The one-row colored $\mathfrak{sl}_3$ Jones polynomial for pretzel links

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Concent

① Introduction

- Results for the colored $\mathfrak{sl}_3$ Jones polynomial
- Main Theorems
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial

② Sketch of the proof of Main Theorem

- Main Theorem 2 $P(2\alpha+1, 2\beta+1, 2\gamma)$

③ Appendix

- $P(3,3,2) = 8_5$, $P(3,7,-2) = 12n242$
- $8_{10}, 8_{15}, 8_{20}, 8_{21}$

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1. Introduction
   - Results for the colored $\mathfrak{sl}_3$ Jones polynomial
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   - The one-row colored $\mathfrak{sl}_3$ Jones polynomial
2. Sketch of the proof of Main Theorem
   - Main Theorem 2 $P(2\alpha+1, 2\beta+1, 2\gamma)$
3. Appendix
   - $P(3,3,2) = 8_5$, $P(3,7,-2) = 12n242$
   - $8_{10}, 8_{15}, 8_{20}, 8_{21}$

Jones polynomial

- An invariant of oriented links

The colored $\mathfrak{sl}_2$ Jones polynomial

(by the $n+1$ irreducible representation of $\mathfrak{sl}_2$)

- An invariant of oriented links associated with n.

The colored $\mathfrak{sl}_3$ Jones polynomial

(by the (n_1, n_2) irruducible representation of $\mathfrak{sl}_3$)

- An invariant of oriented links associated with (n_1, n_2).

The colored $\mathfrak{sl}_3$ Jones polynomial

- The colored $\mathfrak{sl}_3$ Jones polynomial for trefoil knot 3_1
  $J^{\mathfrak{sl}_3}_{(n_1,n_2)}(3_1; q)$ (2003, Lawrance)
- The colored $\mathfrak{sl}_3$ Jones polynomial for $(2, 2m+1)$ torus knots
  $J^{\mathfrak{sl}_3}_{(n_1,n_2)}(T(2,2m+1); q)$ (2013, Garoufalidis, Morton and Young)
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial for 2-bridge links.
  $J^{\mathfrak{sl}_3}_{(n,0)}(L[2a_1, 2a_2, \cdots, 2a_l]; q)$ (2017, Yuasa)
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial for $(2,2m)$ torus links
  $J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m); q)$ (2021 Yuasa)

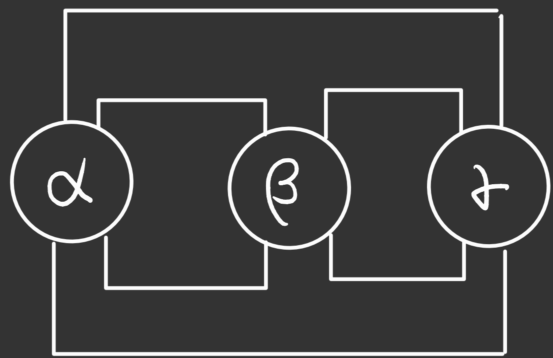
The colored $\mathfrak{sl}_3$ Jones polynomial

- The colored $\mathfrak{sl}_3$ Jones polynomial for thefoil knot 3_1
  $J^{\mathfrak{sl}_3}_{(n_1,n_2)}(3_1; q)$ (2003, Lawrance)
- The colored $\mathfrak{sl}_3$ Jones polynomial for $(2, 2m+1)$ tours knot
  $J^{\mathfrak{sl}_3}_{(n_1,n_2)}(T(2,2m+1); q)$ (2013, Garoufalidis, Morton and Toung)
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial for 2-bridge links.
  $J^{\mathfrak{sl}_3}_{(n,0)}(L[2a_1, 2a_2, \cdots, 2a_l]; q)$ (2017, Yuasa)
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial for $(2,2m)$ tours links
  $J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m); q)$ (2021 Yuasa)

Problem

What is $J^{\mathfrak{sl}_3}_{(n,0)}(P(\alpha, \beta, \gamma); q)$?

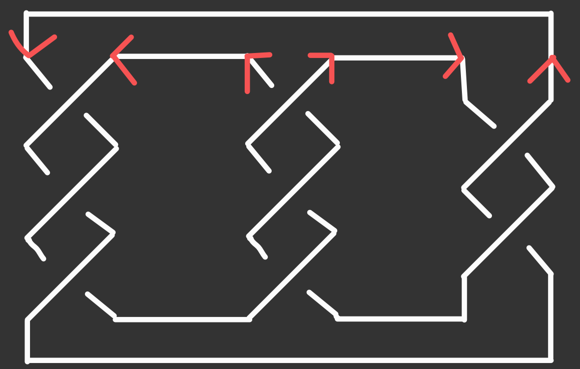
We mainly consider the 3-parameter family of pretzel link $p(\alpha, \beta, \gamma)$ for integers α, β, γ.

$p(\alpha, \beta, \gamma) =$ 

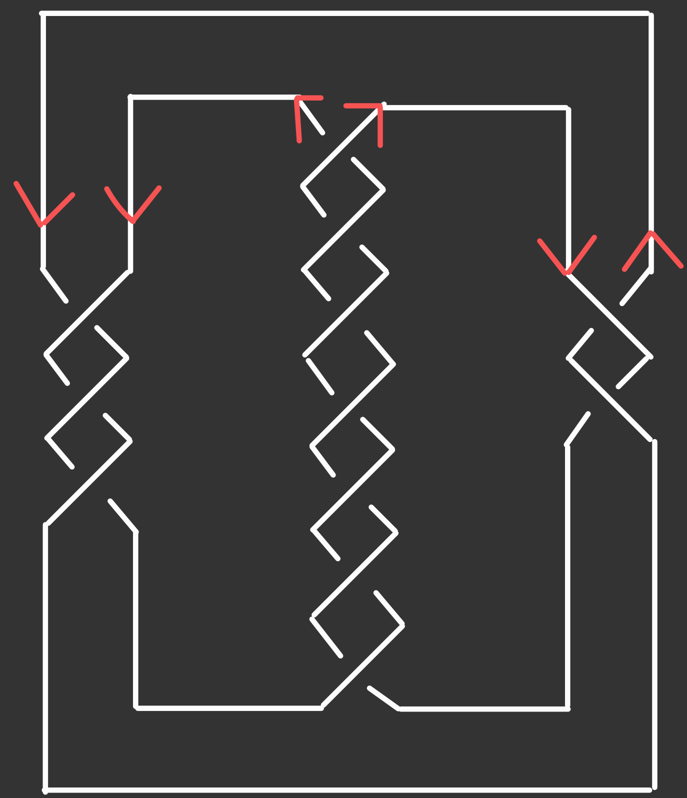
where

 $= \alpha \Bigg\{$  $(\alpha \geq 0)$,

 $= \alpha \Bigg\{$  $(\alpha < 0)$



P(↓3↓, ↑3↑, ↓2↑) = 8_5



P(↓3↓, ↑7↑, ↓-2↑) = 12n242

(Fintushel-Stern knot)

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Main Theorem 1 (Kawasoe)

α, β, γ are integers.

$$\tilde{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha\uparrow, \downarrow 2\beta\uparrow, \downarrow 2\gamma\uparrow); q)$$

$$= \sum_{0\le k_{|\alpha|}\le k_{|\alpha-1|}\le\cdots\le k_1\le n} \ \sum_{0\le l_{|\beta|}\le l_{|\beta-1|}\le\cdots\le l_1\le n} \ \sum_{0\le m_{|\gamma|}\le m_{|\gamma-1|}\le\cdots\le m_1\le n} \ \sum_{s=\max\{k_{|\alpha|}, l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|}, n\}}$$

$$\sum_{t=\max\{s, m_{|\gamma|}\}}^{\min\{s+m_{|\gamma|}, n\}} \left(q^{\frac{n^2+3n}{3}}\right)^{-(2\alpha+2\beta+2\gamma)} \phi(n, k_1, k_2, \cdots, k_{|\alpha|})_q^{\varepsilon_\alpha} \phi(n, l_1, l_2, \cdots, l_{|\beta|})_q^{\varepsilon_\beta}$$

$$\phi(n, m_1, m_2, \cdots, m_{|\gamma|})_q^{\varepsilon_\gamma} \psi(n, s, k_{|\alpha|}, l_{|\beta|}) \psi(n, t, s, m_{|\gamma|}) \, q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}$$

$$\tilde{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha\downarrow, \uparrow 2\beta\uparrow, \downarrow 2\gamma\uparrow); q) = \cdots$$

Main Theorem 2 (Kawasoe)

α, β, γ are integers.

$$J^{\mathfrak{sl}_3}_{(n,0)}\left(P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta+1 \uparrow, \downarrow 2\gamma \uparrow); q\right)$$

$$= \sum_{0 \le k_{|2\alpha+1|} \le k_{|2\alpha|} \le \cdots \le k_1 \le n} \ \sum_{0 \le l_{|2\beta+1|} \le l_{|2\beta|} \le \cdots \le l_1 \le n} \ \sum_{0 \le m_{|\gamma|} \le m_{|\gamma-1|} \le \cdots \le m_1 \le n} \ \sum_{s = \mathrm{Max}\{k_{|2\alpha+1|}, l_{|2\beta+1|}\}}^{\min\{k_{|2\alpha+1|} + l_{|2\beta+1|}, n\}}$$

$$\sum_{a=s}^{n} \ \sum_{t = \mathrm{Max}\{a, m_{|\gamma|}\}}^{\min\{a + m_{|\gamma|}, n\}} \left(q^{\frac{n^2+3n}{3}}\right)^{-(2\alpha+2\beta-2\gamma+2)} \chi_{\mathrm{sign}(2\alpha+1)}(n, k_1, k_2, \dots, k_{|2\alpha+1|})$$

$$\chi_{\mathrm{sign}(2\beta+1)}(n, l_1, l_2, \dots, l_{|2\beta+1|}) \, \phi(n, m_1, m_2, \dots, m_{|\gamma|}) \, q^{\varepsilon_\gamma} \, \Omega(n, s, k_{|2\alpha+1|}, l_{|2\beta+1|})$$

$$\psi(n, t, a, m_\gamma) \, q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}$$

Main Theorem 3 (Kawasoe)

α, β, γ are integers.

