向き付け不可能曲面の
fine curve graphのGromov双曲性

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Plan

§ 1. Background

§ 2. Bowden - Hensel - webb's idea

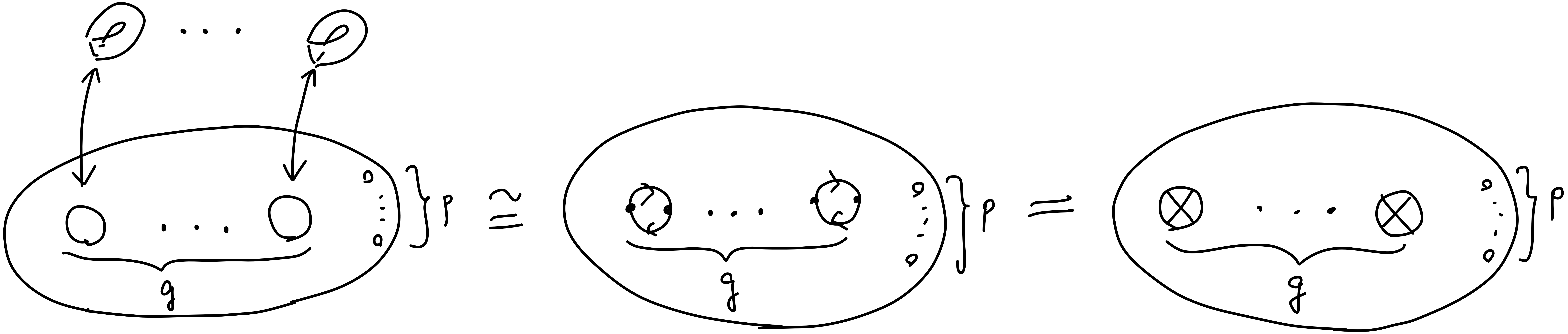
§ 3. Our solution for nonorientable surfaces

§ 1. Background

$S = S_{g,p}$: an ori. surf. of genus g with p punctures.

$N = N_{g,p}$: a nonori. surf. —— // ——.

$F = S$ or N.



(X, d): a metric sp., $x, y, w \in X$

$\langle x, y \rangle_w := \frac{1}{2}\left(d(w,x) + d(w,y) - d(x,y) \right)$: Gromov product

Def.

- A metric sp. (X, d) is δ-hyperbolic $(\delta \geq 0)$
  $\overset{\text{def}}{\iff} \forall w, \forall x, \forall y, \forall z \in X,$

$$\langle x, z \rangle_w \geq \min\{ \langle x, y \rangle_w, \langle y, z \rangle_w \} - \delta.$$

- A metric sp. (X, d) is Gromov hyperbolic
  $\overset{\text{def}}{\iff} \exists \delta \geq 0$ s.t. X is δ-hyp.

