コード図の数え上げと doodle 不変量による Milnor の 3 重絡み数の表示

平田　遼介

信州大学大学院総合理工学研究科　理学専攻数学分野

2024 年 12 月 25 日

結び目の数理VII
早稲田大学

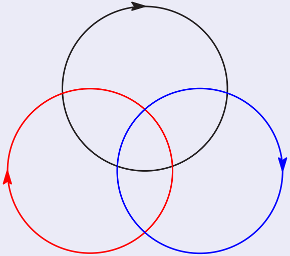
From the results in X. -S. Lin and Z. Wang(1996), the Casson knot invariant can be expressed as the sum of the Gauss diagram term and the Arnold invariant.
In this talk, I will show that the Milnor invariants can be expressed as the sum of the invariants of plane curves and the counting terms of chord diagrams formed from the link.

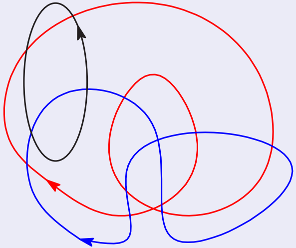
$$\bar{\mu}_L(123) \equiv -\mu(C_1, C_2, C_3) - \sum_{(i,j,k)\in\mathfrak{A}_3} \langle {}_{\mathrm{j}}\otimes_{\mathrm{i}}\, , G_{L_k} \rangle \mod \Delta_L(123). \tag{1}$$

Definition 1 (doodle)

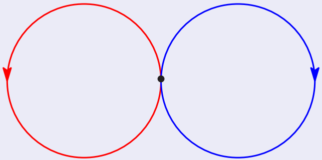
A *doodle* is a union of plane curves that has only transversal double intersections.

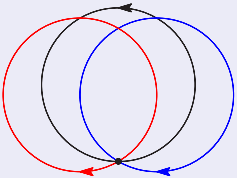
Doodles





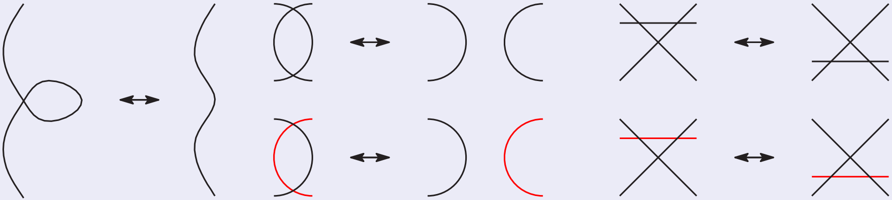
Not doodles





Definition 2

Two doodles are *equivalent* if one doodle is transformed to the other by following moves.



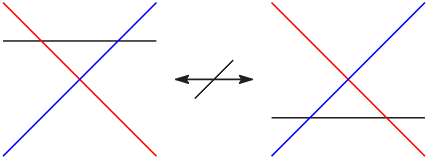
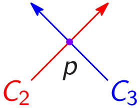
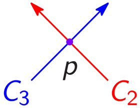


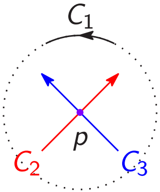
Figure : Forbidden move



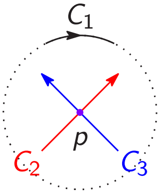
$\delta(p; C_2, C_3) = +1$



$\delta(p; C_2, C_3) = -1$



$\epsilon(p; C_1, C_2, C_3) = \delta(p; C_2, C_3)$



$\epsilon(p; C_1, C_2, C_3) = -\delta(p; C_2, C_3)$

Definition 3 (R. Fenn and P. Taylor, 1979)

Consider doodle $D = (C_1, C_2, C_3)$ with no self-intersections. Consider all intersections of C_2 and C_3 inside C_1 and define $\mu(C_1, C_2, C_3)$ by

$$\mu(C_1, C_2, C_3) := \sum_p \epsilon(p; C_1, C_2, C_3). \qquad (2)$$

Example 4

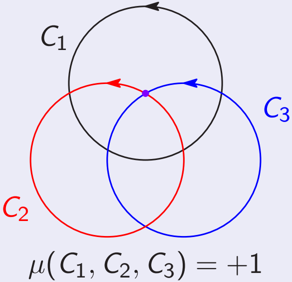


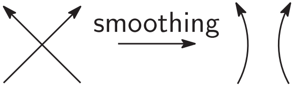
Figure : Examples of μ-invariants

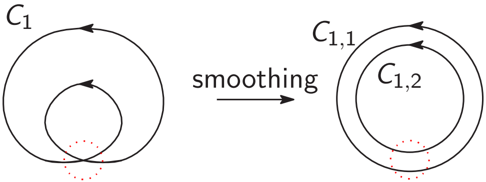
Define μ-invariant for doodle with self-intersections.

