A presentation of the pure cactus group of degree four

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Definition [Henrique-Kaminitzer]

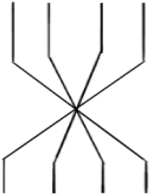
For $n \in \mathbb{Z}_{\geq 2}$, the cactus group J_n of degree n is defined by the following presentation.

Generators: $s_{p,q}$ with $1 \leq p < q \leq n$

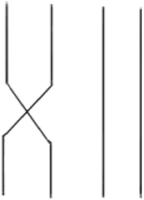
Relations:

- $s_{p,q}^2 = e$
- $s_{p,q} s_{m,r} = s_{m,r} s_{p,q}$ $([p,q] \cap [m,r] = \varnothing)$
- $s_{p,q} s_{m,r} = s_{p+q-r,p+q-m} s_{p,q}$ $([m,r] \subset [p,q])$

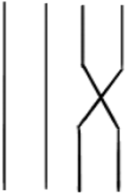
Here, $[p,q]$ denotes the set $\{p, p+1, \ldots, q-1, q\}$ of integers for positive integers p, q with $p < q$.



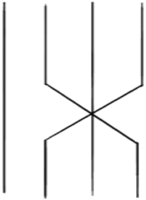
s_{14}



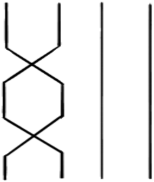
s_{12}



s_{34}

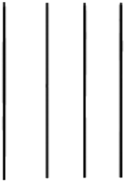


s_{24}

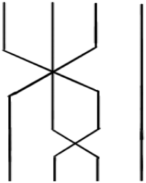


s_{12}^2

$=$

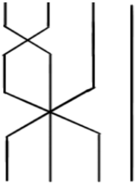


e



$s_{13}s_{23}$

$=$



$s_{12}s_{13}$

We can consider the natural surjection π from J_n to the symmetric group S_n of degree n.

Definition

The pure cactus group PJ_n of degree n is defined by the kernel of the natural surjection π.

Known facts

- $PJ_4 \cong \pi_1(\overline{M_0}^5(\mathbb{R}))$ [Henriques-Kamnitzer, 2006]
- $PJ_4 \cong \pi_1\left(\overset{5}{\#}\mathbb{RP}^2\right) \cong < \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 >$ [Cf. Genevois, 2022] (without direct proof)
- $PJ_4 \cong < \alpha, \beta, \gamma, \delta, \epsilon \mid \alpha\gamma\epsilon\beta\epsilon\alpha^{-1}\delta^{-1}\beta\gamma\delta^{-1} >$ [Bellingeri-Chemin-Lebed, 2022] (by Reidemeister-Schreier method.)

Theorem [Hama-Ichihara]

PJ_4 has the following presentation.

$$\left\langle g_1, \cdots, g_{10} \;\middle|\; \begin{array}{l} g_1 g_{10}^{-1} g_2^{-1}, g_9 g_5^{-1} g_4, g_5 g_1 g_6^{-1}, \\ g_8 g_{10} g_7^{-1}, g_8 g_3^{-1} g_4, \\ g_2 g_9 g_7^{-1} g_6 g_3^{-1} \end{array} \right\rangle$$

This presentation is transformed to the next one.

$$\langle g_2, g_4, g_8, g_9, g_{10} \mid g_2 g_9 g_{10}^{-1} g_8^{-1} g_4 g_9 g_2 g_{10} g_8^{-1} g_4^{-1} \rangle$$

Cororally

The above presentation is equivalent to

$$< \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \alpha_1^2 \alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5^2 > \cong \pi_1(\overset{5}{\#} \mathbb{RP}^2).$$

Remark [Bellingeri-Chemin-Lebed, 2022]

The above presentation is also equivalent to

$$< \alpha, \beta, \gamma, \delta, \epsilon \mid \alpha\gamma\epsilon\beta\epsilon\alpha^{-1}\delta^{-1}\beta\gamma\delta^{-1} >.$$

Poincaré's polygon theorem [Cf. Maskit, 1971]

Let D be a polygon with side-identifications on $\mathbb{H}^2$. Let G be the group generated by the side-identifications. Then G is discontinuous, D is a fundamental polygon for the action $G \curvearrowright \mathbb{H}^2$, and the cycle relations form the complete set of relations for G.

