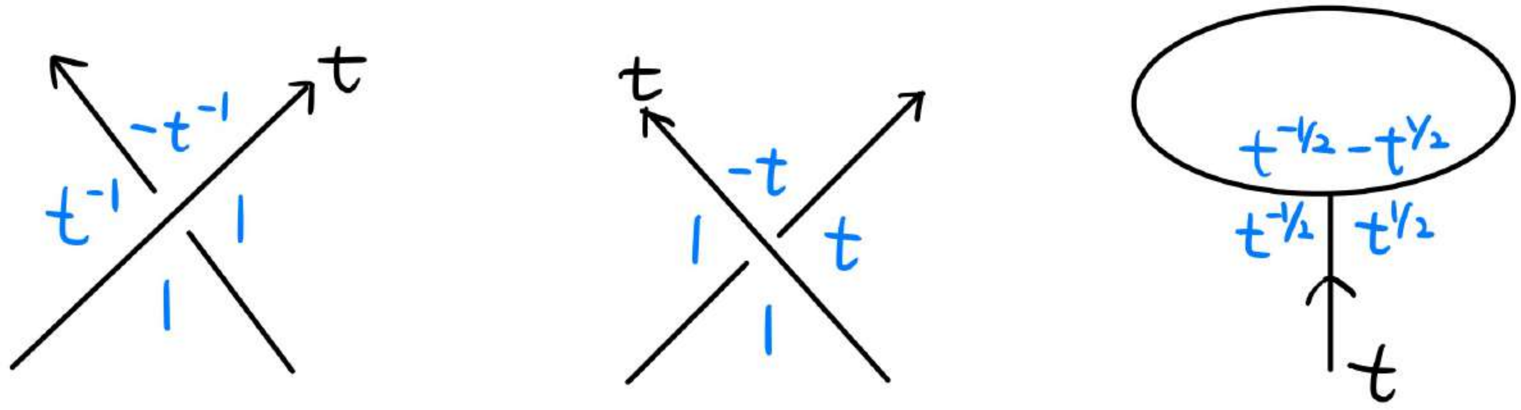




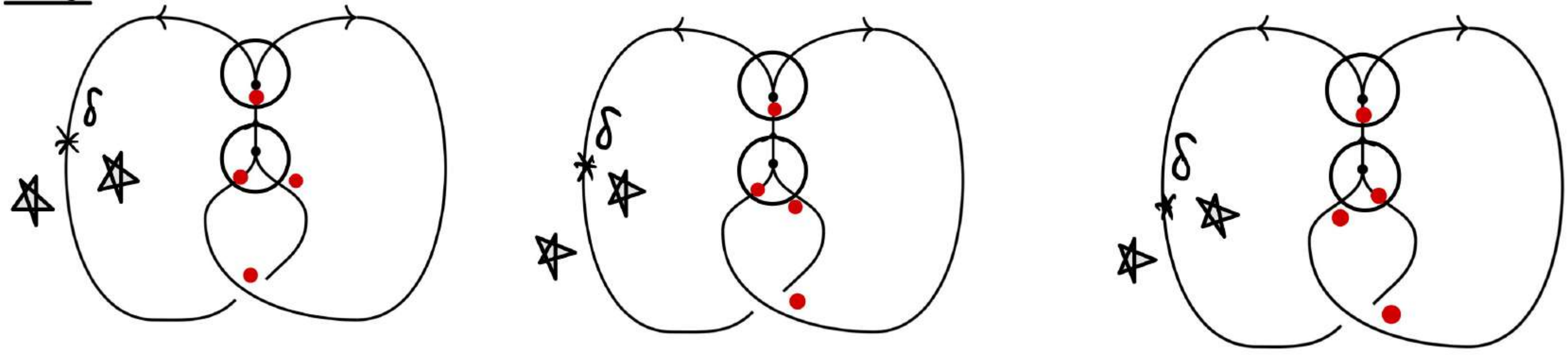
E.g.



Contribution of each state



Eg.



