On characterization of a multivariable polynomial invariant of twisted links

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• Alexander numbering of virtual links (1/2)

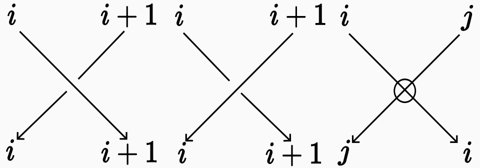
D : a virtual link diagram

A **semi-arc** of D :=

an arc of D between two classical crossings or loop without classical crossing of D.

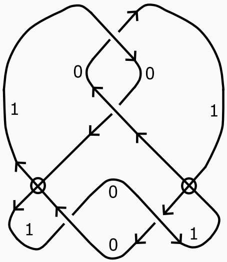
Alexander numbering of D :=

an assignment of a number of $\mathbb{Z}$ to each semi-arc of D depicted as in figure for each crossings.



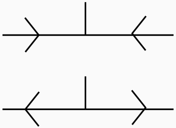
- **Alexander numbering of virtual links (2/2)**

D is **almost classical** :=
D admit an Alexander numbering
A virtual link L is **almost classical** :=
L has an almost classical virtual link diagram

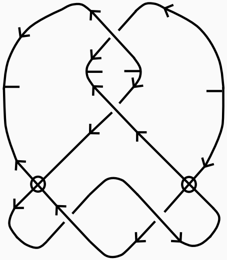


• A multivariable polynomial invariant of virtual links (1/5)

A **pole diagram** :=
A link diagram with possibly some poles on its edges

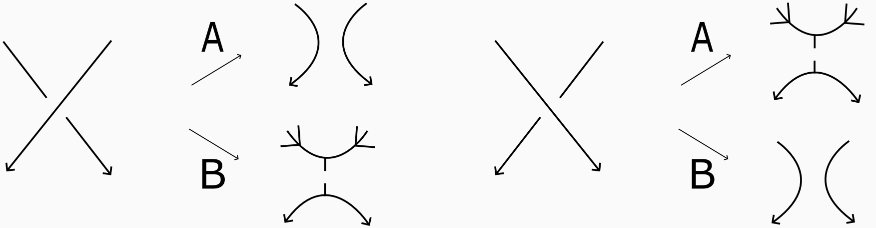


pole



• A multivariable polynomial invariant of virtual links (2/5)

An **A-splice** (or a **B-splice**) :=
A local replacement at real crossings of a link diagram as in figure



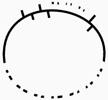
A-splice and B-splice

• A multivariable polynomial invariant of virtual links (3/5)

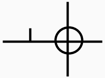
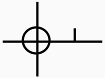
A **state** of a link diagram D :=
A pole diagram obtain from D by applying an A-splice or a B-splice at each real crossing of D

- **An index $\iota(l)$**

l : A loop of a state of D

(I) $\iota\Big($  $\Big) = r$, where $2r$ poles appear on both sides alternately, and the dotted line may have some virtual crossings.

(II) $\iota\Big($  $\Big) = \iota\Big($ $\Big)$

(III) $\iota\Big($  $\Big) = \iota\Big($  $\Big)$

• A multivariable polynomial invariant of virtual links (4/5)

S : A state of D

$$\langle\langle D \rangle\rangle \coloneqq \sum_S A^{\natural S}(-A^2 - A^{-2})^{\sharp S} d_1^{\tau_1(S)} d_2^{\tau_2(S)} \cdots$$

$\natural S \coloneqq \sharp$(A splice obtaining S) − $\sharp$(B splice obtaining S)

$\sharp S \coloneqq \sharp$(loops in S)

$\tau_i(S) \coloneqq \sharp$(loops in S whose indices by ι are i)

• A multivariable polynomial invariant of virtual links (5/5)

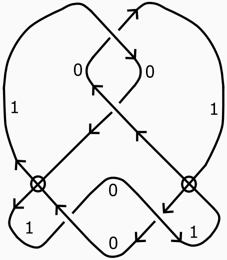
$$R_D \coloneqq (-A^3)^{-w(D)} \langle\langle D \rangle\rangle \quad \in \mathbb{Z}[A^{\pm 1}, d_1, d_2 \cdots]$$

$w(D) = \sharp$(positive crossings of D) − $\sharp$(negative crossings of D)

Theorem [H. A. Dye, L.H. Kauffman and Y. Miyazawa]

The polynomial R_D is an invariant of virtual links.