Definition 5 (μ-invariant)

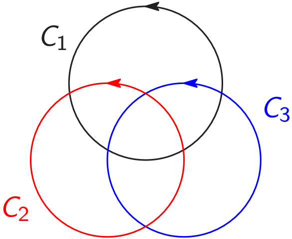
Suppose that C_1 can be decomposed by smoothing into a union of simple closed curves $(C_{1,1}, \dots, C_{1,a})$. Then, define the *$\mu$-invariant* of doodle $D = (C_1, C_2, C_3)$ as

$$\mu(C_1, C_2, C_3) := \sum_{i=1}^{a} \sum_{p} \epsilon(p; C_{1,i}, C_2, C_3) \tag{3}$$

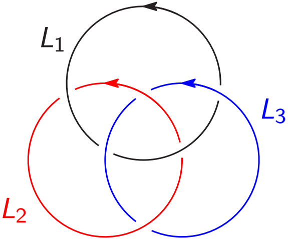




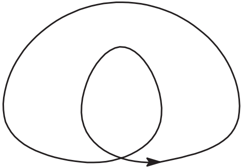
Construction of the link homotopy type $L(D) = (L_1, L_2, L_3)$ from a doodle $D = (C_1, C_2, C_3)$.



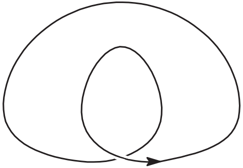
D



$L(D)$



D'

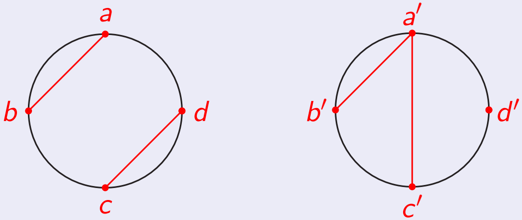


$L(D')$

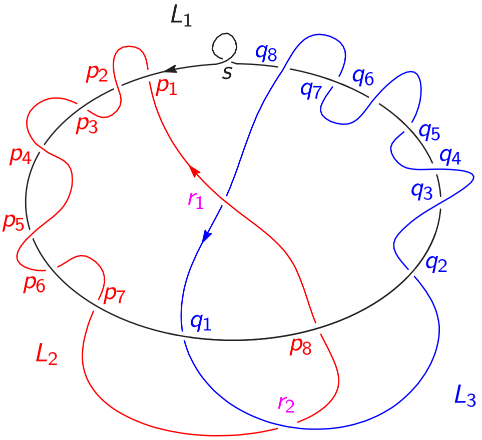
Note that $\mathrm{Link}(L_i, L_j) = 0$.

Definition 6

A *generalized chord diagram* is a chord diagram like below.



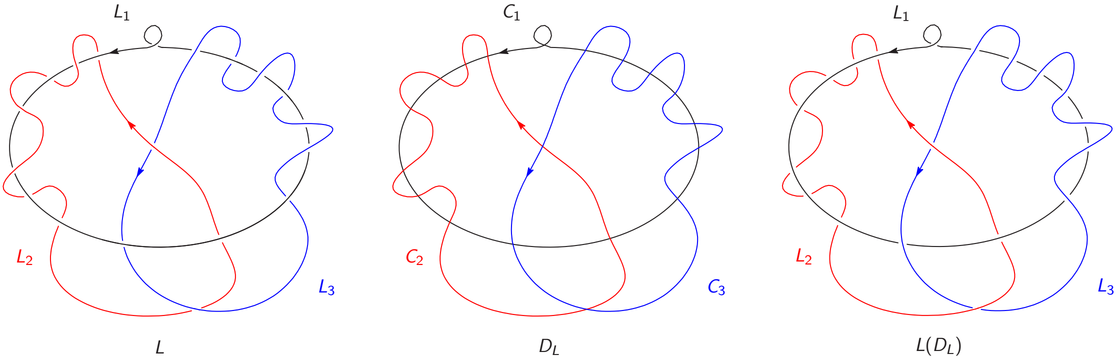
Let $L = (L_1, L_2, L_3)$ be a 3-component link.