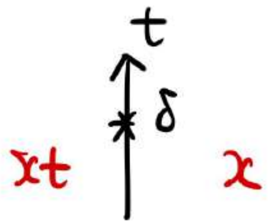
$$(t^{-1/2}s^{-1/2} - t^{1/2}s^{1/2})(s^{-1/2}-s^{1/2})\,t^{1/2}(-s)$$
$$= -(1-s)(1-ts)$$

$$(t^{-1/2}s^{-1/2} - t^{1/2}s^{1/2})(s^{-1/2}-s^{1/2})\cdot t^{-1/2}s$$
$$= t^{-1}(1-ts)(1-s)$$

$$(t^{-1/2}s^{-1/2} - t^{1/2}s^{1/2})(t^{-1/2}-t^{1/2})\,s^{1/2}s$$
$$= st^{-1}(1-t)(1-ts)$$

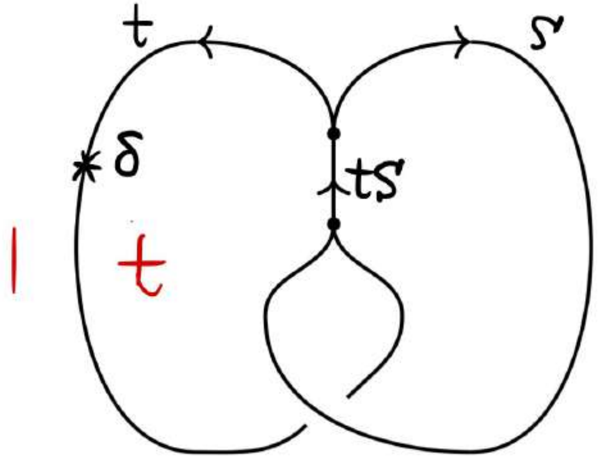
Definition (B.-Wu '24)

$$\langle D\rangle = \frac{\sum_{s:\text{state}} \text{contribution of } s}{|\delta|} \in \mathbb{Q}(t_1^{1/2}, t_2^{1/2}, \dots, t_n^{1/2})$$



$|\delta| = x - xt$

E.g.

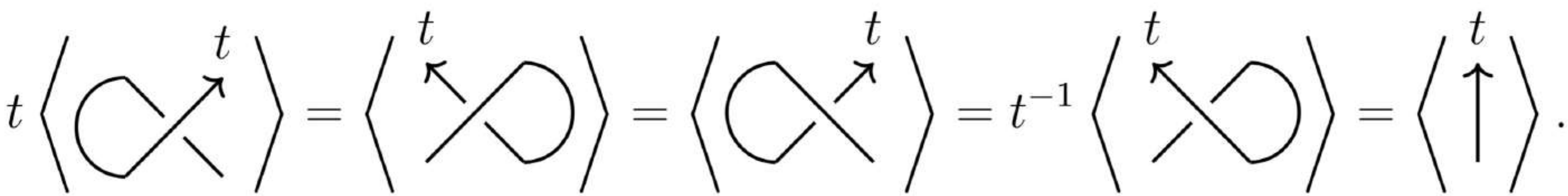


$|\delta| = 1 - t$

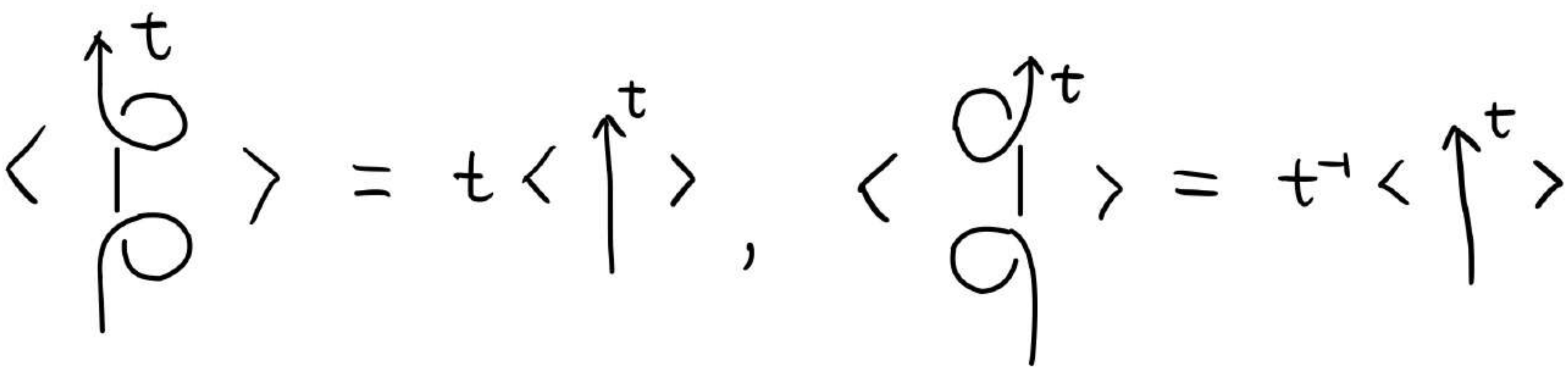
$$\langle D\rangle = \frac{-(1-s)(1-ts) + t^{-1}(1-ts)(1-s) + st^{-1}(1-t)(1-ts)}{1-t}$$