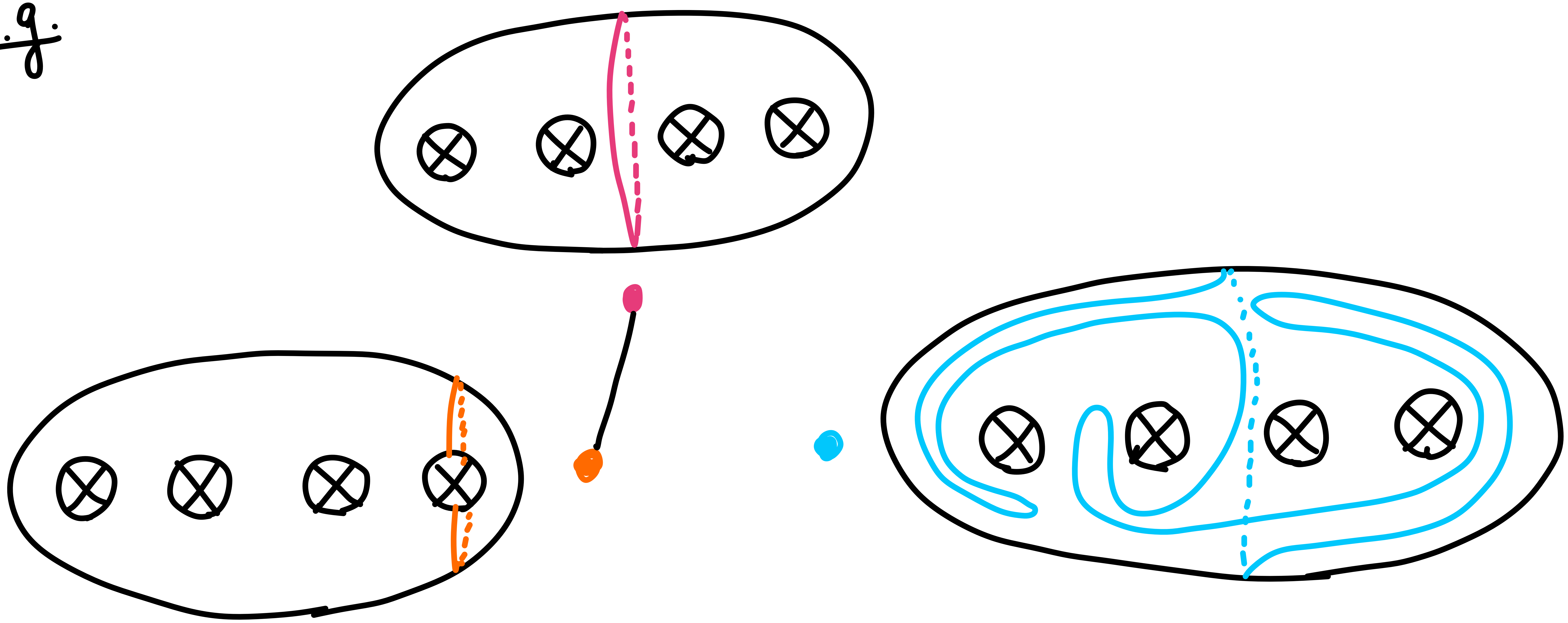
Def. (by Bowden – Hensel – Webb, 2022)

The fine curve graph $\mathcal{C}^{\dagger}(F)$ of F

$\overset{\text{def}}{\Longleftrightarrow}$
- a vertex: a (smoothly embedded) essential s.c.c. on F,
- an edge $\{\alpha, \beta\}$: α and β are disjoint on F.

NOT:

e.g.



<u>cf.</u>

(classical) curve graph $C(F)$ of F

$\underset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \bullet \text{ a vertex: an isotopy class of an ess. s.c.c. on } F, \\ \bullet \text{ an edge } \{a, b\}\text{: they can be realized disjointly on } F. \end{cases}$

- $\underline{\underline{\mathrm{Mod}}}(F) \curvearrowright C(F)$
  mapping class gp.

- $\mathrm{Diff}(F) \curvearrowright C^{\dagger}(F)$, $\mathrm{Homeo}(F) \curvearrowright C^{\dagger}(F)$

Today's subject

Thm. (Bowden–Hensel–Webb, 2022)

$g \geq 1$, $C^\dagger(S_g)$ is uniformly hyperbolic.

we can choose δ independent of top. type. of surfaces.

We prove that:

Thm. (Kimura–K.)

$g \geq 2$, $C^\dagger(N_g)$ is uniformly hyperbolic.

※ We endows each edge with length 1 in all graphs in this talk.