$J^{sl_3}_{(n,0)}(P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$

$$= \sum_{0 \le k_{|2\alpha+1|} \le k_{|2\alpha|} \le \cdots \le k_1 \le n} \; \sum_{0 \le l_{|2\beta|} \le l_{|2\beta-1|} \le \cdots \le l_1 \le n} \; \sum_{0 \le m_{|\gamma|} \le m_{|\gamma-1|} \le \cdots \le m_1 \le n} \; \sum_{s = \mathrm{Max}\{k_{|2\alpha+1|}, l_{|2\beta|}\}}^{\min\{k_{|2\alpha+1|}+l_{|2\beta|}, n\}}$$

$$\sum_{a=s}^{n} \; \sum_{t = \mathrm{Max}\{a, m_{|\gamma|}\}}^{\min\{a+m_{|\gamma|}, n\}} \left(q^{\frac{n^2+3n}{3}} \right)^{-(2\alpha+2\beta-2\gamma-1)} \chi_{\mathrm{sign}(2\alpha+1)}(n, k_1, k_2, \cdots, k_{|2\alpha+1|})$$

$$\chi_{\mathrm{sign}(2\beta+1)}(n, l_1, l_2, \cdots, l_{|2\beta|})\, \phi(n, m_1, m_2, \cdots, m_{|\gamma|})\, q^{\varepsilon}\, \Omega(n, k_{|2\alpha+1|}, l_{|2\beta|}, t)\, \psi(n, t, a, m_{|\gamma|})$$

$$q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}$$

$J^{sl_3}_{(n,0)}(P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta \downarrow, \uparrow 2\gamma \uparrow); q) = \cdots$

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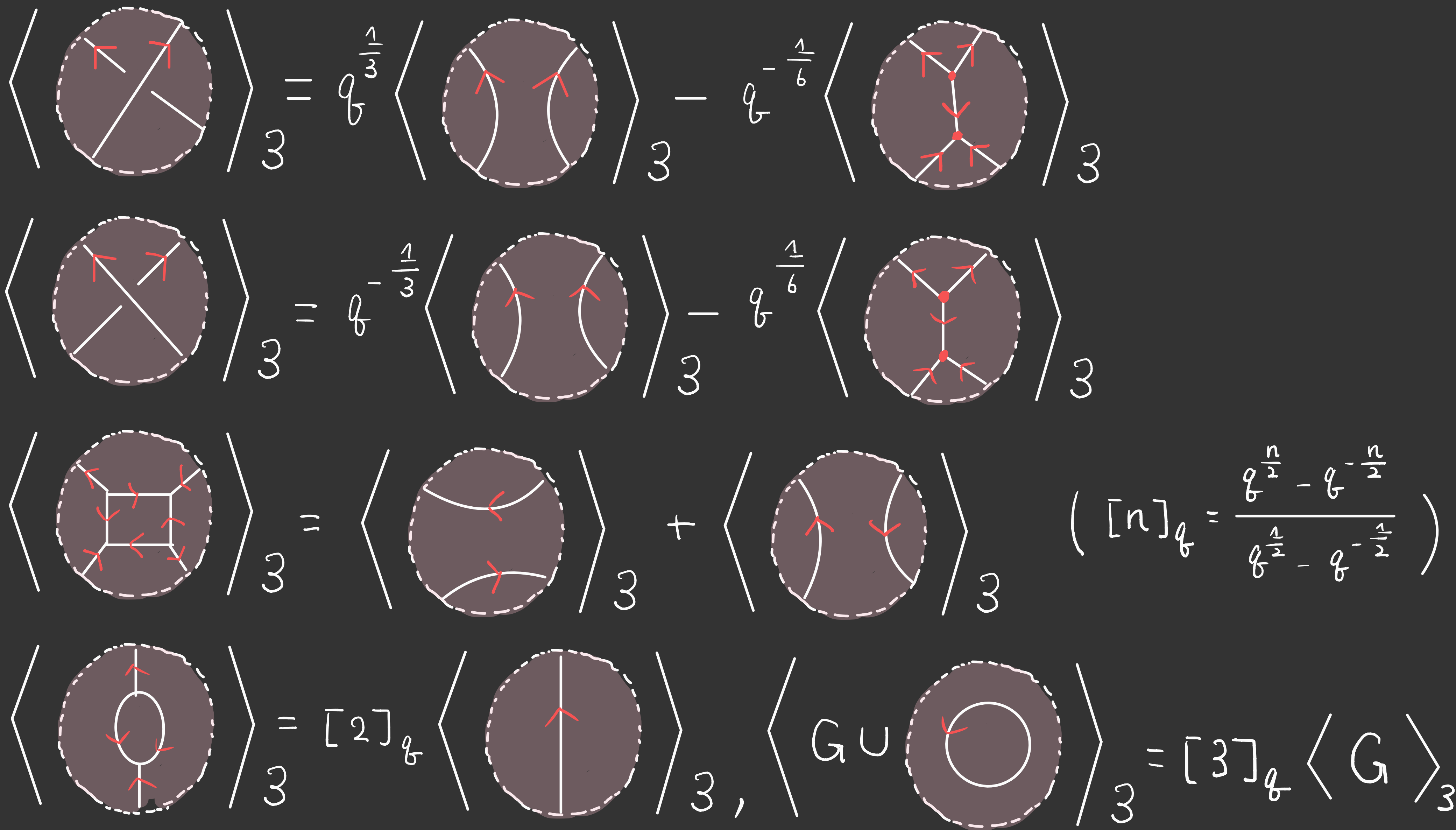
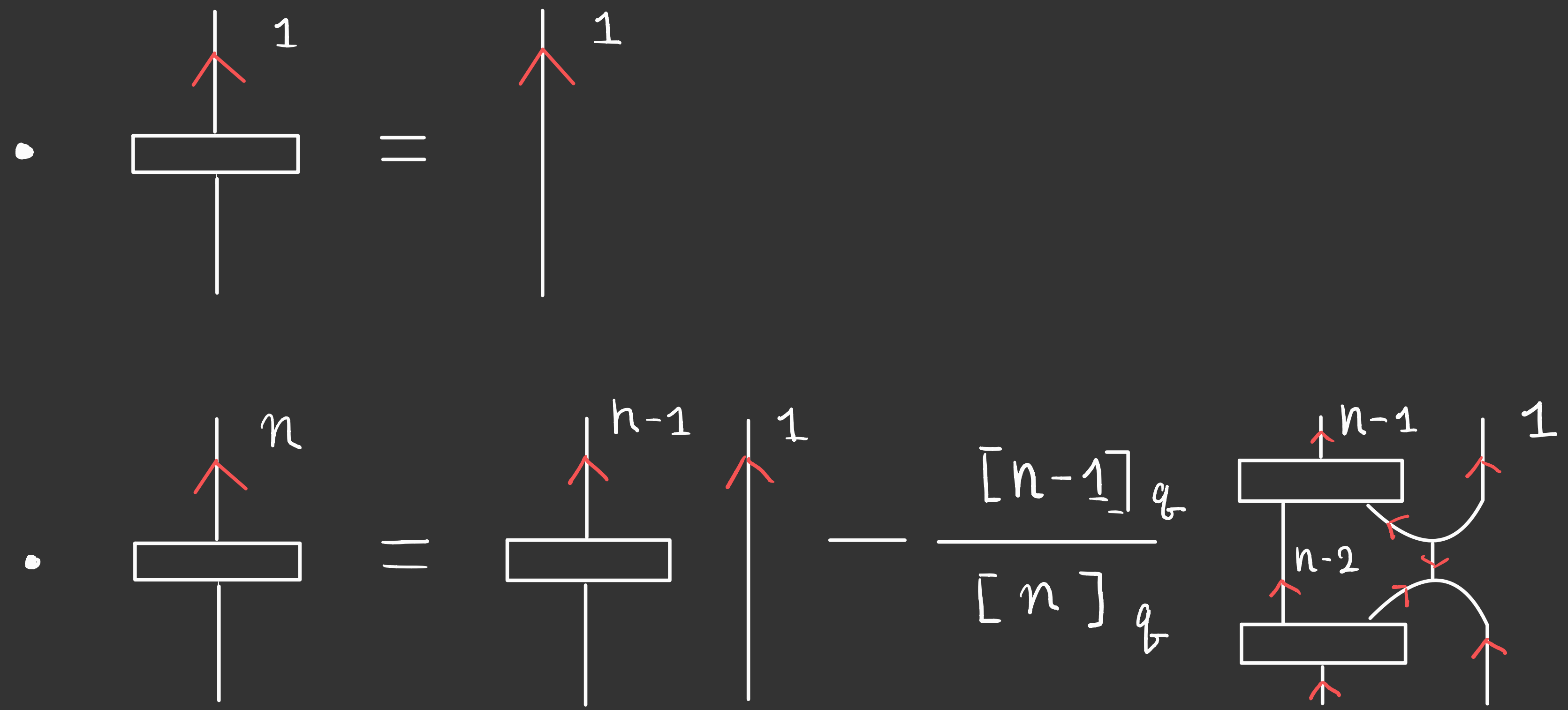
Definition (96, Kurperberg)

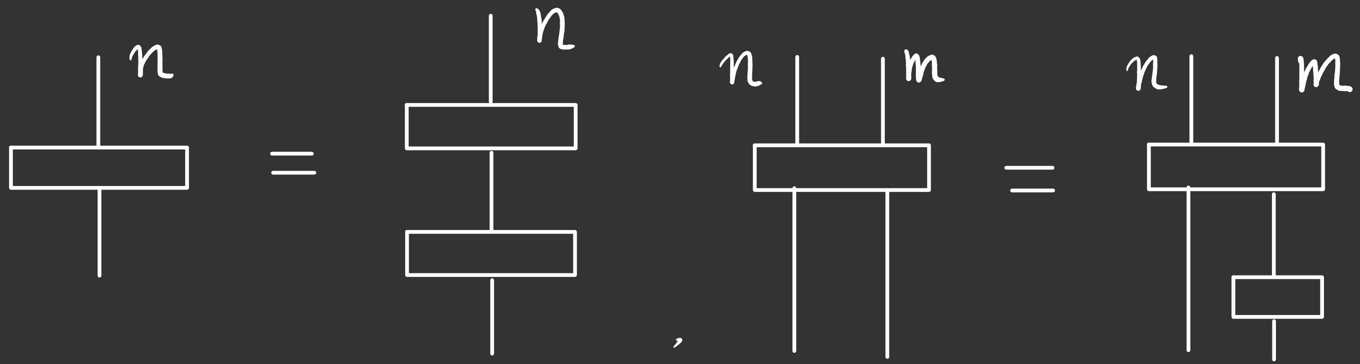


$$\left([n]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right)$$

Definition (The A_2 claps of type $(n, 0)$).



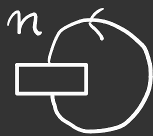
Lem



Proposition (The one-row colored $\mathfrak{sl}_3$ Jones polynomial)

L is a link.

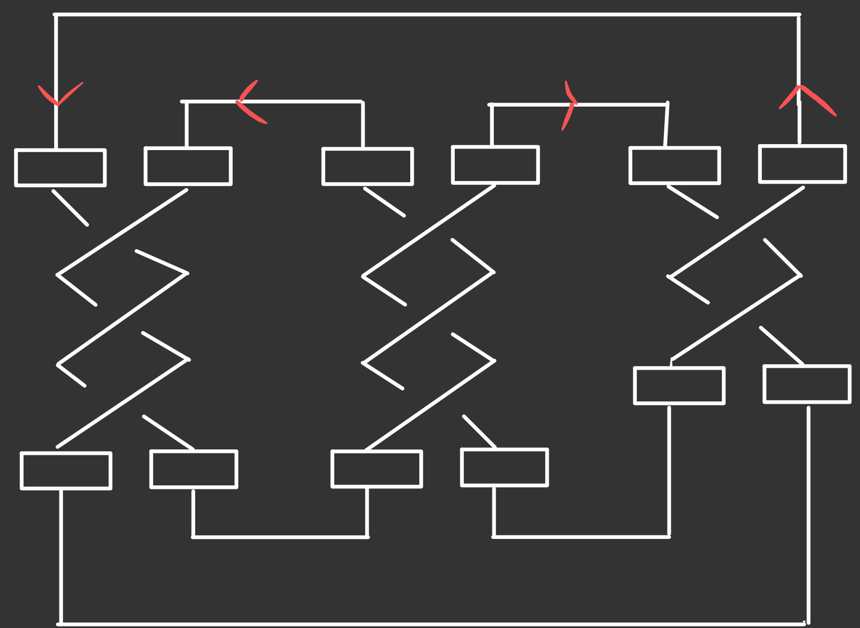
$$J^{\mathfrak{sl}_3}_{(n,0)}(L;q) = \left(q^{\frac{n^2+3n}{3}}\right)^{-w(L)} \langle L(n,0) \rangle_3 \Big/ \Big\langle$$



$$\Big\rangle_3$$

Where $w(L)$ is writhe number of L.

