Right-left equivalent maps of simplified $(2,0)$-trisections with different configurations of vanishing cycles

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結び目の数理 IV
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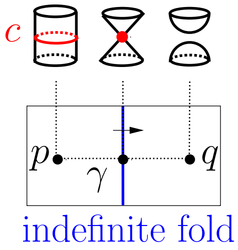
fold/cusp singularities

X: a closed oriented connected smooth 4-manifold,
$f : X \to \mathbf{R}^2$: smooth map and $\mathbf{p} \in \mathrm{Sing}(f)$,

- $\mathbf{p}$ is an indefinite fold singularity
  $\Leftrightarrow \exists$ local coord. (t,x,y,z) around $\mathbf{p}$ s.t. $f(t,x,y,z) = (t, -x^2 - y^2 + z^2)$.
- $\mathbf{p}$ is a definite fold singularity
  $\Leftrightarrow \exists$ local coord. (t,x,y,z) around $\mathbf{p}$ s.t. $f(t,x,y,z) = (t, -x^2 - y^2 - z^2)$.
- $\mathbf{p}$ is a cusp singularity
  $\Leftrightarrow \exists$ local coord. (t,x,y,z) around $\mathbf{p}$ s.t. $f(t,x,y,z) = (t, x^3 - 3xt + y^2 - z^2)$.



- c is called a vanishing cycle associated with γ.
- γ is called a reference path.

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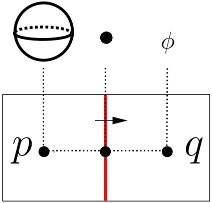

definite fold

fold/cusp singularities

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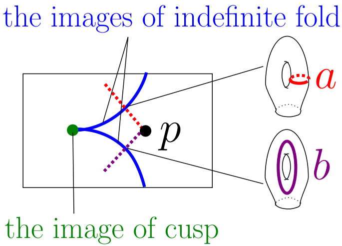
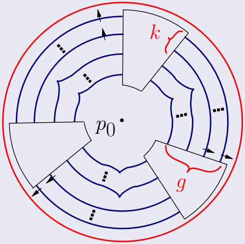
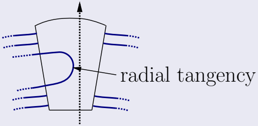


- the vanishing cycles a and b intersect at one point transversely.

(g,k)-trisection mappings [Gay-Kirby, 2016]

a smooth mapping $f : X \to \mathbf{R}^2$ is a (g,k)-trisection map. $(g \geq k \geq 0)$
$\Leftrightarrow f(\mathrm{Sing}(f))$ is as in the figure.





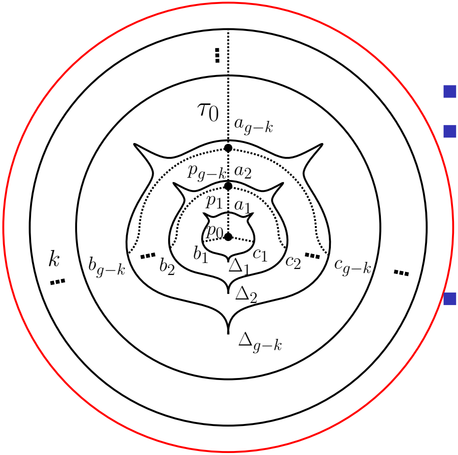
- the most outer red circle$\cdots$definite fold singular value.
- the solid curves$\cdots$indefinite fold singular value.
- the cusped points$\cdots$cusp singular value.
- three white boxes$\cdots$consist of indefinite fold images with transverse double points but without “radial tangencies".
- $f^{-1}(p_0)$ is a closed oriented surface of genus g.

Theorem (Gay-Kirby, 2016)

For any 4-manifolds X, there is a (g,k)-trisection map.

Simplified (g,k)-trisection mappings [Baykur-Saeki, 2017]

$f : X \to \mathbf{R}^2$: a (g,k)-trisection map is simplified.
$\Leftrightarrow f(\mathrm{Sing}(f))$ is as in the figure.