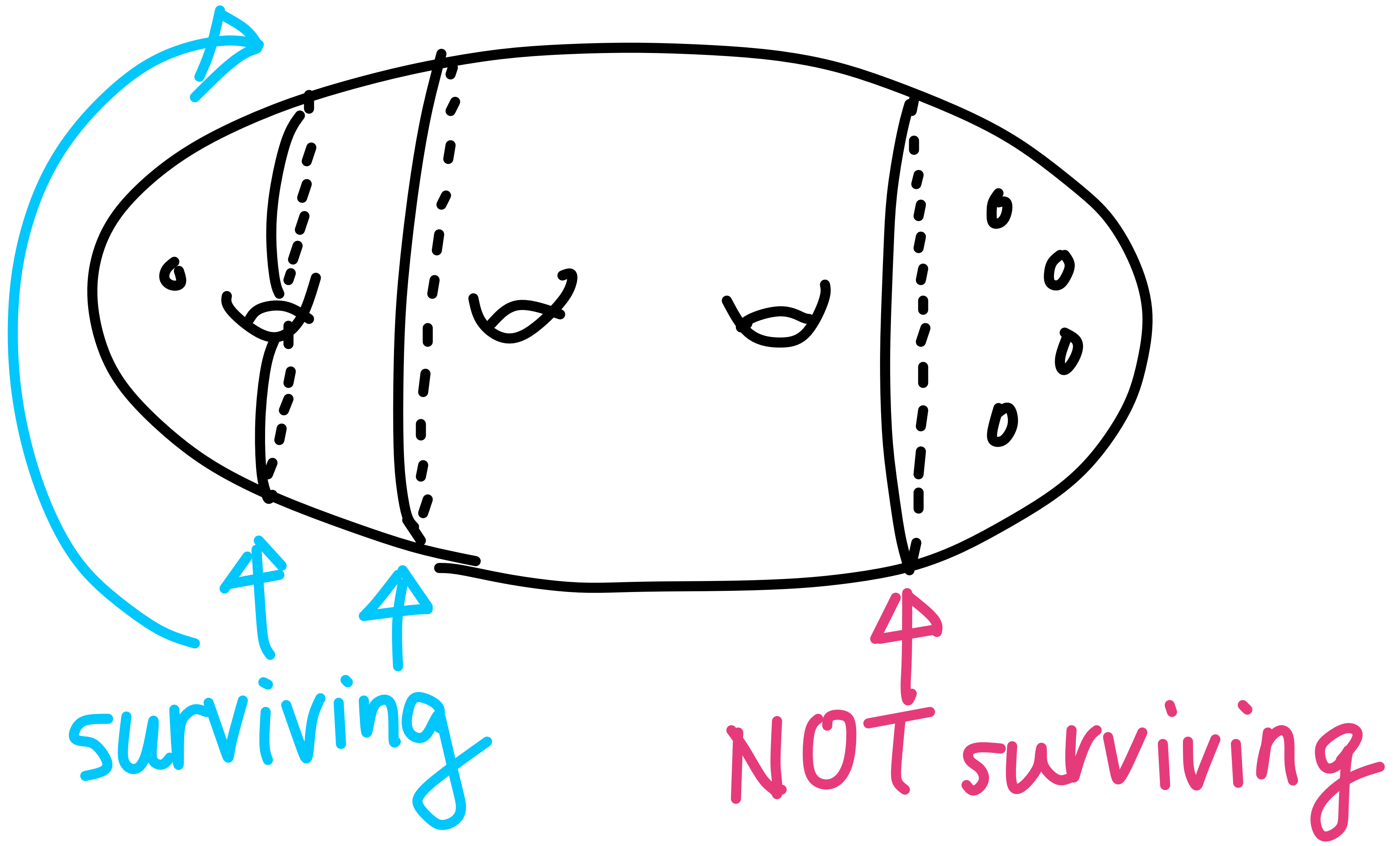
§ 2. Bowden-Hensel-Webb's idea

$S = S_g \ (g \geq 2)$, $P = \{x_1, x_2, \cdots, x_p\} \subset S$

Def

The surviving curve graph $C^s(S-P)$ is the full subgraph of the curve graph $C(S-P)$ spanned by all vertices which are essential even after filling in the punctures.

e.g.



Lem. 1

$$C^s(S-P) \underset{q.i.}{\sim} \underbrace{\mathcal{NC}}_{\text{nonseparating curve graph}}(S-P)$$

↳ uniformly hyp. by Rasmussen

$\therefore$ $C^s(S-P)$ is δ'-hyp.