$P(\downarrow 3 \downarrow, \uparrow 3 \uparrow, \downarrow 2 \uparrow)(n,0) =$



Concent

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- About colored Jones polynomial
- Main Theorem
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial

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Main Theorem 2 (Kawasoe)

α, β, γ are integers.

$$\widetilde{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha+1\downarrow, \uparrow 2\beta+1\uparrow, \downarrow 2\gamma\uparrow); q)$$

$$= \sum_{0 \le k_{|2\alpha+1|} \le k_{|2\alpha|} \le \cdots \le k_1 \le n} \ \sum_{0 \le l_{|2\beta+1|} \le l_{|2\beta|} \le \cdots \le l_1 \le n} \ \sum_{0 \le m_{|\gamma|} \le m_{|\gamma-1|} \le \cdots \le m_1 \le n} \ \sum_{s = \mathrm{Max}\{k_{|2\alpha+1|}, l_{|2\beta+1|}\}}^{\min\{k_{|2\alpha+1|} + l_{|2\beta+1|}, n\}} \ \sum_{a=s}^{n} \ \sum_{t = \mathrm{Max}\{a, m_{|\gamma|}\}}^{\min\{a + m_{|\gamma|}, n\}} \left(q^{\frac{n^2+3n}{3}}\right)^{-(2\alpha+2\beta-2\gamma+2)} \chi_{\mathrm{sign}(2\alpha+1)}(n, k_1, k_2, \cdots, k_{|2\alpha+1|})$$

$$\chi_{\mathrm{sign}(2\beta+1)}(n, l_1, l_2, \cdots, l_{|2\beta+1|})\, \phi(n, m_1, m_2, \cdots, m_{|\gamma|})\, q^{\varepsilon_\gamma}\, \Omega(n, s, k_{|2\alpha+1|}, l_{|2\beta+1|})$$

$$\psi(n, t, a, m_\gamma)\, q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}$$

Definition

$$\phi(n, k_1, k_2, \dots, k_m)_{q^{\varepsilon_m}}$$

$$= \frac{(q^{\varepsilon_m})^{-\frac{2m}{3}(n^2+3n)} (q^{\varepsilon_m})^{n-k_m} (q^{\varepsilon_m})^{\sum_{i=1}^{m}(k_i^2+2k_i)} (q^{\varepsilon_m})_n}{(q^{\varepsilon_m})_{n-k_1} (q^{\varepsilon_m})_{k_1-k_2} \cdots (q^{\varepsilon_m})_{k_{m-1}-k_m} \{(q^{\varepsilon_m})_n\}^2}$$

$$\psi(n, t, k, l)$$

$$= \frac{q^{(t+1)(t-k-l)+kl} (q)_k (q)_l (q)_{n-k}^2 (q)_{n-l}^2 (q)_{2n-t+2}}{(q)_n^2 (q)_{n-t}^2 (q)_{t-k} (q)_{t-l} (q)_{2n-k-l+2} (q)_{-t+k+l}}$$

Where

$$q^{\varepsilon_m} = q^{\frac{m}{|m|}} \quad , \quad (q)_n = \prod_{i=1}^{n} (1-q^i)$$

Definition

$\chi_+(n, k_1, k_2, \dots, k_m)$

$$= (-1)^{nm} q^{-\frac{1}{6}(n^2+3n)m} \frac{q^{\frac{1}{2}(n-k_m)} (-1)^{\sum_{i=1}^{m} k_i} q^{\sum_{i=1}^{m} \frac{1}{2}(k_i^2+k_i)} (q)_n}{(q)_{n-k_1} (q)_{k_1-k_2} \cdots (q)_{k_{m-1}-k_m} (q)_m}$$

$\chi_-(n, k_1, k_2, \dots, k_m)$

$$= (-1)^{nm} q^{\frac{1}{6}(n^2+3n)m} \frac{q^{-\frac{1}{2}(n-k_m)} (-1)^{\sum_{i=1}^{m} k_i} q^{\sum_{i=1}^{m} \frac{1}{2}(k_i^2-k_i)} q^{\sum_{i=1}^{m} k_{i-1}k_i}}{(q)_{n-k_1} (q)_{k_1-k_2} \cdots (q)_{k_{m-1}-k_m} (q)_m}$$

$\Omega(n, t, k, l)$

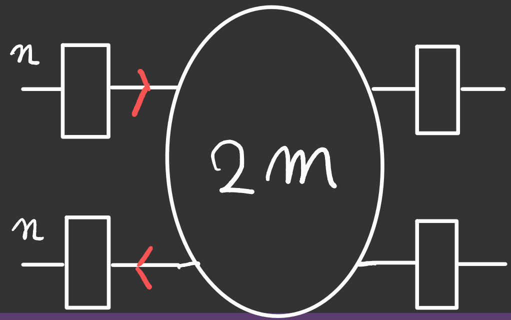
$$= \frac{q^{(t+2)(t-k-l)+kl+\frac{1}{2}(k+l)} (1-q^{n+1-k})(1-q^{n+1-l}) (q)_k (q)_l (q)_{n-k}^2 (q)_{n-l} (q)_{2n-t+2}}{(1-q^{n+1-t})^2 (q)_n^2 (q)_{n-t}^2 (q)_{t-k} (q)_{t-l} (q)_{2n-k+2} (q)_{-t+k+l}}$$

where $(q)_n = \prod_{i=1}^{n} (1-q^i)$

Theorem (2017 Yuasa)

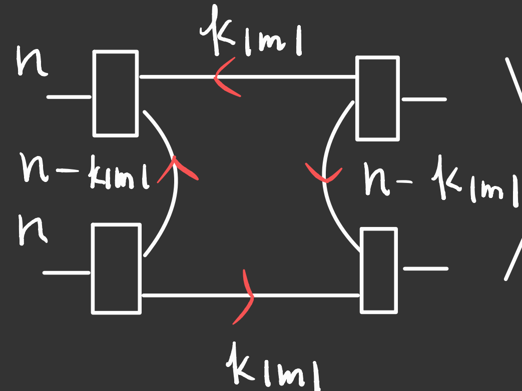
2m-times twists

(1)

$\Big\langle$ 

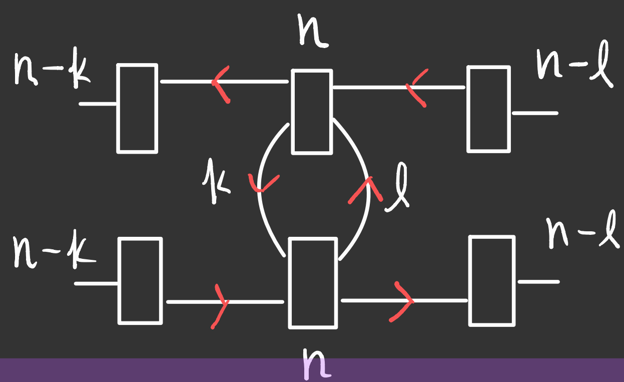
$\Big\rangle_3$

$$= \sum_{0 \le k_{|m|} \le k_{|m-1|} \le \cdots \le k_1 \le n} \phi(n, k_1, k_2, \cdots, k_{|m|})\, q^{\varepsilon_m}$$

$\Big\langle$ 

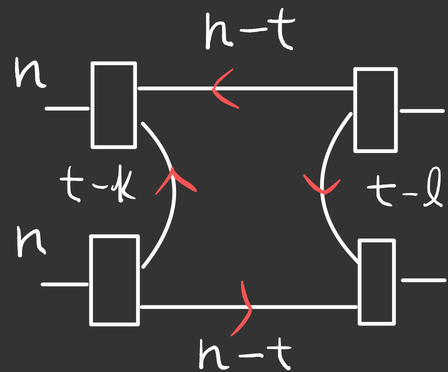
$\Big\rangle_3$

(2)

$\Big\langle$ 

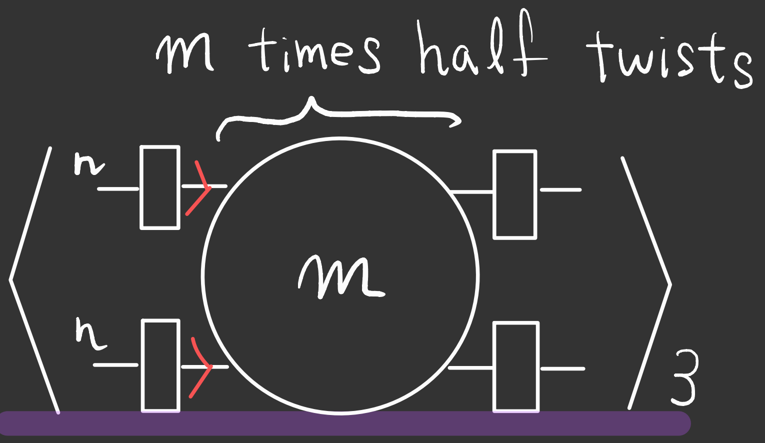
$\Big\rangle_3$

$$= \sum_{t=\max\{k,l\}}^{\min\{k+l,n\}} \psi(n,t,k,l)$$

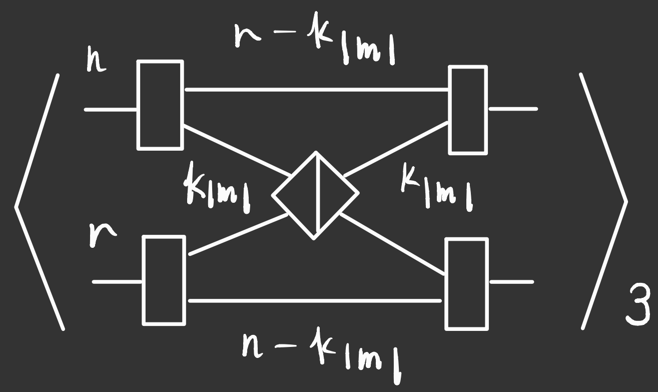
$\Big\langle$ 

$\Big\rangle_3$

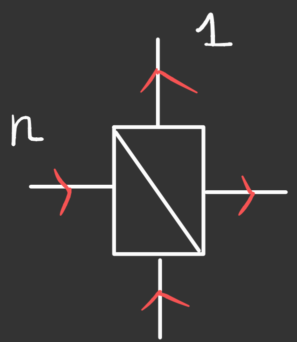
Proposition

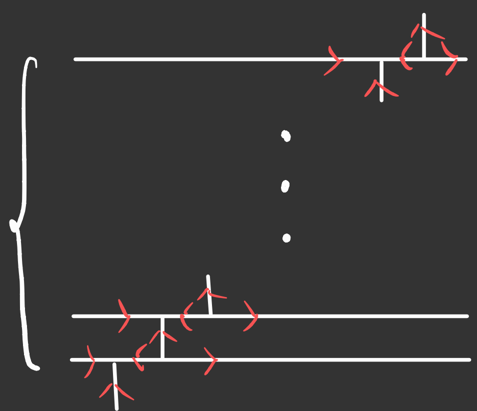
(3) 

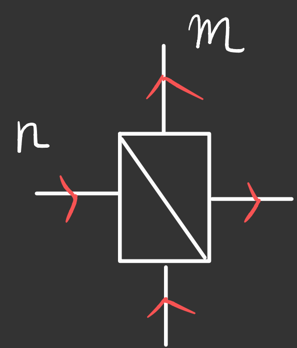
$$= \sum_{0 \le k_{|m|} \le k_{|m-1|} \le \cdots \le k_1 \le n} \chi_{\mathrm{sign}(m)}(n, k_1, k_2, \cdots, k_{|m|})$$

