奈良女子大学
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Virtualized n-gon moves for virtual knots

奈良女子大学 人間文化総合科学研究科 修士２年

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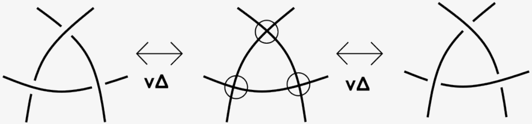
結び目の数理 Ⅶ
2025/02/04 修正

Outline

Background

- Not every virtual knot can be unknotted by crossing changes. [D.Hrencein-L.H.Kauffman(2003)]
- Every virtual knot can be unknotted by virtualized Δ-moves. [Nakamura-Nakanishi-Satoh-Wada(2024)]

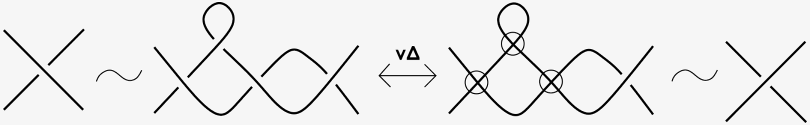


Background

- Virtualized Δ-move is more elemental than crossing change.

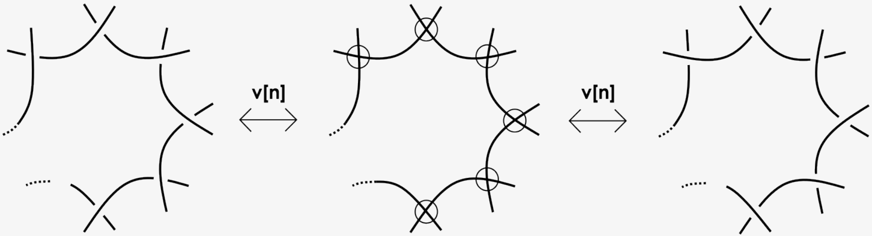
The crossing change at a real crossing is realized by a combination of a virtualized Δ-move and generalized Reidemeister moves.

Nakamura-Nakanishi-Satoh-Wada(2024)



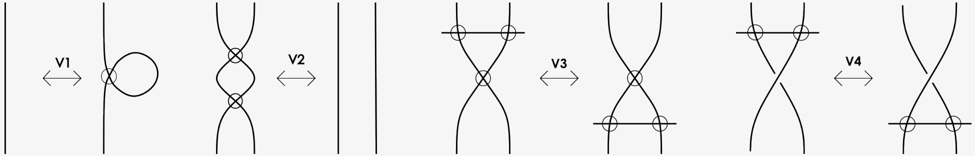
Background

- We introduce a local deformation called the virtualized n-gon move ($v[n]$-move) as generalization of virtualized Δ-move.



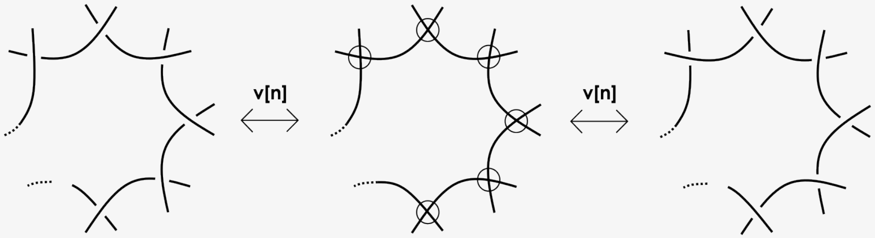
Virtual knots

- A virtual knot diagram is a knot diagram possibly with some ⊗ 's.
- A virtual knot is an equivalence class of virtual knot diagrams under generalized Reidemeister moves R1-R3 and V1-V4.
- {classical knot in S^3} ⊂ {virtual knots} [Goussarov-Polyak-Viro(2000)]