$$= \frac{(1-ts)(t^{-1}-1)}{1-t} = t^{-1}(1-ts)$$

Properties of $\langle D \rangle$

- $\langle D \rangle$ does not depend on the choice of δ
- $\langle D \rangle$ is invariant under Reid. moves (II)~(IV)
- 

Corollary :



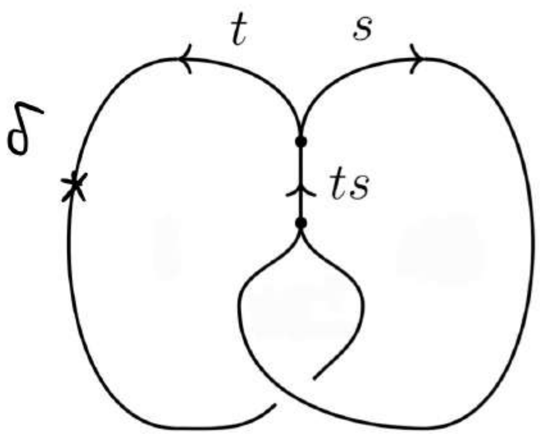
§3 Normalization

Theorem & Definition (B.-Wu '24)

$$\Delta_G = \mathrm{Rot}(G)^{1/2} \langle D \rangle$$

is invariant under Reid. moves (I) $\sim$ (IV), and thus is an invariant of a framed trivalent graph.

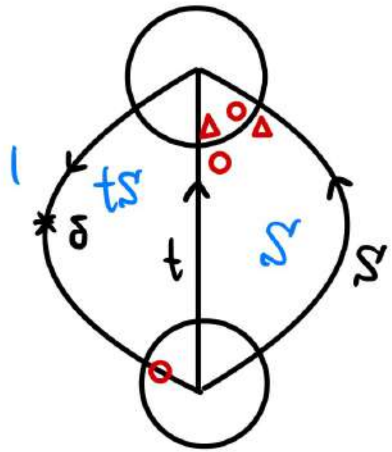
E.g.



$$\begin{aligned}\Delta_G &= (ts^{-1})^{1/2} \cdot t^{-1}(1-ts) \\ &= (ts)^{-\frac{1}{2}}(1-ts) \\ &= (ts)^{-\frac{1}{2}} - (ts)^{\frac{1}{2}}\end{aligned}$$

Example :

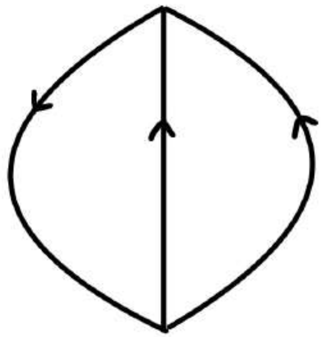
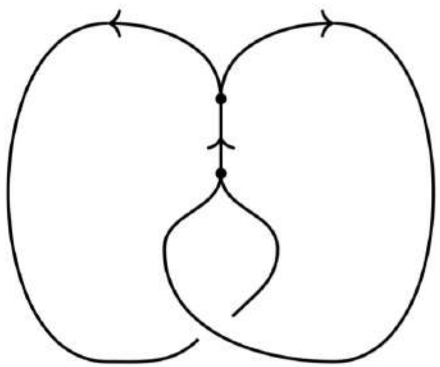
(1) Trivial Θ-curve