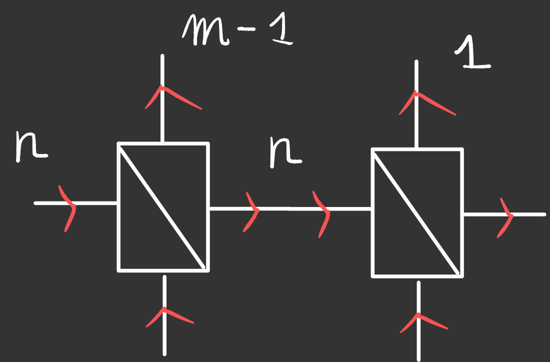


Definition

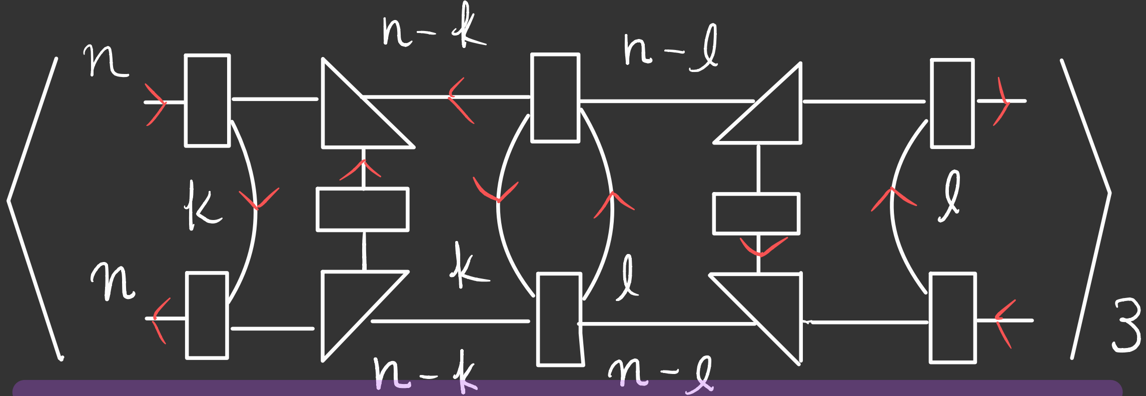


$= n$  ,

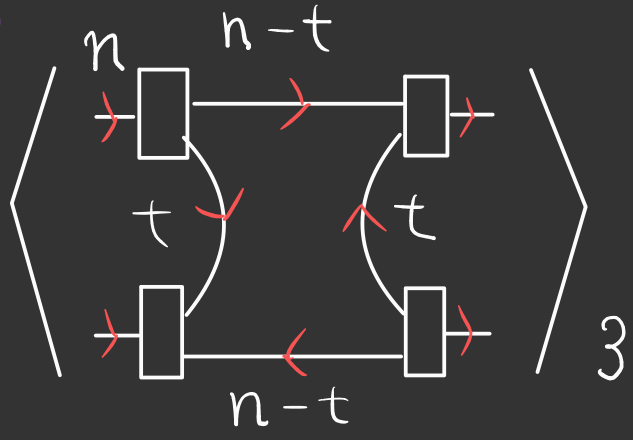


$=$ 

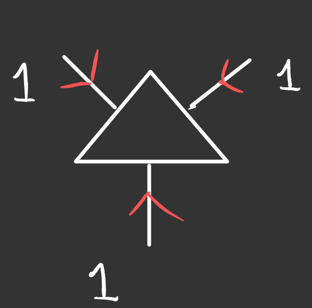
Proposition

(4) 

$$= \sum_{t=\max\{k,l\}}^{\min\{k+l,n\}} \sum_{a=t}^{n} \Omega(n,t,k,l)$$



Definition



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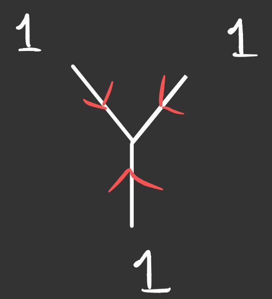

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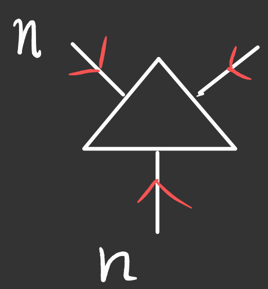

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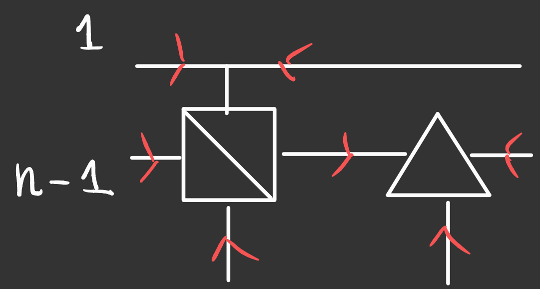
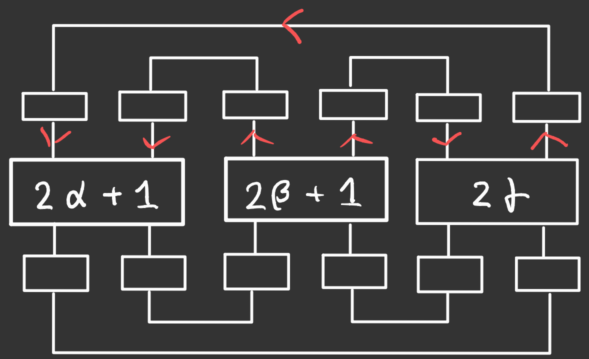


Sketch of the proof



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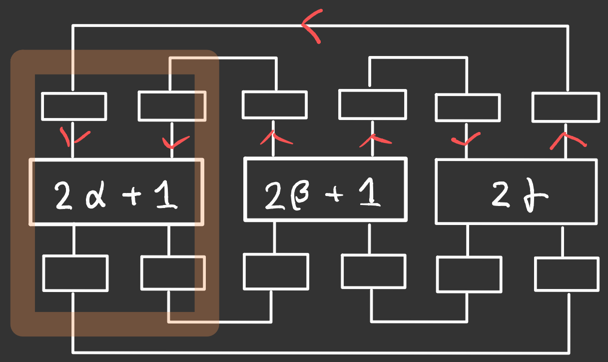

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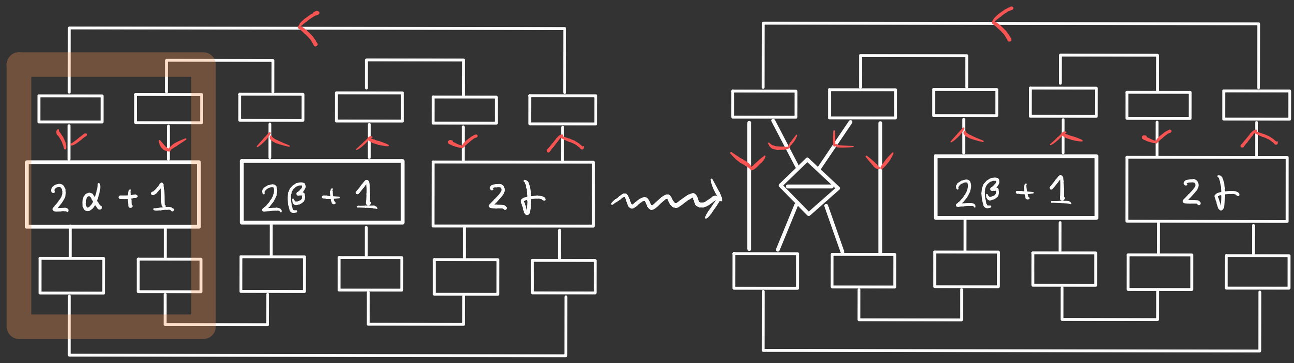

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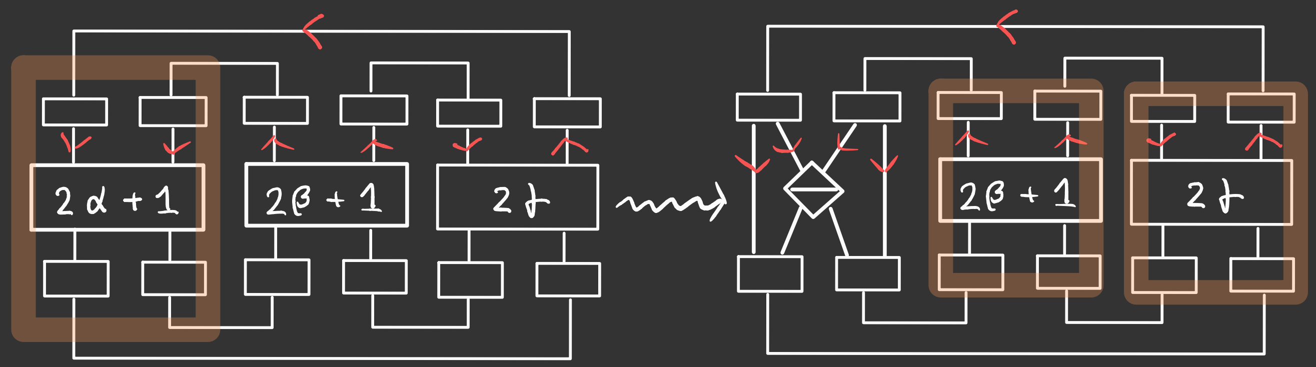