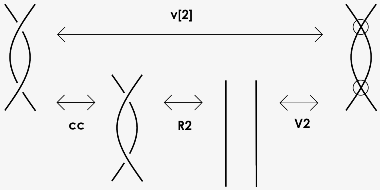
Virtualized n-gon moves (Recall)

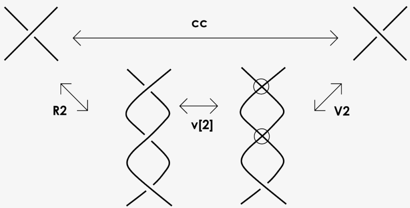
- Now, we think about this local deformation :



$n = 2$

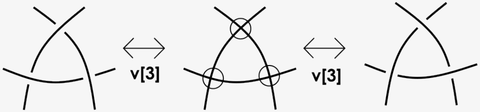
$v[2]$-move ⇔ crossing change





$n = 3$

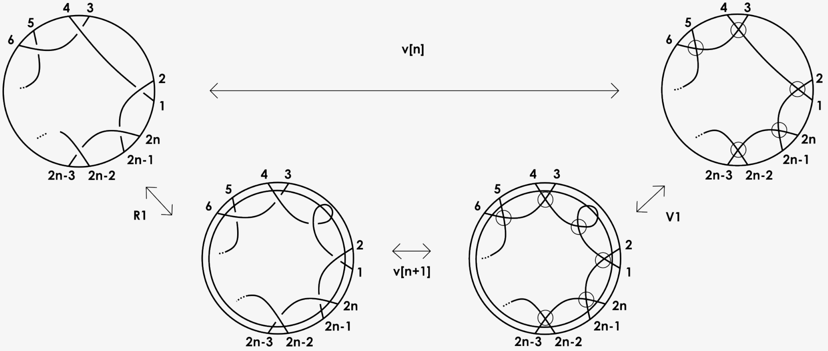
$v[3]$-move=$v\Delta$-move



$v[n]$ and $v[n+1]$

Proposition 1

A $v[n]$-move is realized by a $v[n+1]$-move. (for $^\forall n \geq 2$)



$v[n]$ unknotting number

Corollary 1

A $v[n]$-move is an unknotting operation when $n \geq 3$.

Definition 1

$d_{v[n]}(K, K')$: the minimal number of virtualized n-gon moves needed to deform a diagram of K into a diagram of K'

In particular, $u_{v[n]}(K) = d_{v[n]}(K, O)$ (O: trivial knot)

Corollary 2

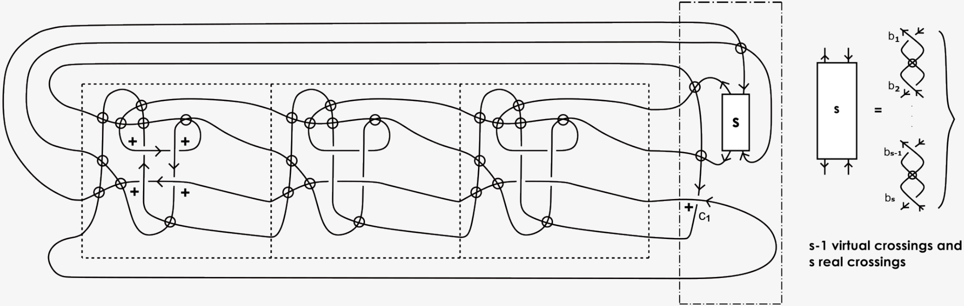
$u_{v[n]}(K) \geq u_{v[n+1]}(K)$ (more generally $d_{v[n]}(K, K') \geq d_{v[n+1]}(K, K')$)