• example and theorem



D

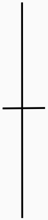
$$R_D = -A^{-10}(A^4+1)(A^{16}+A^{12}+A^8+A^4+1)$$

Theorem [T. Nakamura, Y. Nakanishi, S. Satoh and Y. Tomiyama]

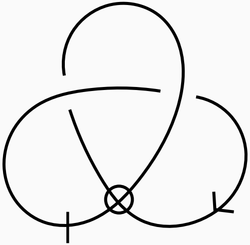
D : an almost classical diagram with $\mu(L)$ compornent $\Rightarrow$ $R_D \in \mathbb{Z}[A^{\pm 4}]A^{2\mu(L)}$

- **A twisted link (1/2)**

A **twisted link diagram** :=
a virtual link diagram possibly with bars on arcs.



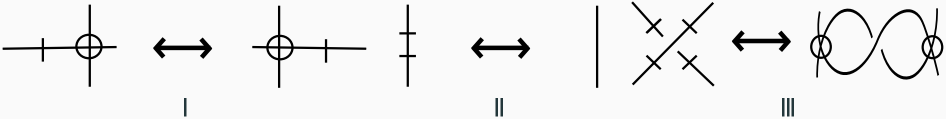
bar



twisted link diagram

- **A twisted link (2/2)**

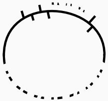
A **twisted link** :=
an equivalence class of twisted links diagram under Generalized Reidemeister move and twisted Reidemeister move



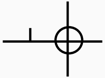
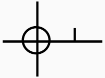
twisted Reidemeister move

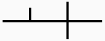
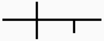
- **An index** $\iota(l)$

l : A loop of a state of D

(I) $\iota\Big($  $\Big) = r$, where $2r$ poles appear on both sides alternately, and the dotted line may have some virtual crossings and some bars.

(II) $\iota\Big($  $\Big) = \iota\Big($ $\Big)$

(III) $\iota\Big($  $\Big) = \iota\Big($  $\Big)$

(IV) $\iota\Big($  $\Big) = \iota\Big($  $\Big)$

• A multivariable polynomial invariant of twisted links (1/2)

S : A state of D

$$\langle\langle D\rangle\rangle \coloneqq \sum_S A^{\natural S}(-A^2 - A^{-2})^{\sharp S} M^{\sharp_0 S} d_1^{\tau_1(S)} d_2^{\tau_2(S)} \cdots$$

$\natural\, S \coloneqq \sharp$(A splice obtaining S) − $\sharp$(B splice obtaining S)

$\sharp\, S \coloneqq \sharp$(loops in S)

$\sharp_0\, S \coloneqq \sharp$(loops in S which have odd numbers of bars)

$\tau_i(S) \coloneqq \sharp$(loops in S whose indices by ι are i)

• A multivariable polynomial invariant of twisted links (2/2)

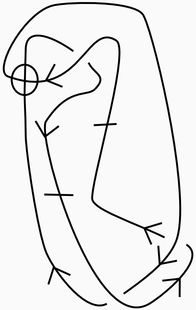
$$X_D \coloneqq (-A^3)^{-w(D)} \langle\langle D \rangle\rangle \quad \in \mathbb{Z}[A^{\pm 1}, M, d_1, d_2 \cdots]$$

$w(D) = \sharp$(positive crossings of D) − $\sharp$(negative crossings of D)

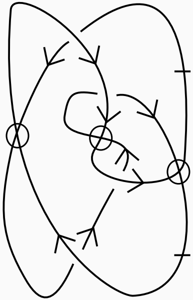
Theorem [N. Kamada]

The polynomial X_D is an invariant of twisted links.