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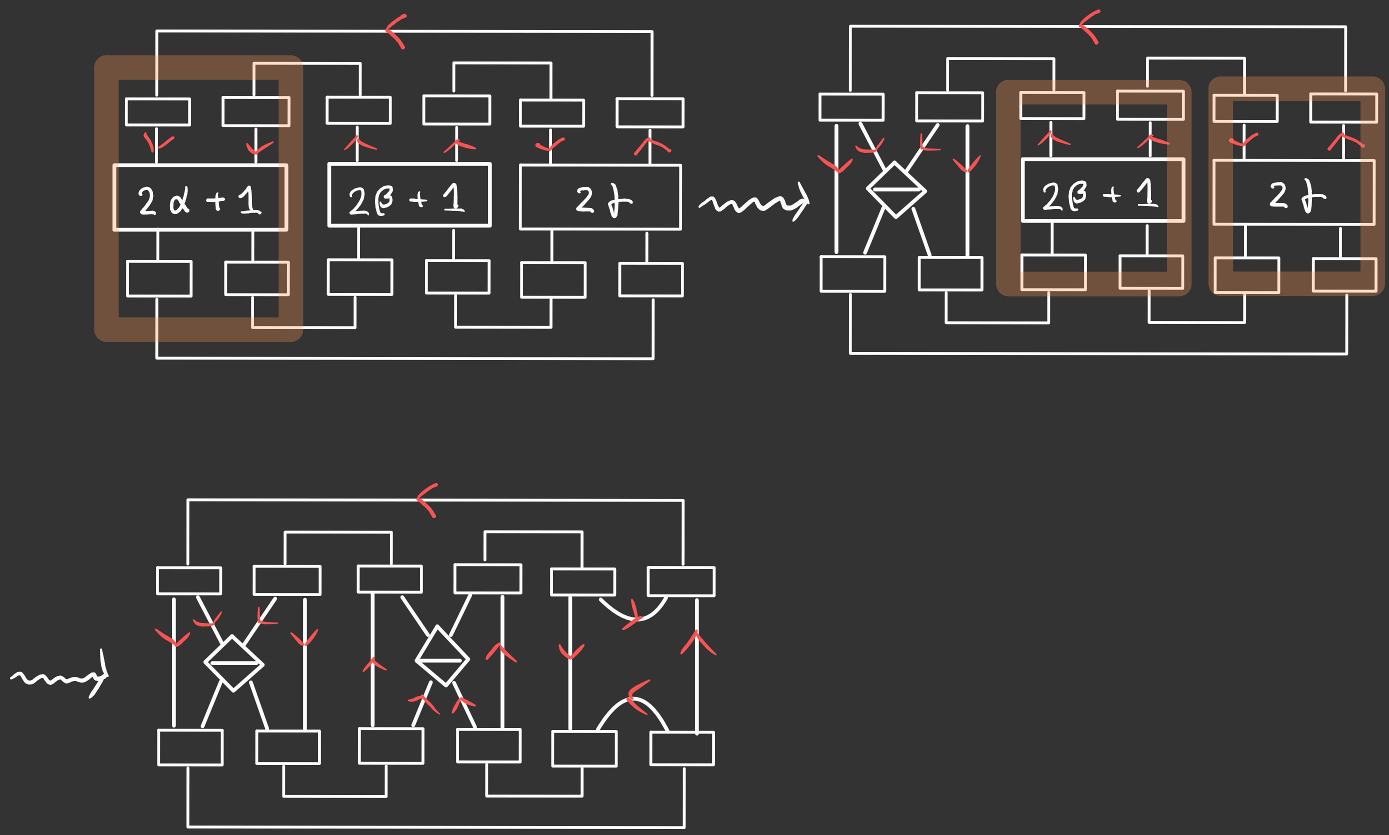
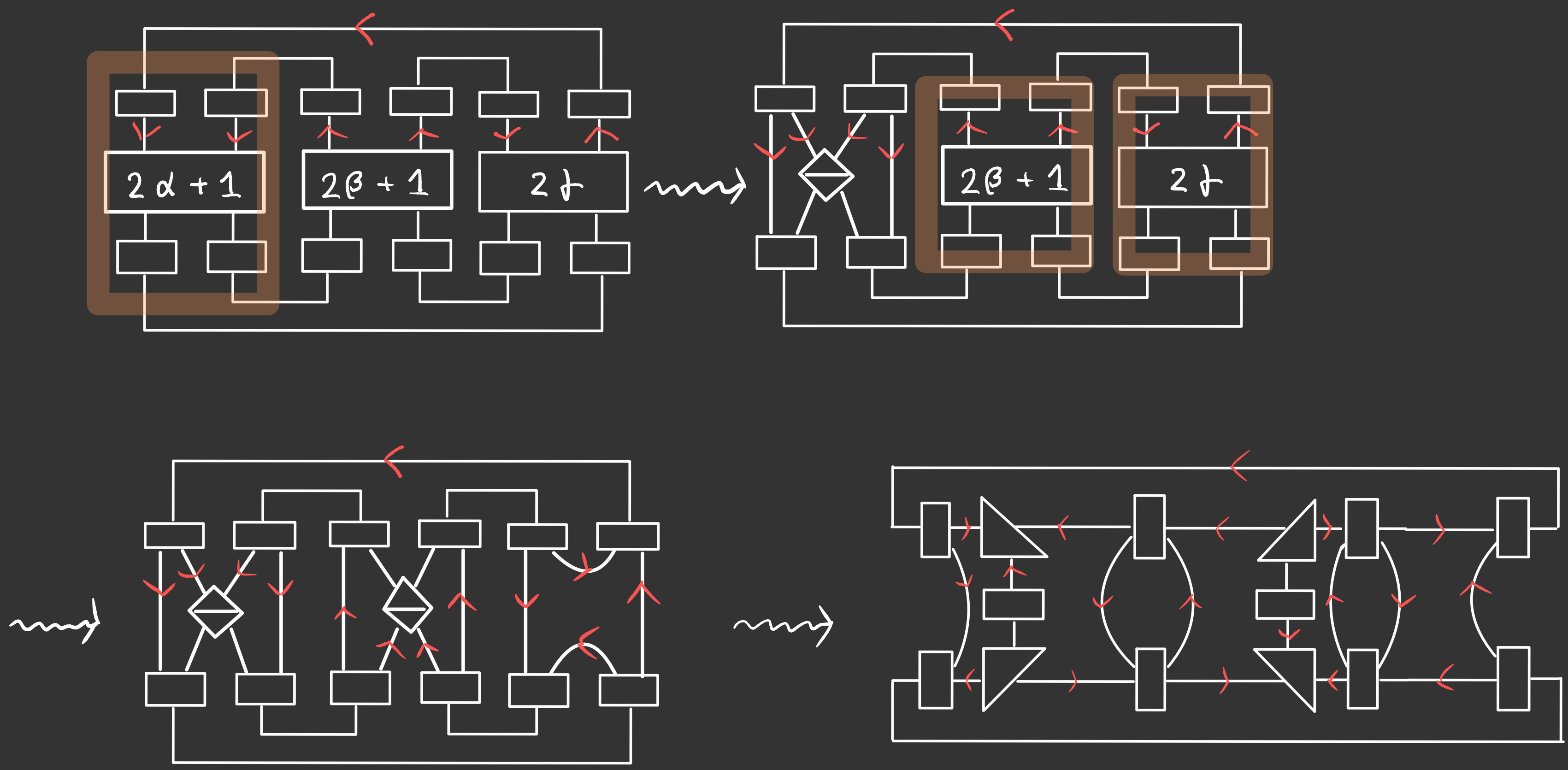
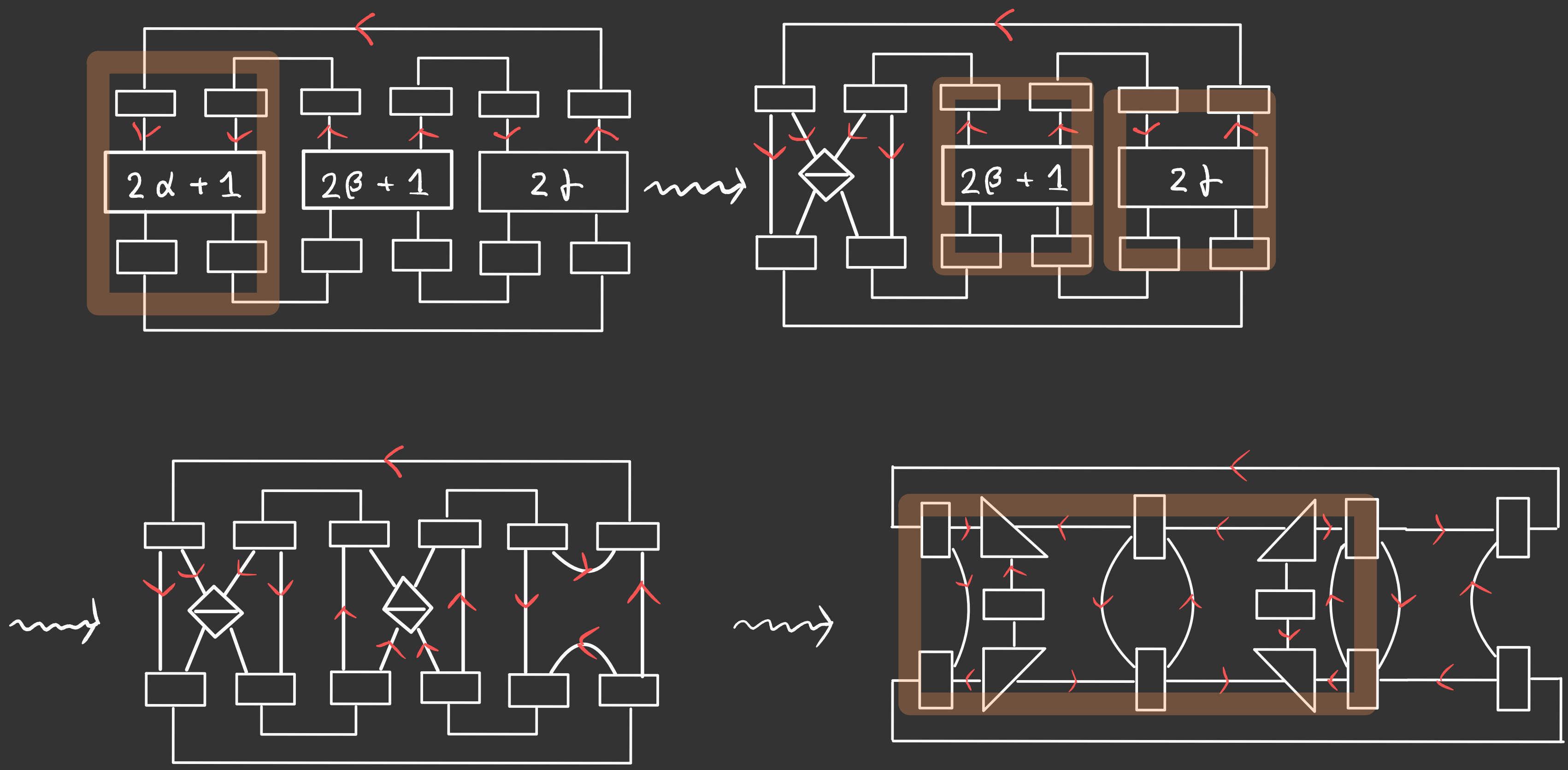