Main Result

Theorem 1

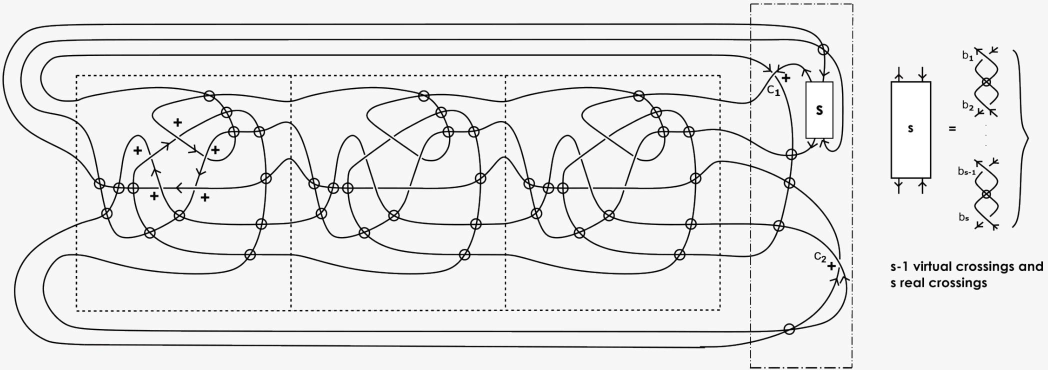
For any integers $n \geq 3$ and $m \geq 1$, there exists an infinite family $\{K_s\}_{s\in\mathbb{N}}$ of virtual knots such that $u_{v[n]}(K_s) = m$.

Construction (ex. $m = 3, n = 4, s \in \mathbb{N}$)

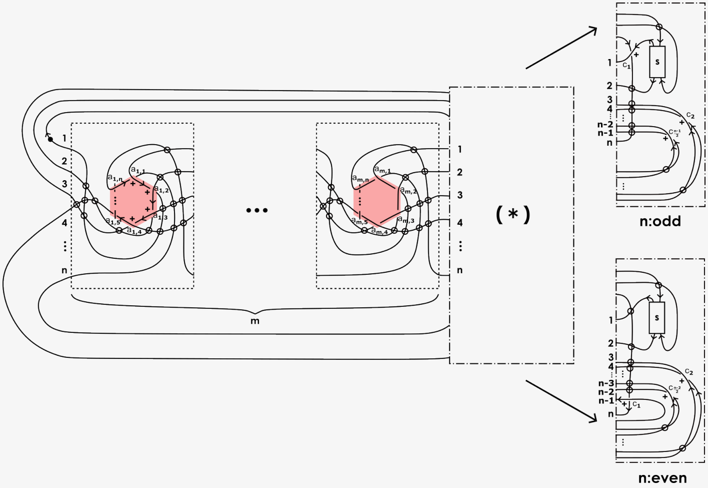


s-1 virtual crossings and
s real crossings

Construction (ex. $m=3, n=5, s\in\mathbb{N}$)



Construction ($m \geq 1, n \geq 3, s \in \mathbb{N}$)



Proof of theorem

Theorem 1

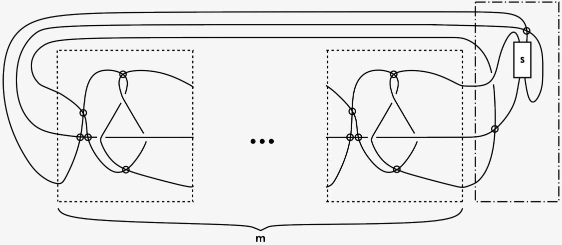
For any integers $n \geq 3$ and $m \geq 1$, there exists an infinite family $\{K_s\}_{s\in\mathbb{N}}$ of virtual knots such that $u_{v[n]}(K_s) = m$.