- $\tau_0 \cdots$ a standard reference path.
- τ_0 induces $3(g-k)+k$ times vanishing cycles on central fiber Σ: $(\Sigma, \{a_1, a_2, \ldots, a_g\}, \{b_1, b_2, \ldots, b_{g-k}\}, \{c_1, c_2, \ldots, c_{g-k}\})$.
- the tuple of Σ and these vanishig cycles is called simplified (g,k)-trisection diagram.

$f, g : M \to N$: smooth map

right-left equivalent

$f \simeq g$ (right-left equivalent)
$\Leftrightarrow \exists \phi : M \to M,\ \exists \psi : N \to N$: self-diffeomorphisms such that the following diagram is commutative.

$$\begin{array}{ccc} M & \xrightarrow{\ \phi\ } & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{\ \psi\ } & N \end{array}$$

stable map

f is a stable map $\Leftrightarrow$ any map g in a small neighborhood of f in the space of smooth maps is right-left equivalent to f.

Roughly speaking, f does not change in a neighborhood up to diffeomorphism.

Question

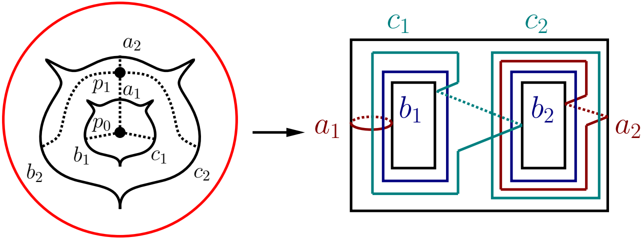
Question.

{ right-left eq. classes of sim. $(2,0)$-tri.} $\overset{1:1?}{\longleftrightarrow}$ { sim. $(2,0)$-tri. diag. }

Recall that vanishing cycles are obtained as follows:

1. Fix a reference path.
2. Draw vanishing cycles on $f^{-1}(p_0)$ according to the reference path.

Note that vanishing cycles are labeled by a_1, b_1, c_1, a_2, b_2 and c_2



upper-triangular handle-slide

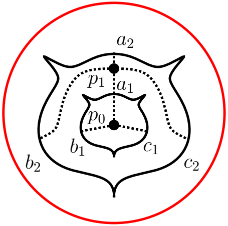
Question.

Find an equivalence relation $\sim$ s. t.

{ right-left eq. classes of sim. $(2,0)$-tri.} $\overset{1:1}{\longleftrightarrow}$ { sim. $(2,0)$-tri. diag. } / $\sim$

$(\Sigma, \{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}) \sim (\Sigma', \{a'_1, a'_2\}, \{b'_1, b'_2\}, \{c'_1, c'_2\}) :=$
$\exists$ a finite seq. of :

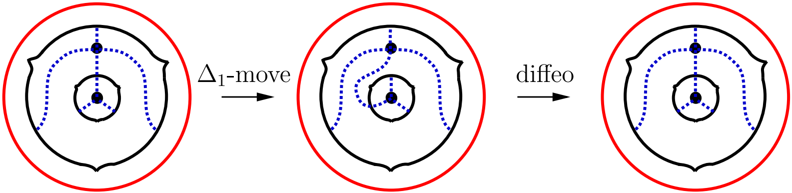
- automorphisms of Σ with labels,
- upper-triangular handle-slide,
- replacing reference paths.



- upper-triangular handle-slide $\cdots$ handle-slide $\{a_{i+1}, b_{i+1}, c_{i+1}\}$ over $a_1, a_2, \dots a_i$ $(i = 1, 2, \dots g-k)$

Replacing reference paths

Let f be a simplified $(2,0)$-trisection.



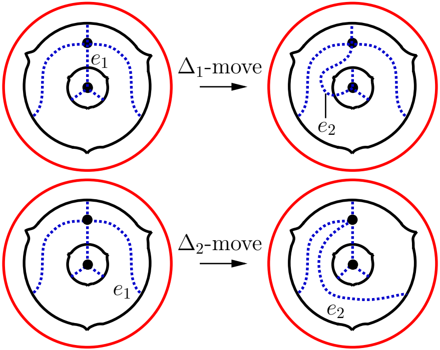
- replacing reference paths does NOT change right-left equivalence class.
- the configurations of simplified trisection diagrams change.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{R}^2 & \xrightarrow{\psi} & \mathbf{R}^2 \end{array}$$

- $f \simeq g \Leftrightarrow$ The left diagram is commutative.
- reference path is changed by $\psi : \mathbf{R}^2 \to \mathbf{R}^2$: homeo.