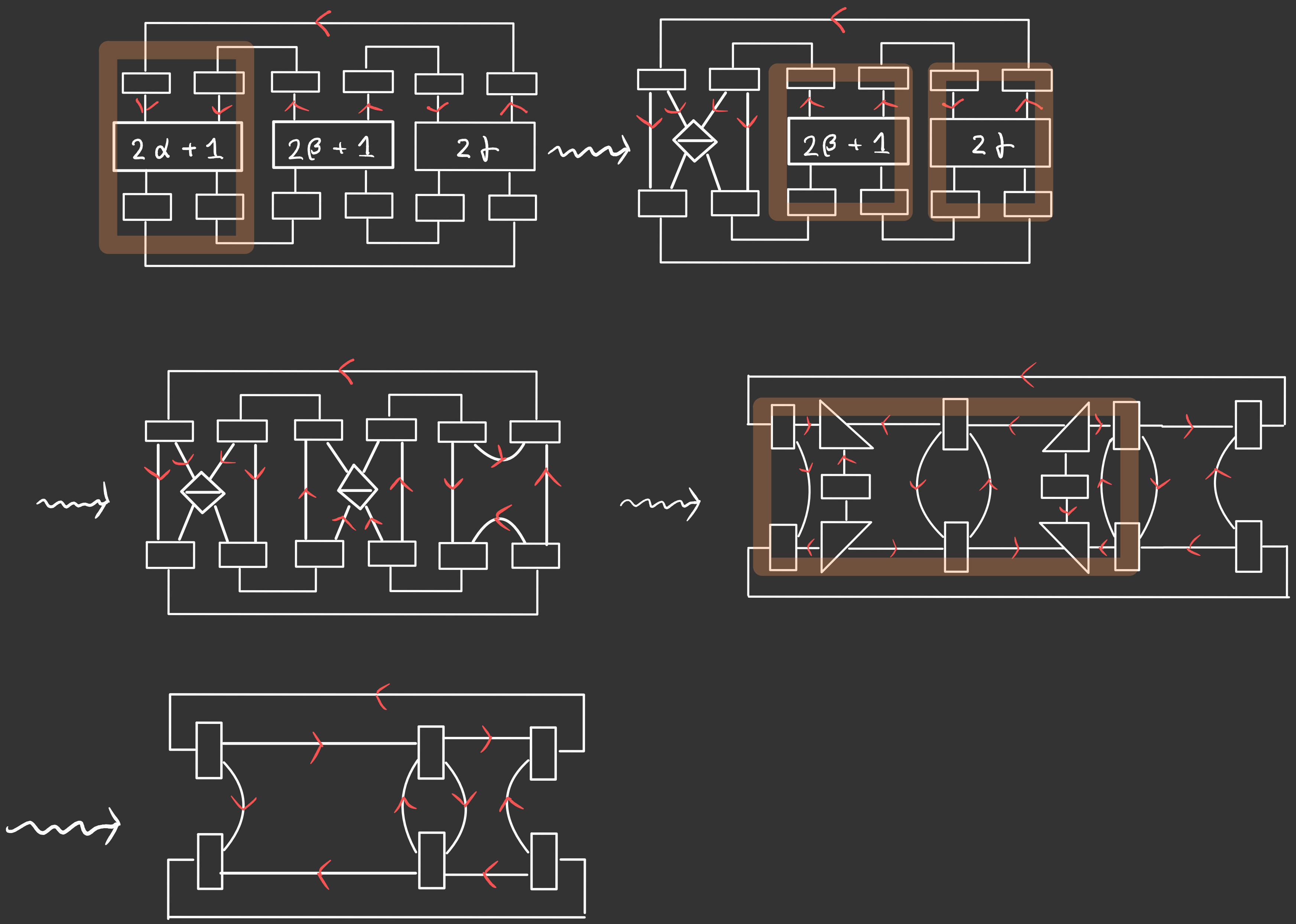
Sketch of the proof



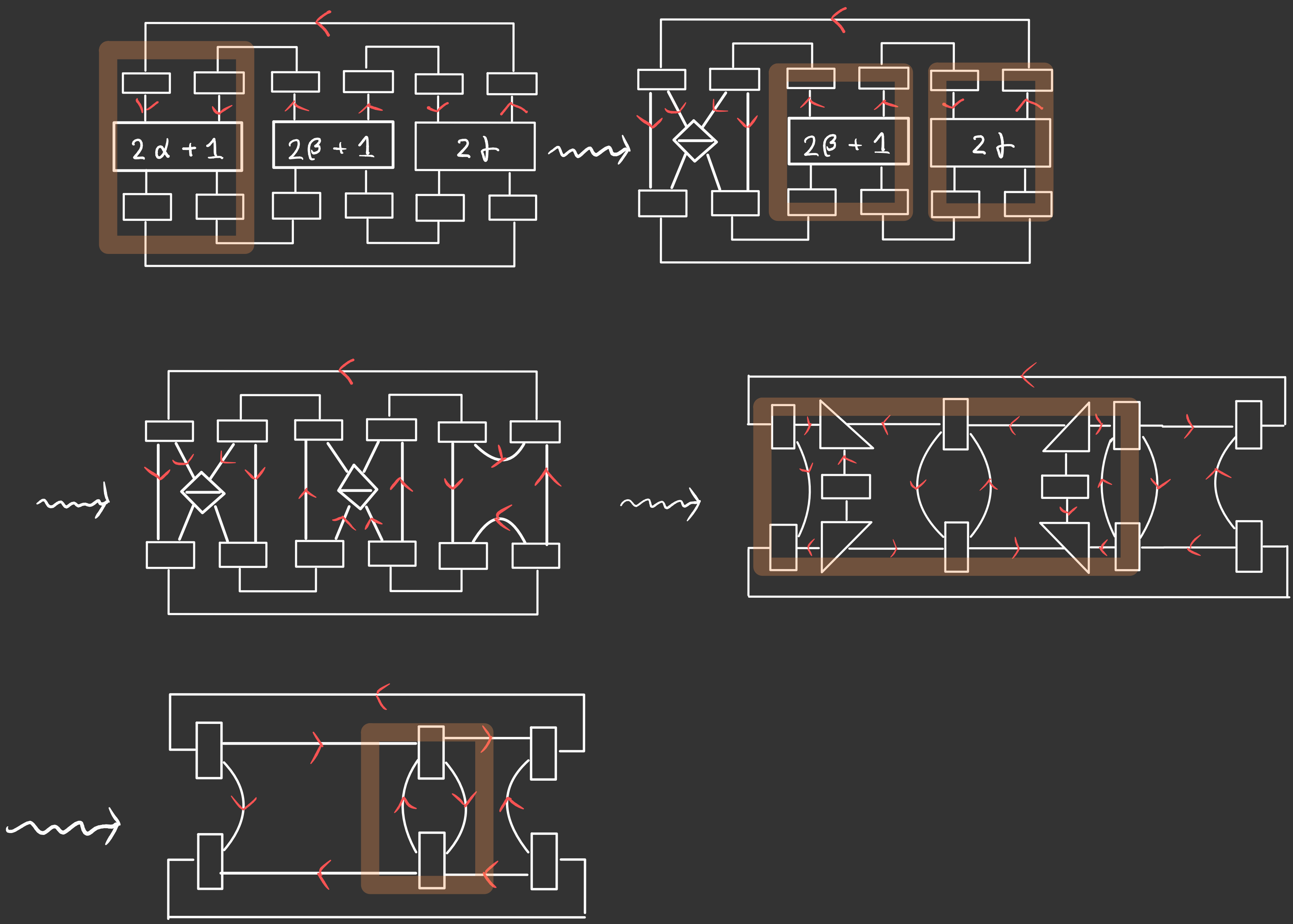
Sketch of the proof



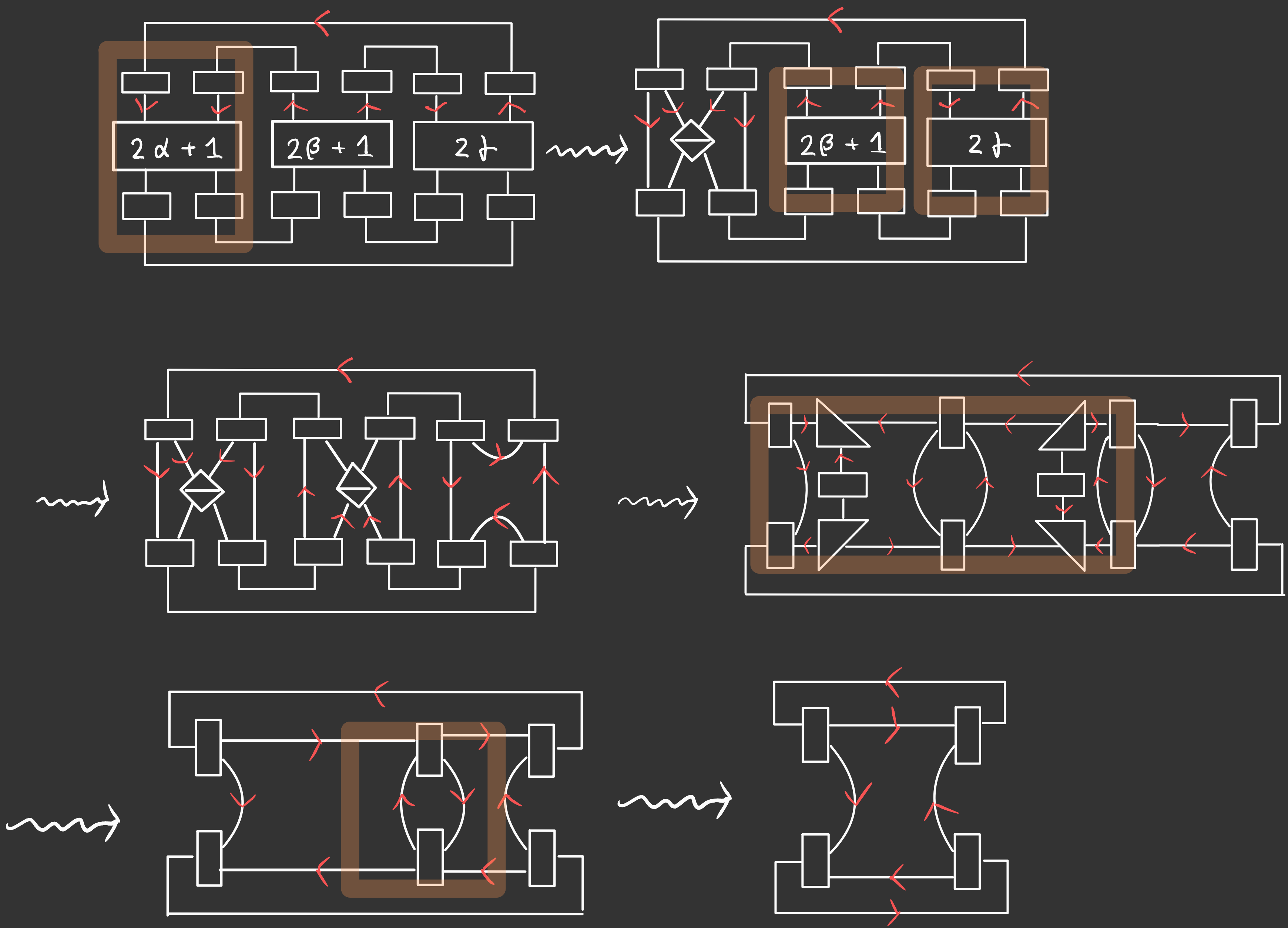
Sketch of the proof



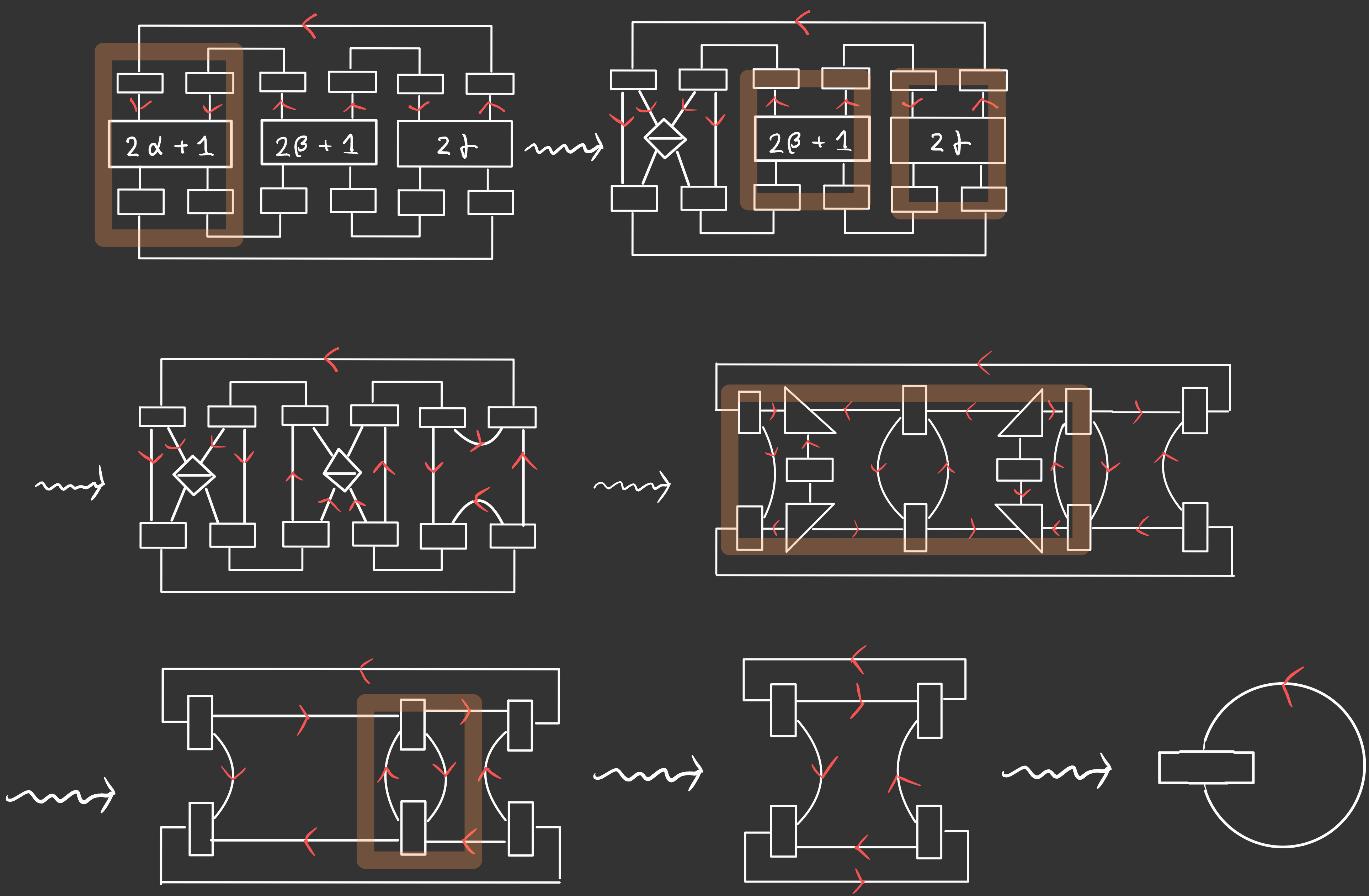
Sketch of the proof



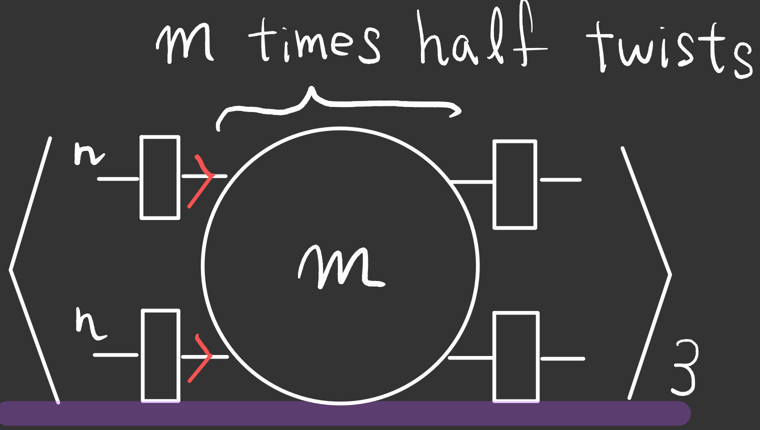
Sketch of the proof



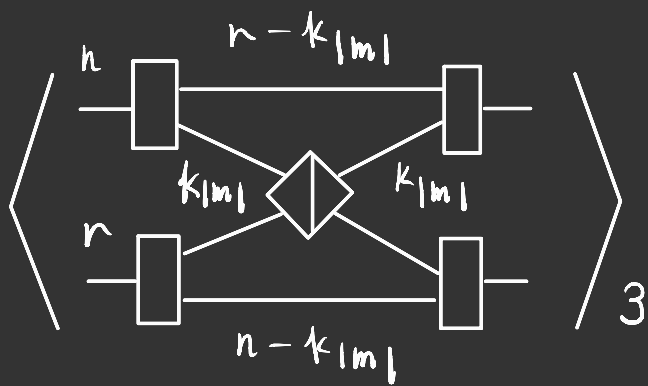
Sketch of the proof



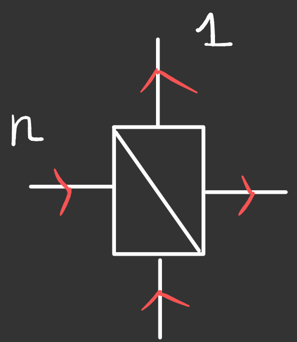
Proposition

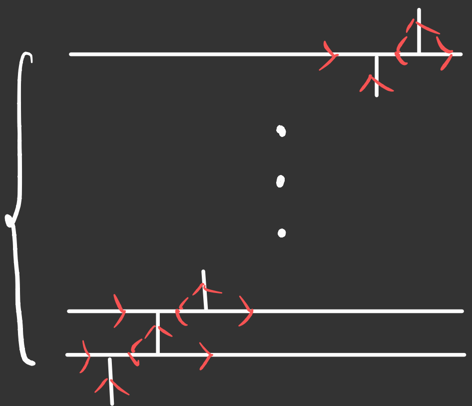
(3) 

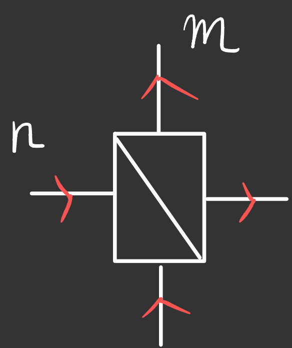
$$= \sum_{0 \leq k_{|m|} \leq k_{|m-1|} \leq \cdots \leq k_1 \leq n} \chi_{\mathrm{sign}(m)}(n, k_1, k_2, \cdots, k_{|m|})$$

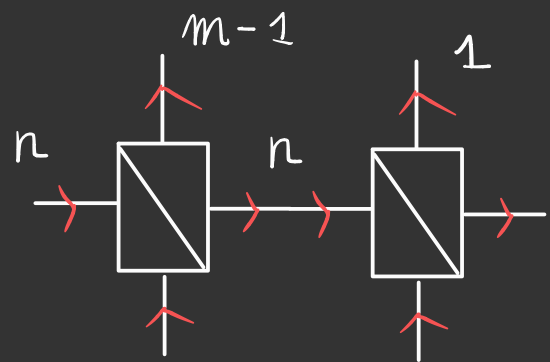


Definition



$= n$  ,



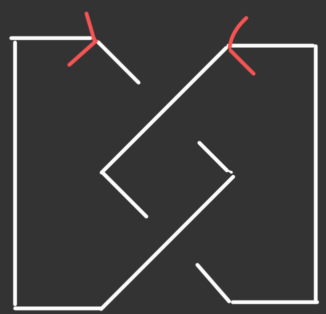
$=$ 

Corollary (Kawasoe)

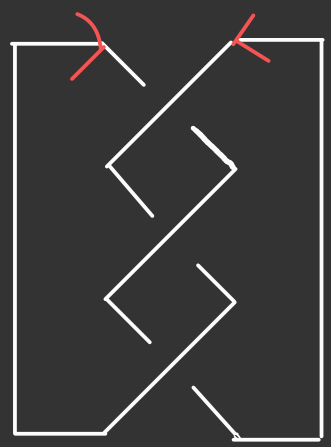
m is an integer.

$$J^{sl_3}_{(n,0)}(T(2,m);q)$$

$$= \left(q^{\frac{n^2+3n}{3}}\right)^{-m} \sum_{0 \le k_{|m|} \le k_{|m|-1} \le \cdots \le k_1 \le n} \chi_{\mathrm{sign}(m)}(n, k_1, k_2, \cdots, k_{|m|})\, q^{-\frac{n+k_{|m|}}{2}} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{n-k_{|m|}+2})(1-q)}$$



$T(2,2)$,



$T(2,3)$

Remark

For $m = 1, 2, 3$, $n = 1, 2, \cdots, 10$,

- $J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m+1);q) = J^{\mathfrak{sl}_3}_{(n,0)}(T(2,-(2m+1));q^{-1})$
- the following are equal

  Corollary (Kawasoe) for $T(2,2m+1)$

  Theorem (Garaoufalidis-Morton-Vuong) for $T(2,2m+1)$

  Theorem (Yuasa) for $T(2,2m+1)$

by using Mathematica.

Remark (Calunculation results by Mathematica)

For $n = 1, 2, \cdots, 8$

- the following are equl

Main Theorem 2 for $P(1,1,2) = 4_1$, $P(1,3,2) = 6_2$ and $P(1,5,2) = 8_2$

Yuasa Theorem for $K[2,2] = 4_1$, $K[2,2,-2,2] = 6_2$ and $K[2,2,-2,2,-2,2] = 8_2$