- **example**



D_1



D_2

$$X_{D_1} = -A^4(A^4+1)(A^{10}-A^6-A^4+A^2+1)$$
$$X_{D_2} = -A^{-10}(A^4+1)\{(A^8-3A^4+1)A^4-(A^{16}-2A^8+1)M^2\}$$

- # Main result (1/3)

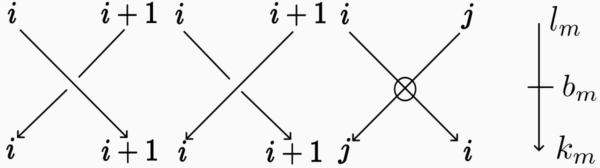
D : a twisted link diagram with bars $b_1, b_2, \cdots, b_r$.

A **bar-edge** of D :=

an arc of D between classical crossings or bars or a loop without classical crossing and bar of D.

twisted pseudo Alexander numbering of D :=

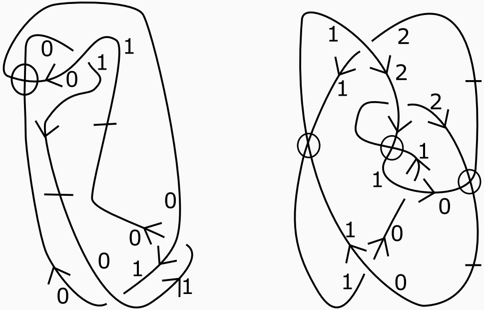
an assignment of a number of $\mathbb{Z}$ to each bar-edge of D depicted as in figure for each crossings and bars.



$$l_1 + k_1 = l_2 + k_2 = \cdots = l_r + k_r = n \ (1 \leq m \leq r)$$

• Main result (2/3)

twisted pseudo odd(or even) Alexander numbering of D
≔ twisted pseudo Alexander numbering of D such that n is odd(or even).



$n = 1$ $n = 2$

- **Main result (3/3)**

Main result 1

D : A twisted link diagram

D admit a twisted pseudo odd Alexander numbering

$\Rightarrow X_D \in \mathbb{Z}[A^{\pm 1}]$

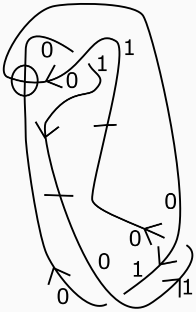
Main result 2

D : A twisted link diagram ($\mu(L)$ compornent)

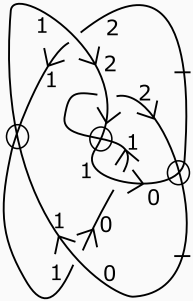
D admit a twisted pseudo even Alexander numbering

$\Rightarrow X_D \in \mathbb{Z}[A^{\pm 4}, M]A^{2\mu(L)}$

- **example**



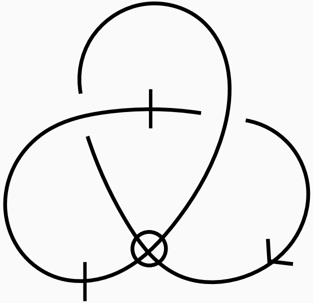
$(n=1)$
D_1



$(n=2)$
D_2

$$X_{D_1} = -A^4(A^4+1)(A^{10}-A^6-A^4+A^2+1)$$
$$X_{D_2} = -A^{-10}(A^4+1)\{(A^8-3A^4+1)A^4-(A^{16}-2A^8+1)M^2\}$$

- **example**



D

$$X_D = -A^{-12}(A^4+1)(A^6+A^4 d_1-(A^4+1)M^2+A^4)$$

- **Remark (1/3)**

Remark

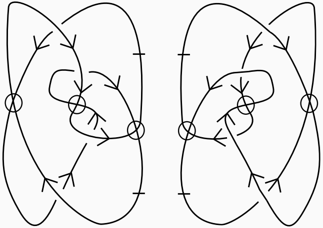
D : A twisted link diagram
$\widetilde{D}$: A double covering diagram of D

D admit a twisted pseudo Alexander numbering
$\Rightarrow \widetilde{D}$ is almost classical.

- **Remark (2/3)**

D : a twisted link diagram which is on the left of the y-axis and all bars are parallel to the x-axis with disjoint y-coordinates.