(1) $u_{v[n]}(K_s) \leq m$

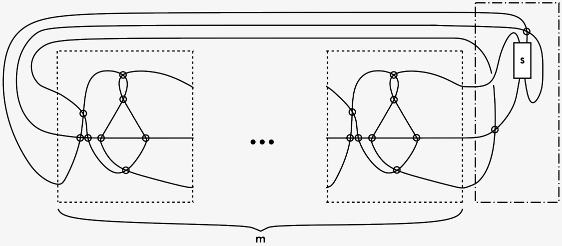
(2) $u_{v[n]}(K_s) \geq m$

(3) $K_s \neq K_{s'} \Leftrightarrow s \neq s'$

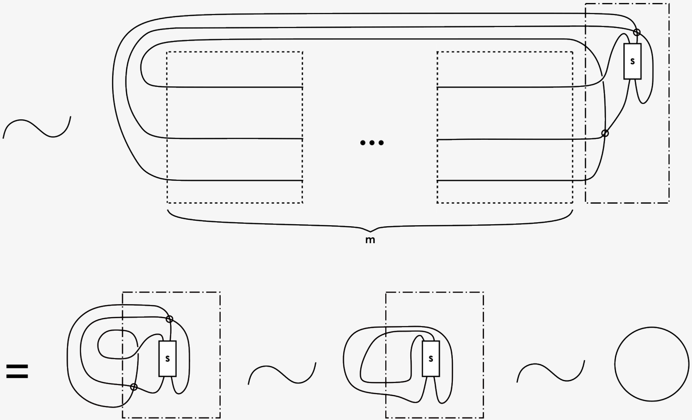
(1) $u_{v[n]}(K_s) \leq m$ (in case of $n = 3$)



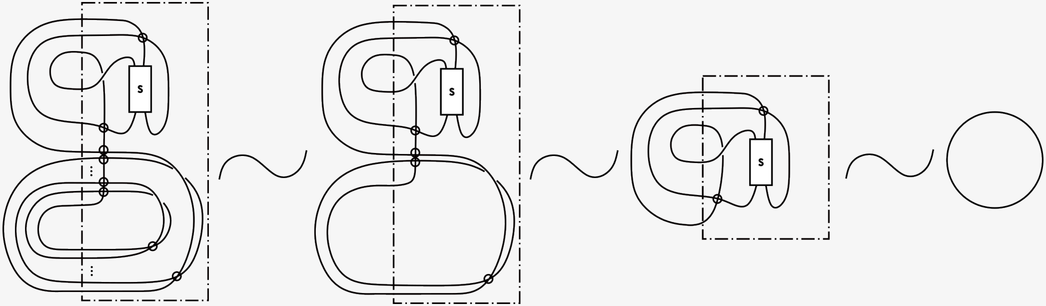
→ m times v[3]-moves



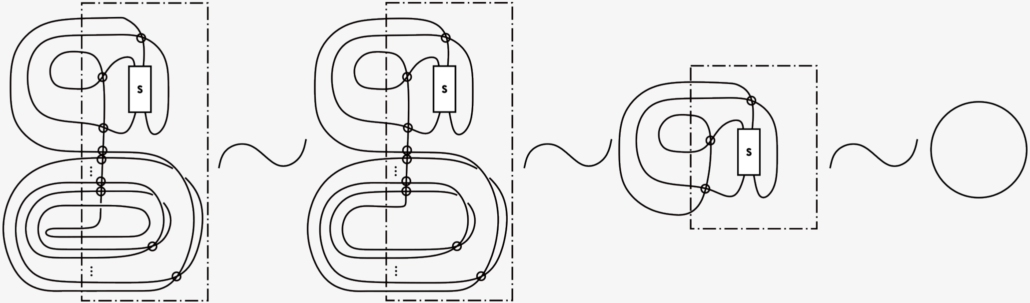
(1) $u_{v[n]}(K_s) \leq m$ (in case of $n = 3$)



(1) $u_{v[n]}(K_s) \leq m$ (n:odd)



(1) $u_{v[n]}(K_s) \leq m$ (n:even)



(2) $u_{v[n]}(K_s) \geq m$

Lemma 1

$$u_{v[n]}(K) \geq \frac{1}{n}|J_*(K)| \quad \left(d_{v[n]}(K,K') \geq \frac{1}{n}|J_*(K) - J_*(K')|\right)$$