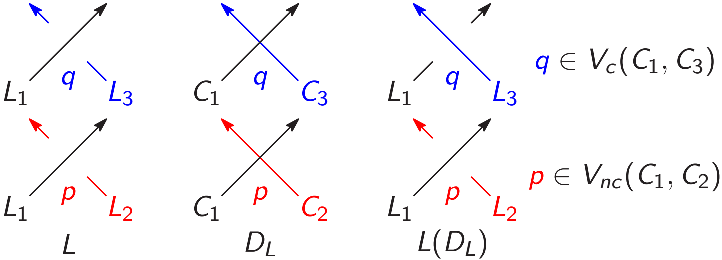
L

Let $V_a(L_1)$ be the set of all crossings that involve L_1 and another component of L where the branch of L_1 is over-crossing. $V_a(L_1) = \{p_1^-, p_3^-, p_6^+, p_7^-, p_8^+, q_1^-, q_2^+, q_5^-, q_7^-\}$.

Now consider a doodle D_L and a link $L(D_L)$.

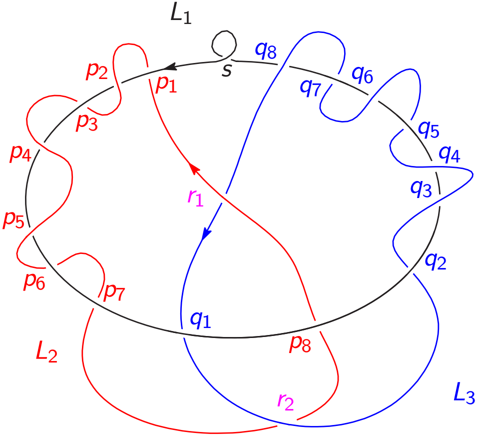


Let $V_c(C_1, C_3)$ and $V_{nc}(C_1, C_2)$ be the set of intersection points such that

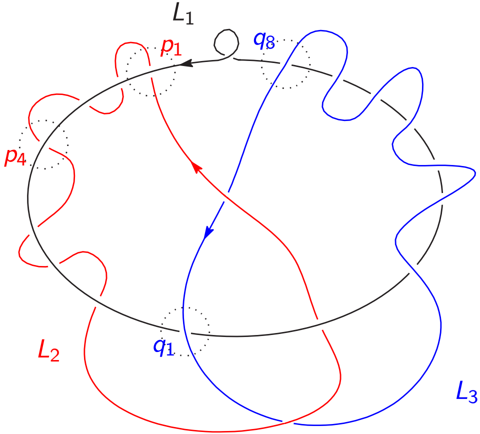


$V_a(L_1)$ can be divided into these two sets,

$$V_a(L_1) = V_c(C_1, C_3) \sqcup V_{nc}(C_1, C_2). \tag{4}$$



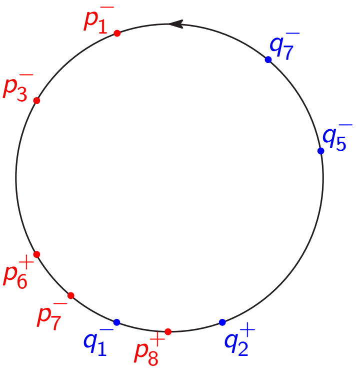
L



$L(D_L)$

$$V_a(L_1) = \{p_1^-, p_3^-, p_6^+, p_7^-, p_8^+, q_1^-, q_2^+, q_5^-, q_7^-\}. \tag{5}$$

The next step is to construct a generalized chord diagram G_{L_1} from $V_a(L_1)$.



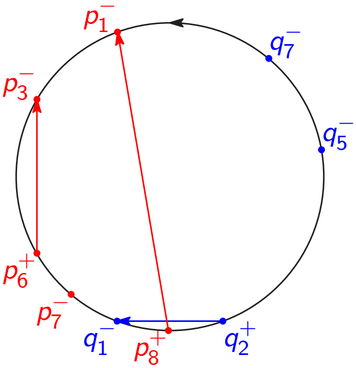
First, the sign divides $V_c(C_1, C_3)$ and $V_{nc}(C_1, C_2)$ into two sets.