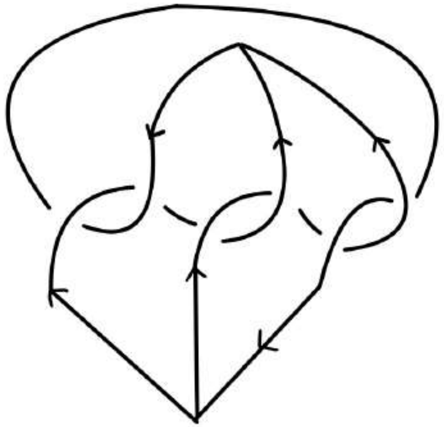
$$\langle D \rangle = \frac{\left[(ts)^{-\frac{1}{2}} - (ts)^{\frac{1}{2}} \right]^2}{1-ts} = t^{-1}s^{-1}(1-ts)$$

$$\mathrm{Rot}(D) = ts$$

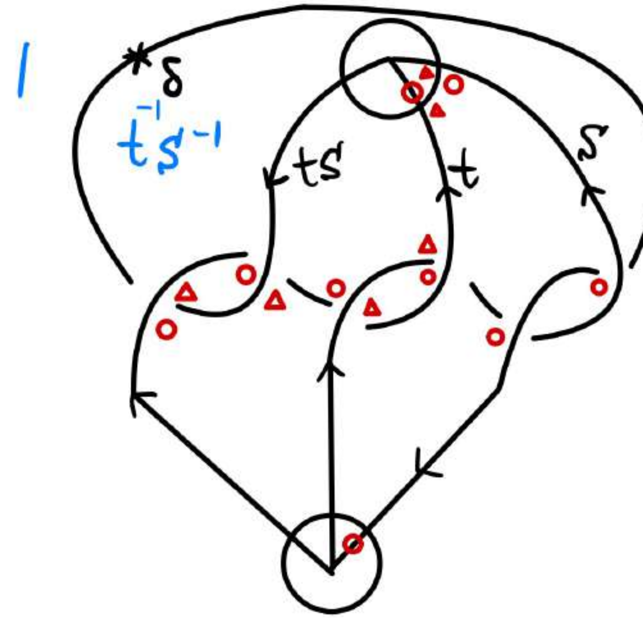
$$\Delta_G = t^{-\frac{1}{2}}s^{-\frac{1}{2}}(1-ts) = (ts)^{-\frac{1}{2}} - (ts)^{\frac{1}{2}}$$

$\therefore$  &  have the same Δ_G .

(2)



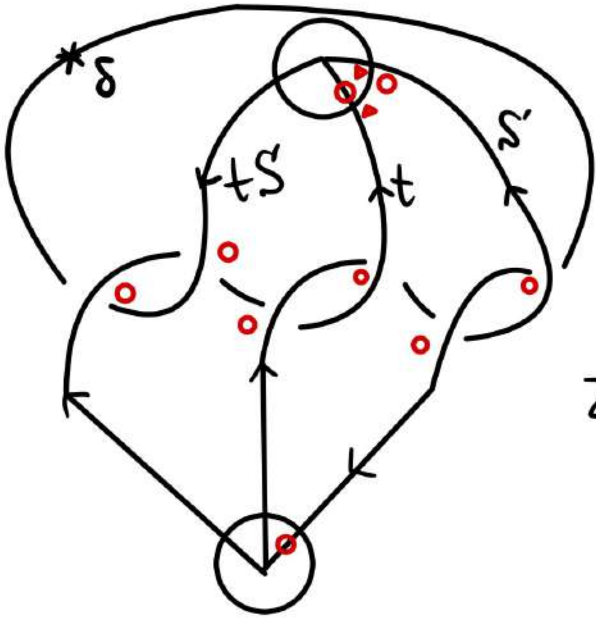
Suzuki's θ-curve



$$|\delta| = t^{-1}s^{-1} - 1$$

8 states, whose contribution is

$$\left[(ts)^{-\frac{1}{2}} - (ts)^{\frac{1}{2}}\right]^2 (1-s)\left[1-(ts)^{-1}\right] s^{-1}$$