by using Mathematica.

$$J^{sl_3}_{(1,0)}(P(1,1,2);q) = q^3 - q + 1 - q^{-1} + q^{-3}$$

$$J^{sl_3}_{(2,0)}(P(1,1,2);q) = q^8 - q^6 - q^5 + q^4 + q^3 - 2q^2 + 3 - 2q^{-2} + q^{-3} + q^{-4} - q^{-5} - q^{-6} + q^{-8}$$

$$J^{sl_3}_{(3,0)}(P(1,1,2);q) = q^{15} - q^{13} - q^{12} - q^{11} + 2q^{10} + q^9 - 2q^7 - q^6 + 4q^5 + 2q^4 - 2q^3$$
$$- 4q^2 + 5 - q^{-2} - 2q^{3} + 2q^{-4} + 4q^{-5} - q^{-6} - 2q^{-7} + q^{-9}$$
$$+ 2q^{-10} - q^{-11} - q^{-12} - q^{-13} + q^{-15}$$

Concent

① Introduction

- About colored Jones polynomial
- Main Theorem
- The one-row colored $\mathfrak{sl}_3$ Jones polynomial

② Sketch of the proof of Main Theorem

- Main Theorem 2 $P(2\alpha+1, 2\beta+1, 2\gamma)$

③ Appendix

- $P(3,3,2) = 8_5$, $P(3,7,-2) = 12n242$
- 8_{10}, 8_{15}, 8_{20}, 8_{21}

8_5

$J^{sl_3}_{(1,0)}(P(3,3,2);q)$

$$= q^{-1} + q^{-3} + q^{-4} - 2q^{-5} + q^{-6} - 2q^{-7} + q^{-9} - q^{-10} + q^{-11}$$

$J^{sl_3}_{(2,0)}(P(3,3,2);q)$

$$= -q^{-2} + q^{-3} + 3q^{-4} - q^{-5} - 3q^{-6} + q^{-7} + 4q^{-8} - 4q^{-9} - 6q^{-10} + 5q^{-11} + 4q^{-12} - 7q^{-13}$$
$$- q^{-14} + 8q^{-15} - 5q^{-17} + q^{-18} + 4q^{-19} - 2q^{-20} - 2q^{-21} + 2q^{-22} - 2q^{-24} + q^{-25} - q^{-27} + q^{-28}$$

$J^{sl_3}_{(3,0)}(P(3,3,2);q)$

$$= q^{3} - q - 1 + q^{-1} + 4q^{-2} + q^{-3} - 4q^{-4} - 5q^{-5} + 2q^{-6} + 9q^{-7} + 3q^{-8} - 9q^{-9} - 12q^{-10} + q^{-11} + 18q^{-12}$$
$$+ 7q^{-13} - 14q^{-14} - 21q^{-15} + 5q^{-16} + 29q^{-17} + 11q^{-18} - 23q^{-19} - 26q^{-20} + 12q^{-21} + 36q^{-22} + 6q^{-23}$$
$$- 33q^{-24} - 25q^{-25} + 18q^{-26} + 36q^{-27} - 4q^{-28} - 34q^{-29} - 16q^{-30} + 26q^{-31} + 26q^{-32} - 12q^{-33} - 25q^{-34}$$
$$- 3q^{-35} + 21q^{-36} + 9q^{-37} - 11q^{-38} - 9q^{-39} + 3q^{-40} + 8q^{-41} - q^{-42} - 4q^{-43} + q^{-44} + q^{-46} - q^{-47} - q^{-50} + q^{-51}$$