$$V_c(C_1, C_3) = \{q_2^+\} \sqcup \{q_1^-, q_5^-, q_7^-\}, \tag{6}$$

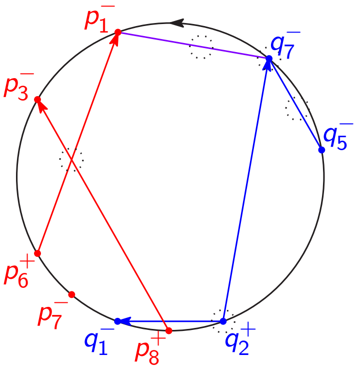
$$V_{nc}(C_1, C_2) = \{p_6^+, p_8^+\} \sqcup \{p_1^-, p_3^-, p_7^-\}. \tag{7}$$

Then connect pairs of points (q_i^+, q_j^-) or (p_i^+, p_j^-) by chords so that two chords with the same color do not have intersections, and two or more chords do not share a single vertex.

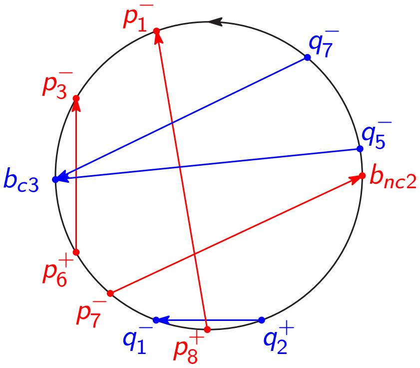
○



×



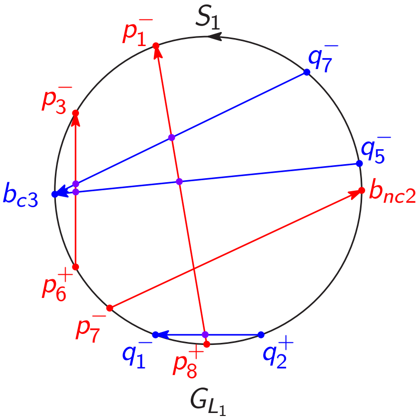
Finally, take two points b_{nc2} and b_{c3} on the circumference and connect each of the extra points.



G_{L_1}

Let G_{L_1} be the chord diagram thus constructed.

Let $T_{1,l}$ be the set of chords whose endpoints are the intersections of L_1 and L_l. We will write S_1 for the skeleton in G_{L_1}, and let $\langle {}_3\bigotimes_2, G_{L_1} \rangle$ stand for the sum of $\epsilon(p; S_1, T_{1,3}, T_{1,2})$ of all intersections of $T_{1,3}$ and $T_{1,2}$.



We define $\langle {}_1\!\bigotimes_3 , G_{L_2} \rangle$ and $\langle {}_2\!\bigotimes_1 , G_{L_3} \rangle$ in the same ways.

Theorem 7

Let $L = L_1 \cup L_2 \cup L_3$ be an oriented 3-component link. Let $D_L = (C_1, C_2, C_3)$ be the 3-component doodle obtained by projecting L to a plane. Also, let $\mathfrak{A}_3$ be the third alternating group. Define

$$\Delta_L(123) := \gcd\{\mathrm{Link}(L_1, L_2), \mathrm{Link}(L_2, L_3), \mathrm{Link}(L_3, L_1)\}. \tag{8}$$

Then we have

$$\bar{\mu}_L(123) \equiv -\mu(C_1, C_2, C_3) - \sum_{(i,j,k)\in\mathfrak{A}_3} \langle {}_j\!\bigotimes_i , G_{L_k} \rangle \mod \Delta_L(123). \tag{9}$$

Outline of proof

Theorem 7 is based on the following Theorem:

Theorem 8 (B. Mellor and P. Melvin, 2003)

Let $L = (L_1, L_2, L_3)$ be a link. Let F_i be a Seifert surface bounded by component L_i and let F be $F_1 \cup F_2 \cup F_3$. For any choice of Seifert surfaces F,

$$\bar{\mu}_L(123) \equiv m_{123}(F) - t_{123}(F) \mod \Delta_L(123). \tag{10}$$

t_{123} represents the sum of the signs of the triple points of F, and m_{123} is an integer determined from the intersection of $\mathrm{Int}\,(F)$ and L. It is possible to choose F so that $m_{123} = 0$. It can be done by first constructing F_i by the Seifert algorithm, and then surgering them as the generalized chord diagrams G_{L_i} indicates.

When considering such a surface, the term of the generalized chord diagram represents the sum of the signs of the triple points on the "tube" and the doodle invariant part represents the sum of the signs of the triple points in other places.