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Sketch of proof

- Find the fundamental polygon as D w.r.t. $PJ_4 \curvearrowright C_4^{\{2,3\}} \cong \mathbb{H}^2$
- Applying Poincaré's polygon theorem.

Cf. [Genevois 2022]

$$J_4^{\{2,3\}} = \left\langle s_{12}, s_{23}, s_{34}, s_{13}, s_{24} \;\middle|\; \begin{array}{l} s_{ij}^2 = e, s_{13}s_{12} = s_{23}s_{13}, \\ s_{24}s_{23} = s_{34}s_{24}, \\ s_{12}s_{34} = s_{34}s_{12} \end{array} \right\rangle$$

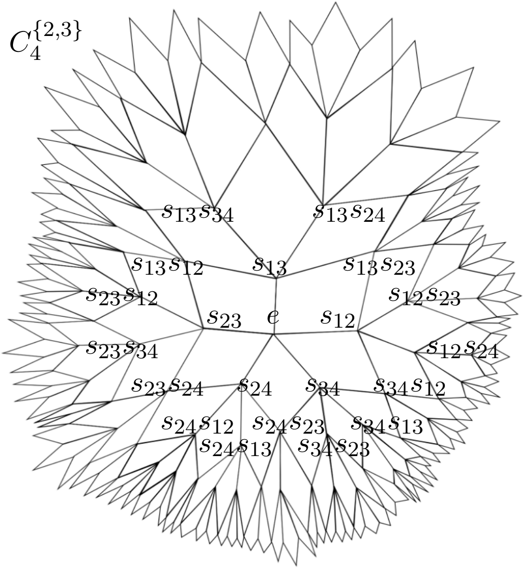
$C_4^{\{2,3\}}$ denotes the Cayley complex of $J_4^{\{2,3\}}$.

Fact [Genevois, 2022]

The map Γ induced from Γ_0 defined by the following implies an action of PJ_4 on ${C_4}^{\{2,3\}}$

$$\begin{aligned} \Gamma_0 : PJ_4 \times \left({C_4}^{\{2,3\}}\right)^{(0)} &\longrightarrow \left({C_4}^{\{2,3\}}\right)^{(0)} \\ (g,h) &\longmapsto \begin{cases} gh & gh \in J_4^{\{2,3\}} \\ ghs_{14} & gh \notin J_4^{\{2,3\}} \end{cases}, \end{aligned}$$

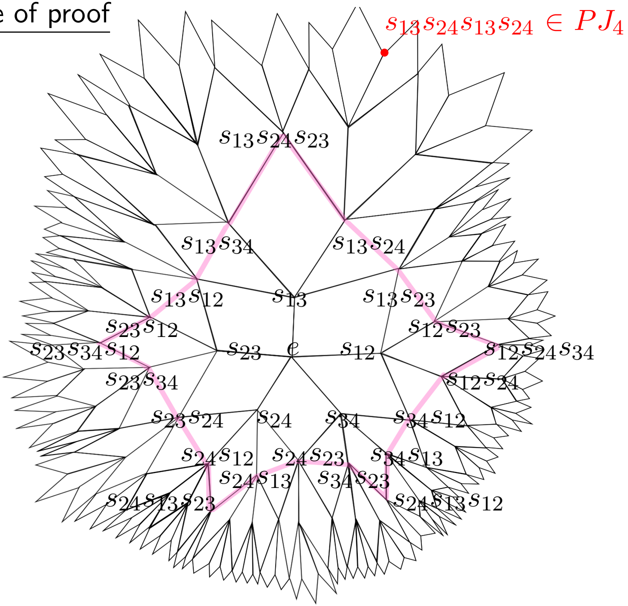
and so, Γ acts on ${C_4}^{\{2,3\}}$ freely and cocompactly.