Prop. (A. 2021)

∀ replacing reference path is obtained by following moves.



Question.

Find an equivalence relation $\sim$ s. t.
{ right-left eq. classes of sim. $(2,0)$-tri.} $\overset{1:1}{\longleftrightarrow}$ { sim. $(2,0)$-tri. diag. } / $\sim$

$(\Sigma,\{a_1,a_2\},\{b_1,b_2\},\{c_1,c_2\}) \sim (\Sigma',\{a'_1,a'_2\},\{b'_1,b'_2\},\{c'_1,c'_2\}) :=$
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Lemma(A. 2021)

f,g : sim. $(2,0)$-tri.,
$(\Sigma,\{a_1,a_2\},\{b_1,b_2\},\{c_1,c_2\}),(\Sigma',\{a'_1,a'_2\},\{b'_1,b'_2\},\{c'_1,c'_2\})$
: sim. $(2,0)$-tri. diagrams
$f \simeq g \Rightarrow (\Sigma,\{a_1,a_2\},\{b_1,b_2\},\{c_1,c_2\}) \sim (\Sigma',\{a'_1,a'_2\},\{b'_1,b'_2\},\{c'_1,c'_2\})$.

Question.

Find an equivalence relation $\sim$ s. t.

{ right-left eq. classes of sim. $(2,0)$-tri.} $\stackrel{1:1}{\longleftrightarrow}$ { sim. $(2,0)$-tri. diag. } / $\sim$

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$\exists$ a finite seq. of :

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Question

Is it necessary to replace reference paths?

Main Theorem

Theorem. (A. , 2021)

$f : X \to \mathbf{R}^2$ be a simplified $(2,0)$-trisection. Suppose that the following holds:

- The monodromy μ_1 is not the identity map.
- b'_2 and c'_2 are not parallel.
- a'_2 and $\mu_1^{-1}(c'_2)$ are not parallel.

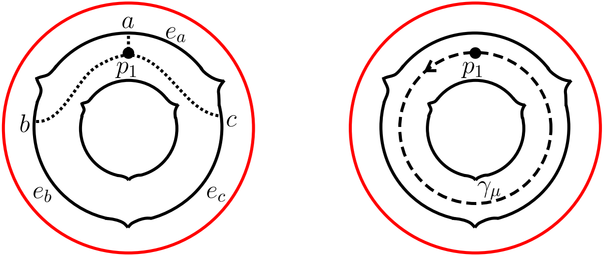
$\Rightarrow \exists\, f', f''$: sim. $(2,0)$-tri. s.t. $f \simeq f' \simeq f''$ but their sim. tri. diag. are NOT related by

- automorphisms of Σ and
- upper-triangular handle-slides over a_1.

Main Theorem

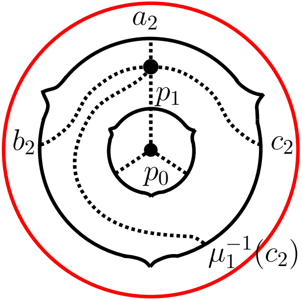
- The monodromy μ_1 is not the identity map.

$p_1 \in \mathbf{R}^2, f^{-1}(p_1) \simeq$ a torus T^2
γ_μ : the circle as shown in the figure.
$\mu_1 : T^2 \to T^2$:monodromy associated with γ_μ.



Main Theorem

- b_2' and c_2' are not parallel.
- a_2' and $\mu_1^{-1}(c_2')$ are not parallel.



- The surface $f^{-1}(p_1)$ is obtained from the genus-2 surface $f^{-1}(p_0)$ on the right by surgerying it along a_1, which is a torus.
- The a_2', b_2' and c_2' are the vanishing cycles on the torus $f^{-1}(p_1)$ corrsponding to a_2, b_2 and c_2, respectively.

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- automorphisms of Σ and
- upper-triangular handle-slides over a_1.