$J^{sl_3}_{(4,0)}(P(3,3,2);q)$

$$
\begin{aligned}
= {} & q^{8} - q^{6} - q^{5} - q^{4} + 2q^{3} + 4q^{2} + 2q - 2 - 7q^{-1} - 6q^{-2} + 3q^{-3} + 12q^{-4} + 10q^{-5} - q^{-6} - 17q^{-7} - 19q^{-8} - q^{-9} \\
& + 20q^{-10} + 28q^{-11} + 7q^{-12} - 26q^{-13} - 39q^{-14} - 16q^{-15} + 31q^{-16} + 52q^{-17} + 27q^{-18} - 31q^{-19} - 69q^{-20} - 35q^{-21} \\
& + 35q^{-22} + 84q^{-23} + 53q^{-24} - 43q^{-25} - 98q^{-26} - 63q^{-27} + 43q^{-28} + 118q^{-29} + 70q^{-30} - 52q^{-31} - 135q^{-32} \\
& - 83q^{-33} + 61q^{-34} + 149q^{-35} + 87q^{-36} - 74q^{-37} - 164q^{-38} - 83q^{-39} + 90q^{-40} + 178q^{-41} + 75q^{-42} \\
& - 108q^{-43} - 174q^{-44} - 57q^{-45} + 122q^{-46} + 164q^{-47} + 29q^{-48} - 126q^{-49} - 140q^{-50} - 7q^{-51} + 123q^{-52} \\
& + 105q^{-53} - 15q^{-54} - 102q^{-55} - 74q^{-56} + 28q^{-57} + 77q^{-58} + 39q^{-59} - 27q^{-60} - 52q^{-61} \\
& - 15q^{-62} + 23q^{-63} + 27q^{-64} + 4q^{-65} - 17q^{-66} - 9q^{-67} + 2q^{-68} + 7q^{-69} + 4q^{-70} - 4q^{-71} \\
& - 2q^{-72} + q^{-74} + 2q^{-75} - 2q^{-76} - q^{-79} + q^{-80}
\end{aligned}
$$

$$J^{\mathfrak{sl}_3}_{(1,0)}(P(3,7,-2);q)$$

$$= q^{-10} + q^{-11} + q^{-12} + q^{-14} - q^{-15} - q^{-16} - q^{-19} - q^{-22} + q^{-23}$$

$$J^{\mathfrak{sl}_3}_{(2,0)}(P(3,7,-2);q)$$

$$= q^{-20} + q^{-22} + 2q^{-23} + q^{-24} + 2q^{-26} - q^{-27} + q^{-28} + 2q^{-29} - 4q^{-30} - 3q^{-31} + 4q^{-32} - q^{-33}$$

$$- 9q^{-34} + 8q^{-36} - 9q^{-38} + 2q^{-39} + 9q^{-40} - 4q^{-41} - 8q^{-42} + 2q^{-43} + 6q^{-44} - 2q^{-45} - 4q^{-46}$$

$$+ 3q^{-47} + 4q^{-48} - 2q^{-50} + q^{-51} - q^{-53} + q^{-55} - q^{-56} - q^{-57} + q^{58}$$

$$J^{sl_3}_{(3,0)}(P(3,7,-2);q)$$

$$\begin{aligned}
= {} & q^{-30} + q^{-33} + 2q^{-34} + q^{-35} + q^{-37} + 3q^{-38} + q^{-39} - 2q^{-40} - q^{-41} + 4q^{-42} + 2q^{-43} - 2q^{-44} - 4q^{-45} \\
& + 2q^{-47} - 2q^{-48} - 2q^{-49} - 2q^{-50} - 3q^{-51} - 5q^{-52} - 4q^{-53} + 8q^{-54} + 3q^{-55} - 8q^{-56} - 8q^{-57} \\
& + 2q^{-58} + 19q^{-59} + 6q^{-60} - 12q^{-61} - 14q^{-62} + 21q^{-64} + 7q^{-65} - 20q^{-66} - 22q^{-67} + q^{-68} + 29q^{-69} \\
& + 14q^{-70} - 16q^{-71} - 19q^{-72} + 4q^{-73} + 28q^{-74} + 13q^{-75} - 18q^{-76} - 20q^{-77} + 3q^{-78} + 24q^{-79} \\
& + 8q^{-80} - 19q^{-81} - 19q^{-82} + q^{-83} + 19q^{-84} + 10q^{-85} - 10q^{-86} - 12q^{-87} + 10q^{-89} + 2q^{-90} \\
& - 7q^{-91} - 4q^{-92} + 2q^{-93} + 4q^{-94} + q^{-96} - 2q^{-99} + q^{-100} + q^{-101} - q^{-103} - q^{-104} + q^{-105}
\end{aligned}$$

$J^{\mathfrak{sl}_3}_{(4,0)}(P(3,7,-2);q)$

$$\begin{aligned}
&= q^{-40} + q^{-44} + 2q^{-45} + q^{-46} + q^{-49} + 4q^{-50} + q^{-51} - 2q^{-53} - q^{-54} + 4q^{-55} + 3q^{-56} - q^{-57} - 4q^{-58} \\
&\quad - 3q^{-59} + 5q^{-60} + 3q^{-61} + q^{-62} - 5q^{-63} - 8q^{-64} + 3q^{-65} + 3q^{-66} + 3q^{-67} - 2q^{-68} - 12q^{-69} - 6q^{-70} \\
&\quad - 2q^{-71} + 3q^{-72} + 2q^{-73} - 6q^{-74} - 2q^{-75} - 4q^{-76} + 2q^{-78} + 2q^{-79} + 7q^{-80} + 6q^{-81} - q^{-82} - 6q^{-83} \\
&\quad + 4q^{-84} + 8q^{-85} + 18q^{-86} + 11q^{-87} - 13q^{-88} - 31q^{-89} - 20q^{-90} + 11q^{-91} + 29q^{-92} + 15q^{-93} \\
&\quad - 21q^{-94} - 39q^{-95} - 12q^{-96} + 26q^{-97} + 49q^{-98} + 24q^{-99} - 19q^{-100} - 29q^{-101} - 16q^{-102} + 20q^{-103} \\
&\quad + 36q^{-104} + 13q^{-105} - 32q^{-107} - 23q^{-108} + 7q^{-109} + 20q^{-110} + 23q^{-111} - 24q^{-112} - 24q^{-113} - 27q^{-114} \\
&\quad - 9q^{-115} + 18q^{-116} + 27q^{-117} + 11q^{-118} - 16q^{-119} - 23q^{-120} - 6q^{-121} + 22q^{-122} + 28q^{-123} - 31q^{-125} \\
&\quad - 41q^{-126} - 13q^{-127} + 26q^{-128} + 42q^{-129} + 23q^{-130} - 19q^{-131} - 35q^{-132} - 19q^{-133} + 16q^{-134} + 33q^{-135} + 20q^{-136} - 9q^{-137} \\
&\quad - 25q^{-138} - 13q^{-139} + 4q^{-140} + 14q^{-141} + 6q^{-142} - 8q^{-143} - 8q^{-144} + 7q^{-146} + 6q^{-147} - q^{-148} - 6q^{-149} - 5q^{-150} - q^{-151} + 3q^{-152} \\
&\quad + 3q^{-153} + 2q^{-154} - 2q^{-156} - q^{-157} + 2q^{-159} - q^{-162} - q^{-163} + q^{-164}
\end{aligned}$$

Concent

① Introduction

- About colored Jones polynomial
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② Sketch of the proof of Main Theorem

- Main Theorem 2 $P(2\alpha+1, 2\beta+1, 2\gamma)$

③ Appendix

- $P(3,3,2) = 8_5$, $P(3,7,-2) = 12n242$
- 8_{10}, 8_{15}, 8_{20}, 8_{21}

Theorem ($P(-3,-2,3,1) = 8_{10}$)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(-3,-2,3,1);q)$$

$$= \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_2\le l_1\le n} \sum_{0\le P_3\le P_2\le P_1\le n} \sum_{m=0}^{n} \sum_{t=\mathrm{Max}\{k_3,l_3\}}^{\min\{k_3+l_3,n\}}$$

$$\sum_{a=t}^{n} \sum_{S=\mathrm{Max}\{P_2,m\}}^{\min\{P_2+m,n\}} \sum_{b=S}^{n} \sum_{u=\mathrm{Max}\{a,b\}}^{\min\{a+b,n\}} \left(q^{\frac{n^2+3n}{3}}\right)^3 \chi_-(n,k_1,k_2,k_3)$$

$$\chi_-(n,l_1,l_2)_{q^{-1}}\, \chi_+(n,P_1,P_2,P_3)\, \chi_-(n,m)\, \Omega(n,t,k_3,l_2)\, \Omega(n,S,P_3,m)$$

$$\phi(n,u,a,b)\, q^{-(n-u)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{u+1})(1-q^{u+2})}$$

Theorem ($P(3,-1,-2,-1,3) = 8_{15}$)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(3,-1,-2,-1,3);q)$$

$$= \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l\le n} \sum_{0\le m_2\le m_1\le n} \sum_{0\le p\le n} \sum_{0\le r_3\le r_2\le r_1\le n} \sum_{t=\mathrm{Max}\{k_3,l\}}^{\min\{k_3+l,n\}} \sum_{a=t}^{n}$$

$$\sum_{s=\mathrm{Max}\{p,r_3\}}^{\min\{p+r_2,n\}} \sum_{b=s}^{n} \sum_{u=\mathrm{Max}\{a,m_2\}}^{\min\{a+m_2,n\}} \sum_{v=\mathrm{Max}\{b,m_2\}}^{\min\{b+m_2,n\}} \left(q^{\frac{n^2+3n}{3}}\right)^{-2} \chi_+(n,k_1,k_2,k_3)$$

$$\chi_-(n,l)\,\psi(n,m_1,m_2)_{q^{-1}}\,\chi_-(n,p)\,\chi_+(n,r_1,r_2,r_3)\,\Omega(n,t,k_3,l)$$

$$\Omega(n,s,p,r_3)\,\phi(n,u,a,m_2)\,\phi(n,v,u,b)\,q^{-(n-v)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{v+1})(1-q^{v+2})}$$

Theorem $P(3,-2,-3,1)=\delta_{20})$

$$J^{sl_3}_{(n,0)}(P(3,-2,-3,1);q)$$

$$= \sum_{0\le k_3\le k_2\le k_1\le n}\ \sum_{0\le l_1\le l_2\le n}\ \sum_{0\le p_3\le p_2\le p_1\le n}\ \sum_{m=0}^{n}\ \sum_{t=\mathrm{Max}\{k_2,l_1\}}^{\min\{k_2+l_1,n\}}\ \sum_{a=t}^{n}$$

$$\sum_{s=\mathrm{Max}\{p_2,m\}}^{\min\{p_2+m,n\}}\ \sum_{b=s}^{n}\ \sum_{u=\mathrm{Max}\{a,b\}}^{\min\{a+b,n\}} q^{\frac{n^2+3n}{3}}\chi_+(n,k_2,k_2,k_3)\,\chi_-(n,l_1,l_2)_{q^{-1}}$$

$$\chi_-(n,p_1,p_2,p_3)\,\chi_+(n,m)\,\Omega(n,t,k_2,l)\,\Omega(n,s,p_3,m)\,\phi(n,u,a,b)$$

$$q^{-(n-u)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{u+1})(1-q^{u+2})}$$

Theorem ($P(3,3,-1,-2) = 8_{21}$)

$$\tilde{J}^{\mathfrak{sl}_3}_{(n,0)}(P(3,3,-1,-2);q)$$

$$= \sum_{0 \le k_3 \le k_2 \le k_1 \le n} \; \sum_{0 \le l_2 \le l_1 \le l_0 \le n} \; \sum_{0 \le p \le n} \; \sum_{0 \le m_2 \le m_1 \le n} \; \sum_{t = \mathrm{Max}\{k_3, l_2\}}^{\min\{k_3 + l_2, n\}}$$

$$\sum_{a=t}^{n} \; \sum_{s = \mathrm{Max}\{p, m_2\}}^{\min\{p + m_2, n\}} \; \sum_{b=s}^{n} \; \sum_{u = \mathrm{Max}\{a, b\}}^{\min\{a+b, n\}} \left(q^{\frac{n^2+3n}{3}}\right)^{-3} \chi_1(n, k_1, k_2, k_3)$$

$$\chi_1(n, l_1, l_2, l_3)\, \chi_2(n, p)\, \iota(n, m_1, m_2)\, q^{-1}\, \Omega(n, t, k_3, l_2)$$

$$\Omega(n, s, p, m_2)\, \phi(n, u, a, b)\, q^{-(n-u)} \frac{(1 - q^{n+1})(1 - q^{n+2})}{(1 - q^{u+1})(1 - q^{u+2})}$$