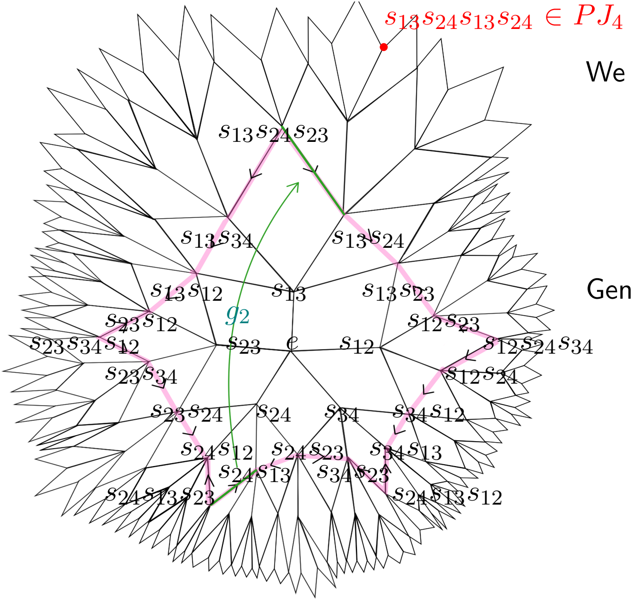


[Fact]
$C_4^{\{2,3\}}$ is isometric to $\mathbb{H}^2$ up to scaling.

$$s_{ij}^2 = e$$
$$s_{13}s_{12} = s_{23}s_{13}$$
$$s_{24}s_{23} = s_{34}s_{24}$$
$$s_{12}s_{34} = s_{34}s_{12}$$

Outline of proof





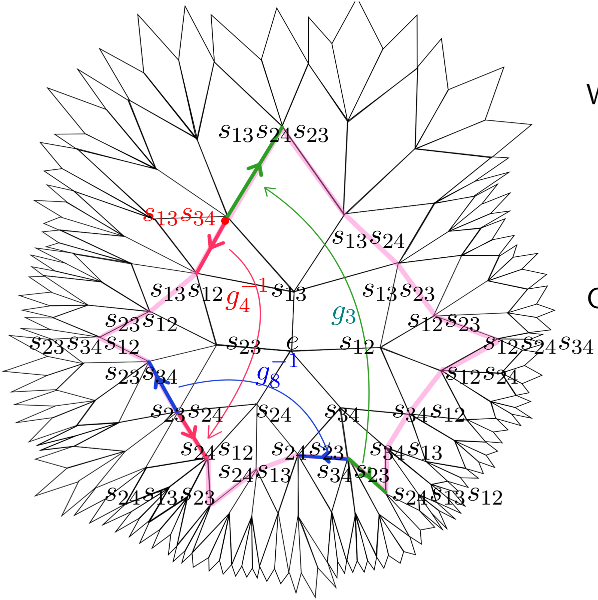
We obtain a generator:

$$g_2$$

$$\downarrow$$

Generators:

$$g_1, g_2, g_3, g_4, g_5,$$
$$g_6, g_7, g_8, g_9, g_{10}$$



We obtain a cycle relation:

$$g_3 g_8^{-1} g_4^{-1}$$

$$\downarrow$$

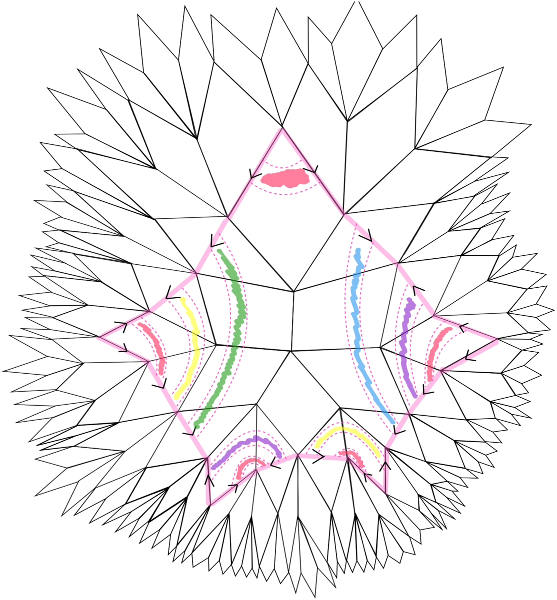
Complete set of relations:

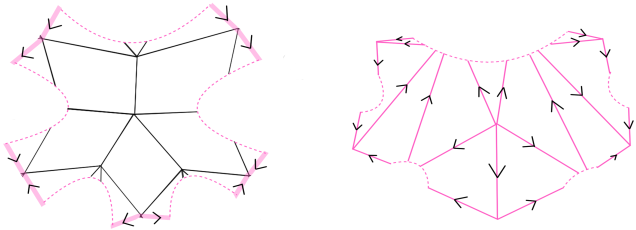
$$g_3 g_8^{-1} g_4^{-1},$$
$$g_5 g_9^{-1} g_4^{-1},$$
$$g_5 g_1 g_6^{-1},$$
$$g_8 g_{10} g_7^{-1},$$
$$g_{10} g_1^{-1} g_2,$$
$$g_3 g_6^{-1} g_7 g_9^{-1} g_2^{-1} \qquad \square$$

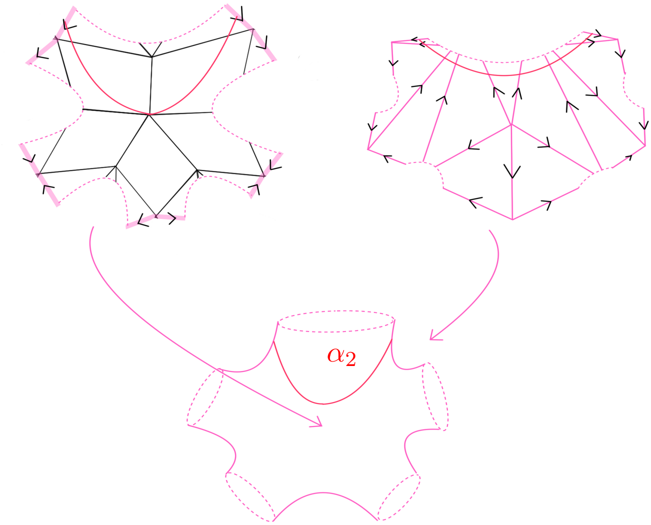
Explicit isomorphism

The following map is isomorphism.

$$
\begin{aligned}
f : &\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 | \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 \rangle \\
\longrightarrow &\left\langle g_1, \cdots, g_{10} \;\middle|\; \begin{array}{l} g_1 g_{10}^{-1} g_2^{-1}, g_9 g_5^{-1} g_4, g_5 g_1 g_6^{-1}, \\ g_8 g_{10} g_7^{-1}, g_8 g_3^{-1} g_4, \\ g_2 g_9 g_7^{-1} g_6 g_3^{-1} \end{array} \right\rangle \\
\alpha_1 &\longmapsto g_1^{-1} = g_{10}^{-1} g_2^{-1} \\
\alpha_2 &\longmapsto g_2 g_{10} g_5^{-1} g_8 g_3^{-1} = g_2 g_{10} g_9^{-1} g_4^{-2} \\
\alpha_3 &\longmapsto g_4 \\
\alpha_4 &\longmapsto g_9 g_{10}^{-1} \\
\alpha_5 &\longmapsto g_8^{-1} g_6 = g_8^{-1} g_4 g_9 g_2 g_{10}
\end{aligned}
$$






α_2

Thank you
for your attention.