D : a diagram of K

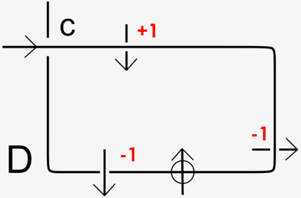
- For a real crossing c of D, sgn$(c) :=$ $\begin{cases} +1 \\ -1 \end{cases}$: the *sign* of c

$+1$ ()

-1 ()

(2) $u_{v[n]}(K_s) \geq m$

- For a real crossings c of D,
  Ind(c) := the sum of following numbers :
  When we encounter a real crossing in walking along D from the over crossing to the under crossing at c, the number is $+1$ if the crossing intersects us transversely from left to right, -1 if the crossing intersects us transversely from right to left.



Ind(c)=-1

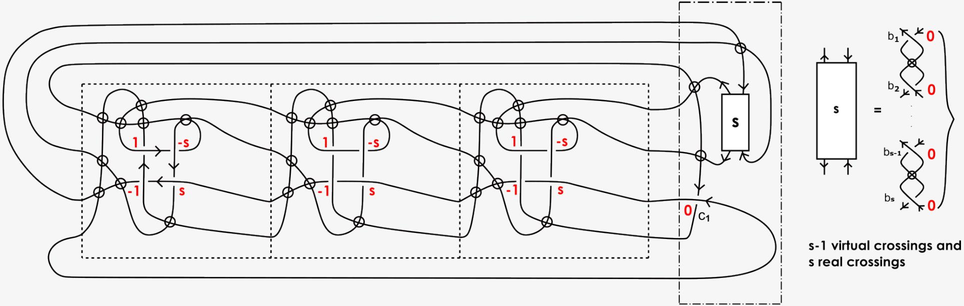
(2) $u_{v[n]}(K_s) \geq m$

- $J_k(D) := \sum_{\mathrm{Ind}(c)=k} \mathrm{sgn}(c)$: *k-writhe* of D
  This is independent of D if $k \neq 0$ (cf.[Satoh-Taniguchi(2014)])
- $J_*(K) := \sum_{k \neq 0} J_k(K)$: *non-zero writhe* of K

(2) $u_{v[n]}(K_s) \geq m$

We can see that $J_*(K_s) = nm$.

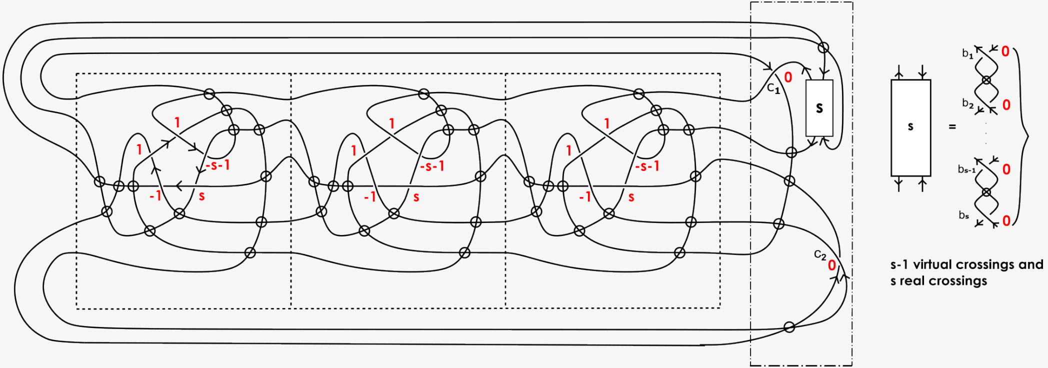
ex.The construction of $m = 3, n = 4, s \in \mathbb{N}$



(2) $u_{v[n]}(K_s) \geq m$

We can see that $J_*(K_s) = nm$.