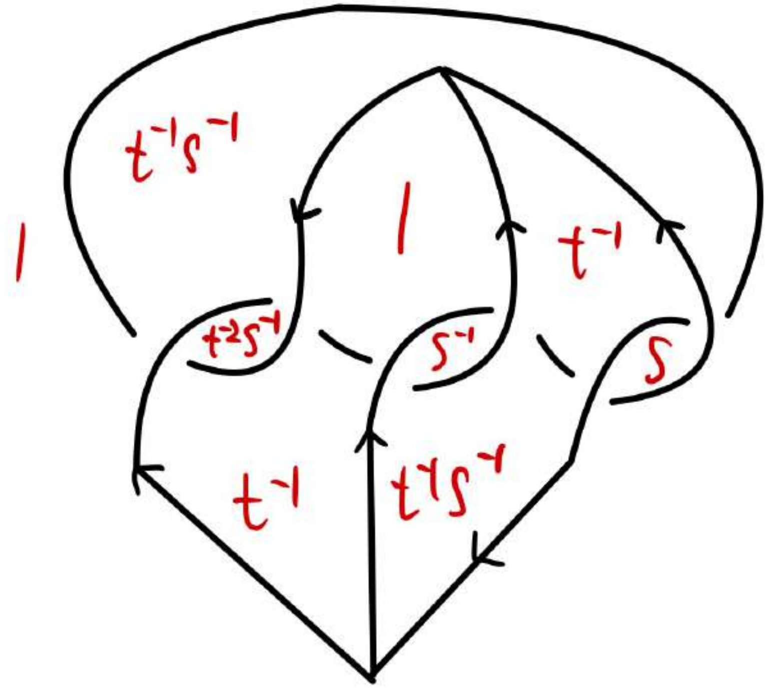


2 states, whose contribution is

$$\left[(ts)^{-\frac{1}{2}} - (ts)^{\frac{1}{2}}\right]^2 s^{-1} \cdot 1 \cdot (ts)^{-1}$$

$$\langle D \rangle = \frac{[(ts)^{-\frac{1}{2}} - (ts)^{\frac{1}{2}}]^2 s^{-1}(1-s+t^{-1})}{t^{-1}s^{-1}-1}$$

$$= (t^{-1}s^{-1}-1)\,t\,(1-s+t^{-1})$$

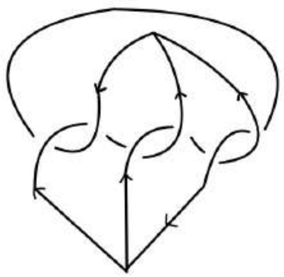
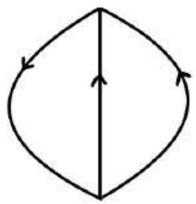


$$\mathrm{Rot}(D) = \frac{t^{-6}s^{-1}}{t^{-1}(ts)^{-1}t^{-2}s^{-1}t^{-1}}$$

$$= t^{-1}s$$

$$\therefore\ \Delta_D = t^{-\frac{1}{2}}s^{\frac{1}{2}}(t^{-1}s^{-1}-1)\,t\,(1-s+t^{-1})$$

$$= (t^{-\frac{1}{2}}s^{-\frac{1}{2}} - t^{\frac{1}{2}}s^{\frac{1}{2}})(1-s+t^{-1})$$

$\therefore$  $\not\cong$  as spatial graphs.

Remark :

- If D is a link, one can use writhe of D to modify Δ_D to get an link inv, which coincides with Conway function of D
- For an MOY graph (D, c),

$$\varphi_c(\Delta_D) = \Delta_{(D,c)}(t)$$

- Conjecture : Δ_D coincides with Viro's $gl(1|1)$ - Alexander polynomial.
- Viro's definition exists for graphs with sources and sinks.