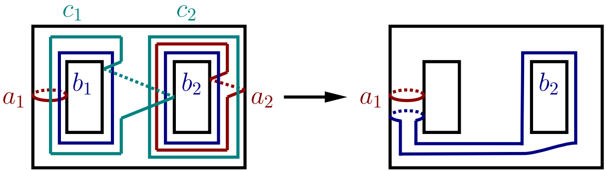
Proposition. (A., 2021)

The following matrix is an upper-triangular handle-slide (over a_1) invariant .

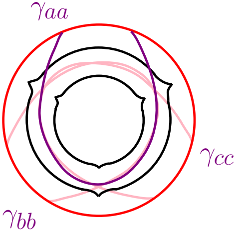
$$I(V) = \begin{pmatrix} |(b_1 + c_1) \cdot a_2| \\ |(b_1 + c_1) \cdot b_2| \\ |(b_1 + c_1) \cdot c_2| \end{pmatrix},$$

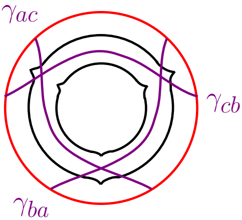
where $b_1 + c_1$ is the union of b_1 and c_1 on Σ.

slide b_2 over a_1



$$I(V) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad I(V') = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$





Set $V_{ij} = f^{-1}(\gamma_{ij})$.

Theorem (A. 2019)

The non-trivial 6-tuple $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix}$ is one of the following up to reflection:

$$\begin{pmatrix} S^3 & S^3 & L((q-1)^2, q-1+\varepsilon) \\ S^1 \times S^2 & L(q-2,\varepsilon) & L(q,-\varepsilon) \end{pmatrix}, \begin{pmatrix} S^3 & L(9,2\varepsilon) & L(4,\varepsilon) \\ L(2,1) & L(5,\varepsilon) & S^3 \end{pmatrix},$$

$$\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & L(1+\varepsilon,1) & S^3 \end{pmatrix}, \begin{pmatrix} S^1 \times S^2 & L(4,1) & L(4,1) \\ S^3 & L(4+\varepsilon,1) & S^3 \end{pmatrix},$$

where $q \neq 1$ and $\varepsilon \in \{-1,1\}$.

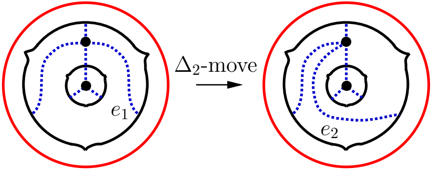
Sketch of proof

f : sim. $(2,0)$-tri. , V: sim. tri. diag. of f
f' : taking a Δ_2-move for f
f'' : taking two times Δ_2-move for f
V', V'' : sim. tri. diag of f', f'' .

Aim

$I(V) \neq I(V') \neq I(V'')$.

We calculate $I(V)$ for each 6-tuple.



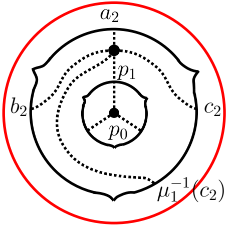
Remark

The Δ_2-move does NOT change right-left eq. classes.

We calculate $I(V)$ for each 6-tuple.

(1) $\begin{pmatrix} S^3 & S^3 & L((q-1)^2, q-1+\varepsilon) \\ S^1 \times S^2 & L(q-2,\varepsilon) & L(q,-\varepsilon) \end{pmatrix}$

We can take vanishing cycles as follows :



$$[a_2'] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2'] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} \pm q - 2 \\ \pm 1 \end{pmatrix},$$

where $q \in \mathbf{Z}$.

We have

$$I(V) = \begin{pmatrix} |(b_1+c_1)\cdot a_2| \\ |(b_1+c_1)\cdot b_2| \\ |(b_1+c_1)\cdot c_2| \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ |1 \mp q| \end{pmatrix}, I(V') = \begin{pmatrix} |(b_1+c_1)\cdot b_2| \\ |(b_1+c_1)\cdot c_2| \\ |(b_1+c_1)\cdot a_2| \end{pmatrix} = \begin{pmatrix} 1 \\ |1 \mp q| \\ 1 \end{pmatrix}.$$

Thus we obtain $I(V) \neq I(V')$ except for the case $|1 \mp q| = 1$. These exceptional cases are excluded in the assertion.