ex.The construction of $m = 3, n = 5, s \in \mathbb{N}$



(2) $u_{v[n]}(K_s) \geq m$

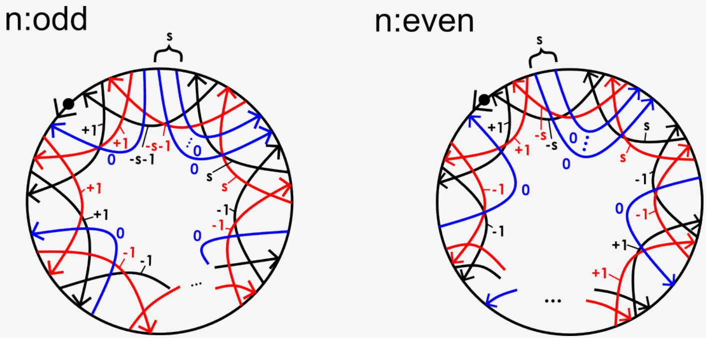
Lemma 1

$$u_{v[n]}(K) \geq \frac{1}{n}|J_*(K)| \quad \left(d_{v[n]}(K, K') \geq \frac{1}{n}|J_*(K) - J_*(K')|\right)$$

Hence we have $u_{v[n]}(K_s) \geq \frac{1}{n}|J_*(K)| = \frac{1}{n} \cdot nm = m$ by Lemma 1.

(2) $u_{v[n]}(K_s) \geq m$

cf. The Gauss diagram of K_s



(3) $K_s \neq K_{s'} \Leftrightarrow s \neq s'$

Lemma [Satoh-Taniguchi(2014)]

If $K \sim K'$ then $J_k(K) = J_k(K')$ for any $k \neq 0$ $(J_k : k-$writhe$)$.

- n:odd

  $J_1(K_s) = \frac{n-1}{2}m, \quad J_{-1}(K_s) = \frac{n-3}{2}m, \quad J_s(K_s) = m, \quad J_{-s-1}(K_s) = m,$

  $J_k(K_s) = 0$ if $k \neq 1, -1, s, -s-1, 0$

- n:even

  $J_1(K_s) = \frac{n-2}{2}m, \quad J_{-1}(K_s) = \frac{n-2}{2}m, \quad J_s(K_s) = m, \quad J_{-s}(K_s) = m,$

  $J_k(K_s) = 0$ if $k \neq 1, -1, s, -s, 0$

(3) $K_s \neq K_{s'} \Leftrightarrow s \neq s'$

s	J_{-5}	J_{-4}	J_{-3}	J_{-2}	J_{-1}	J_1	J_2	J_3	J_4
1				m	$\frac{n-3}{2}m$	$\frac{n+1}{2}m$			
2			m		$\frac{n-3}{2}m$	$\frac{n-1}{2}m$	m		
3		m			$\frac{n-3}{2}m$	$\frac{n-1}{2}m$		m	
4	m				$\frac{n-3}{2}m$	$\frac{n-1}{2}m$			m

Table: n:odd

(3) $K_s \neq K_{s'} \Leftrightarrow s \neq s'$

s	J_{-4}	J_{-3}	J_{-2}	J_{-1}	J_1	J_2	J_3	J_4
1				$\frac{n}{2}m$	$\frac{n}{2}m$			
2			m	$\frac{n-2}{2}m$	$\frac{n-2}{2}m$	m		
3		m		$\frac{n-2}{2}m$	$\frac{n-2}{2}m$		m	
4	m			$\frac{n-2}{2}m$	$\frac{n-2}{2}m$			m

Table: n:even

Remark and Future work

- Examples of virtual knots with $v[3]$ unknotting number m are given by Nakamura-Nakanishi-Satoh-Wada.

- We are trying to give a necessary and sufficient condition for two virtual links to be related by a finite sequence of virtualized n-gon moves.
- For any virtual knot K, is there an integer n such that $u_{v[n]}(K) = 1$?

Theorem [Aida(1992)]

For any knot K, there exist an integer n such that $u_n(K) = 1$.

Thank you for your attention.