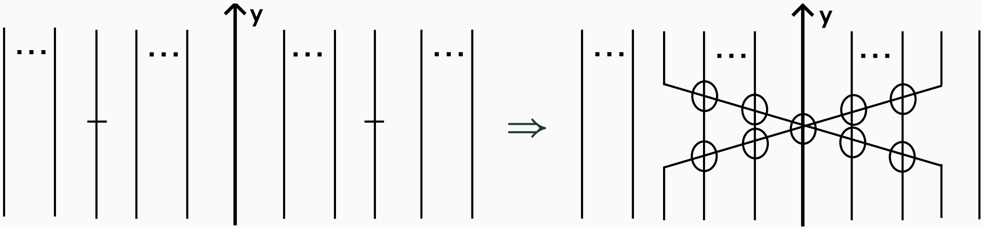
$s(D)$: a twisted link diagram which is obtained from D by reflection with respect to the y-axis and switching all real crossings of D.



D $\qquad$ $s(D)$

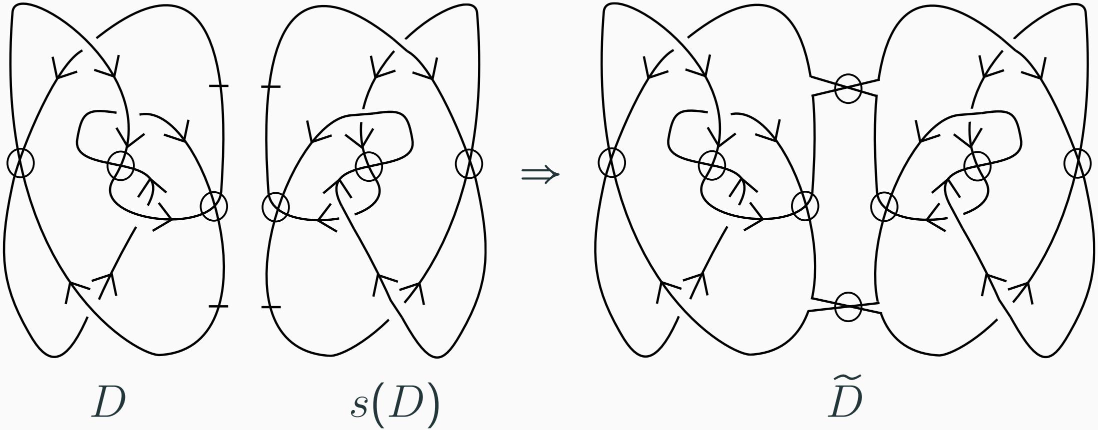
- **Remark (3/3)**

A **double covering diagram** of $D :=$
a virtual link diagram which is obtained from D and $s(D)$ as in figure.

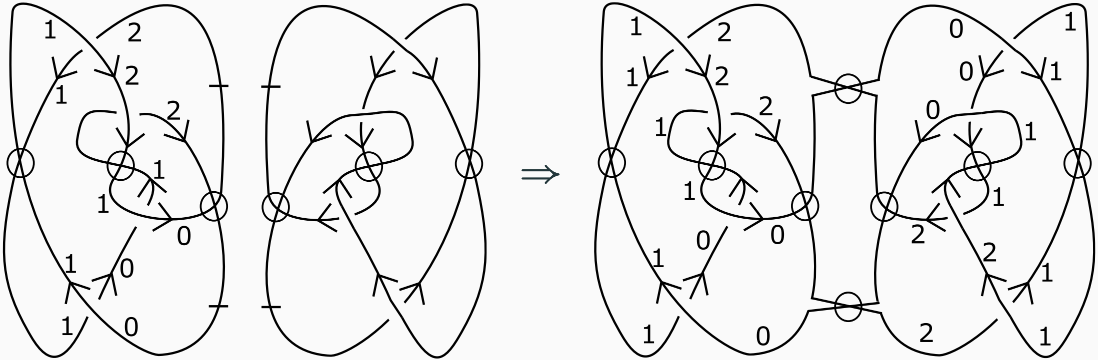


D $s(D)$ $\widetilde{D}$

- **example**



- **example**



D $s(D)$ $\widetilde{D}$

Thank you for your attention !