Lem. 2

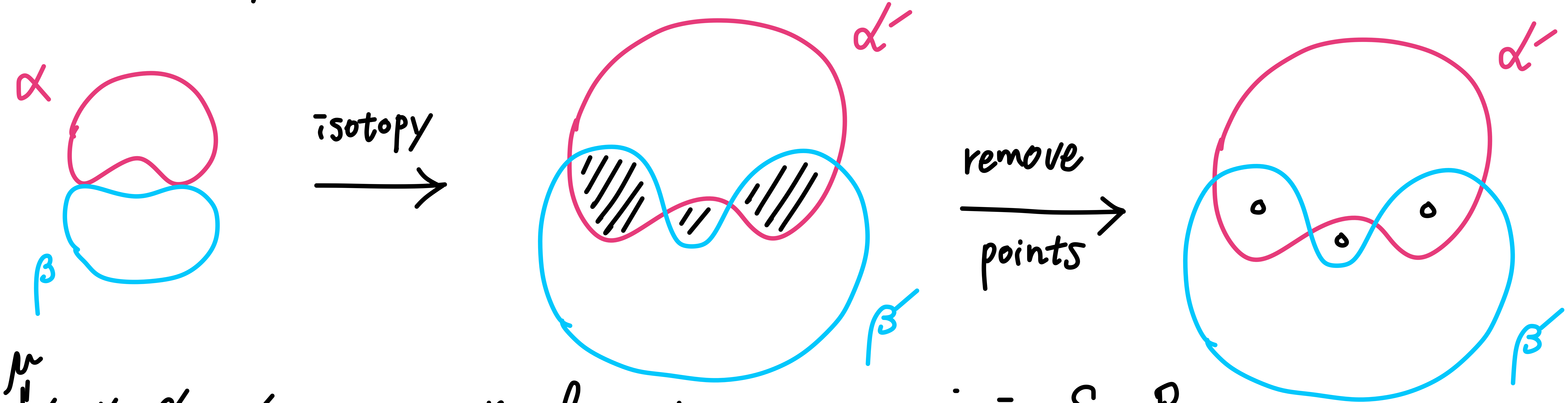
$\alpha, \beta \in C^t(S_g)$: transversal and in minimal position in $S-P$.

$$\Rightarrow d_{C^t(S_g)}(\alpha, \beta) = d_{\underbrace{C^s}_{\delta'\text{-hyp.}}(S_g - P)}\left(\underbrace{[\alpha]_{S_g-P}}_{\text{isotopy class in } S_g - P}, [\beta]_{S_g-P}\right)$$

Idea of proof

Goal: to prove $\forall \mu, \alpha, \beta, \gamma \in C^{\dagger}(S_g)^0$,

$$\langle \alpha, \gamma \rangle_\mu \geqslant \min\{\langle \alpha, \beta \rangle_\mu, \langle \beta, \gamma \rangle_\mu\} - \delta' - 4.$$



$\overset{\mu}{\overset{\shortparallel}{\mu'}}, \alpha', \beta', \gamma'$: transversal and in mini. posi. in $S - P$

s.t. $d_{C^{\dagger}(S)}(\alpha, \alpha'), d_{C^{\dagger}(S)}(\beta, \beta'), d_{C^{\dagger}}(\gamma, \gamma') \leq 1.$

$\Rightarrow$ we can apply Lem. 2 to $\alpha', \beta', \gamma', \mu$

$$\langle \alpha, \gamma \rangle_\mu \geq \langle \alpha', \gamma' \rangle_\mu - 2$$

$$\geq \min\{\langle \alpha', \beta' \rangle_\mu, \langle \beta', \gamma' \rangle_\mu\} - \delta' - 2$$

($\because$ $C^s(S-P)$ is δ'-hyp.)

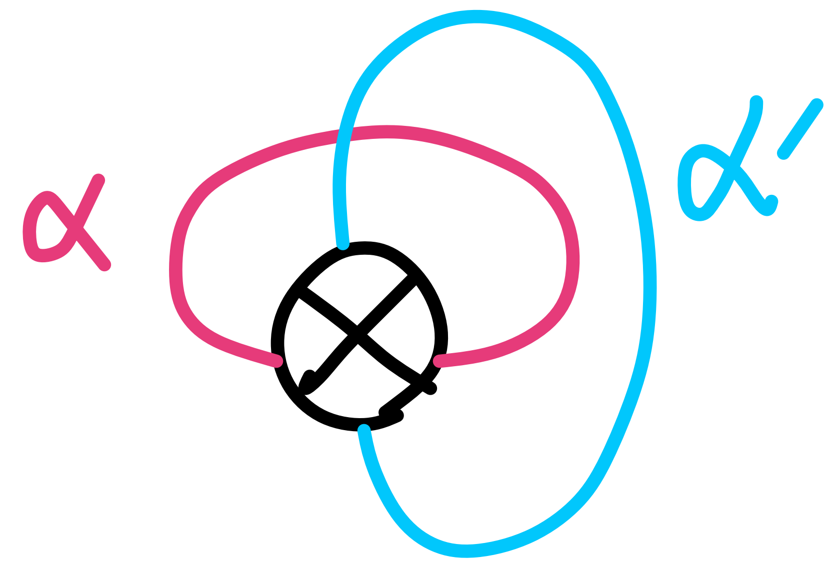
$$\geq \min\{\langle \alpha, \beta \rangle_\mu, \langle \beta, \gamma \rangle_\mu\} - 2 - \delta' - 2.$$

§ 3. Our solution for nonorientable surfaces

Issue for nonori. surf.

Isotopic one-sided curves always intersect

:⇔ the reg. nbd. is a Möbius band.



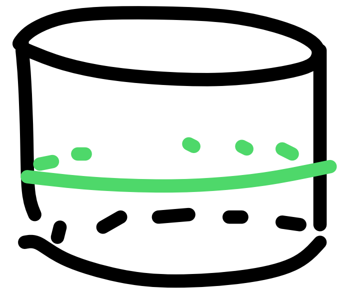
one-sided

Our solution

$N = N_g \ (g \geq 3)$.

$C_{two}(N) < C(N)$: spanned by only two-sided s.c.c.s

:⇔ the reg. nbd. is an annulus.



two-sided

$C^{\pm}(N)$: "extended" curve graph

vertices {
- the usual vertices
- + the isotopy classes of s.c.c.s bounding a Möbius band.

Prop. 1

$N = N_{g,p}$ $(g \geq 3,\ p \geq 0)$,

$C^s(N)$, $C^{\pm s}(N)$, $C^{\pm s}_{two}(N)$ are path-conn., and

$$\underline{\underline{\mathcal{NC}}}(N) \underset{q.i.}{\sim} C^s(N) \underset{q.i.}{\sim} C^{\pm s}(N) \underset{q.i.}{\sim} C^{\pm s}_{two}(N).$$

unif. hyp. by K.

$\therefore$ $C^{\pm s}_{two}(N)$ is δ-hyp.

Prop. 2

$N = N_g$ $(g \geq 3)$.

$C^{+}(N)$, $C^{\pm +}(N)$, $C^{\pm +}_{\mathrm{two}}(N)$ are path-conn., and

$C^{+}(N) \underset{q.i.}{\simeq} C^{\pm +}(N) \underset{q.i.}{\simeq} C^{\pm +}_{\mathrm{two}}(N)$.

we will prove $C^{\pm +}_{\mathrm{two}}(N)$ are unif. hyp.

Lem. 3 $N = N_g$ $(g \geq 3)$.

$P \subset N$: a finite set.

$\alpha, \beta \in C^{\pm +}_{\mathrm{two}}(N)^{0}$: transversal and in mini. posi. in $N - P$.

$$\Rightarrow d_{C^{\pm +}_{\mathrm{two}}(N)}(\alpha, \beta) = d_{C^{\pm s}_{\mathrm{two}}(N-P)}([\alpha]_{N-P}, [\beta]_{N-P}).$$

Outline of proof

$\mu, \alpha, \beta, \gamma \in C^{\pm t}_{\underline{\underline{two}}}(N)^0$

$\Longrightarrow$ we can take $\mu', \alpha', \beta', \gamma' \in C^{\pm t}_{two}(N)^0$

by the same way.