3-manifold invariant derived from $\mathfrak{gl}(1|1)$-Alexander polynomial

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§ Background

connected, compact, oriented, closed

$\{3\text{-manifolds } M\} \longleftrightarrow \{\text{framed links in } S^3\} / \text{Kirby moves}$

- **Witten-Reshetikhin-Turaev 型** (WRT)

M : 3-mfd, L: surgery presentation of M. $S^3_L = M$

$\mathfrak{g}$: simple Lie alg. (finite dim)

$$\tau_r(M) := \sum_{\text{color}} \left(\prod_i q\text{-dim}\, \underbrace{V_i}_{\text{color. 表現}} \right) Q^{\text{color}}(L) \Big|_{q=\zeta_r}$$

- 半単純
- q-dim ≠ 0

- **Costantino, Geer, Patureau-Mirand (2014)** (CGP)

$\mathcal{C}$: strict ribbon Ab-category

G : abelian gp so that $\mathcal{C}$ has a G-modular str.

M : 3-mfd, Γ : $\mathcal{C}$-colored ribbon graph

ω : $H_1(M \setminus \Gamma, \mathbb{Z}) \to G$ satisfying certain condition.

(M, Γ, ω) : compatible triple

L : surgery presentation for (M, Γ, ω).

$$N(M, \Gamma, \omega) := \frac{F'(L \cup \Gamma)}{\Delta_+^p \Delta_-^s} \quad \text{is a topol. invariant.}$$

WRT CCGP

§ Viro's $\mathfrak{gl}(1|1)$-Alexander polynomial for a colored framed graph

B : a field of char. 0

G : subgp of the multiplicative gp of B

Λ : a framed trivalent graph $\subset S^3$

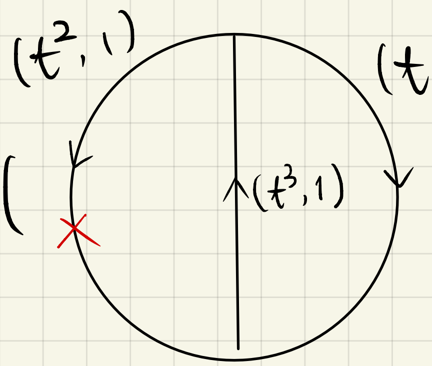
- each edge is colored by an element $(t, N) \in (G, N)$
- The coloring satisfies admissibility conditions

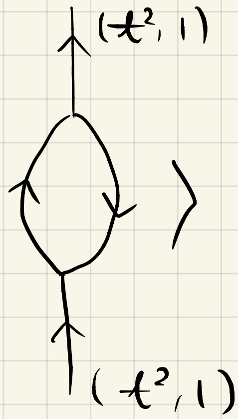
$$(t, N) \rightsquigarrow U^1\text{- module } B^{(1|1)} = B\langle e_0, e_1\rangle$$

U^1 : subalg of $U_q(\mathfrak{gl}(1|1))$

$\Longrightarrow$ $\Delta(\Lambda)$: $\mathfrak{gl}(1|1)$-Alex. polynomial

Example: $B=\mathbb{Q}(t)$. $G=\mathbb{Z}\langle t\rangle$

$\Delta\Big($ 

$\Big) = \dfrac{1}{t^4-t^{-4}} \Big\langle$ 

$\Big\rangle$

$$= \frac{1}{t^4-t^{-4}}\left(t^4-t^{-4}\right) = 1$$

View this as a morphism from $B^{(1|1)}$ to $B^{(1|1)}$, which is a scalar of Id

§ Main Result.

Compatible triple

(B, G) : as above

M : compact, connected, oriented, closed 3-manifold

Γ : colored framed graph in M.

ω : $H_1(M\setminus\Gamma, \mathbb{Z}) \longrightarrow G$. ← cohomology class

Def : (CGPの論文に類似)

(M, Γ, ω) is a compatible triple

$G \ni \omega([m_e]) =$ color of e

where e: edge of Γ, m_e: meridian of e

L: surgery presentation of M.

Def : L is computable for (M, Γ, ω) if

$G \ni \omega([m_K]) \neq 1$ ← unit of G

where $K \subset L$, m_K: meridian of K

Main Theorem (B.-Ito)

(B, G): as above, G contains $\mathbb{Z}$ but no $\mathbb{Z}/2\mathbb{Z}$ as subgp.

(M, Γ, ω) : compatible triple

L : computable surgery presentation

Then

$$\Delta(M,\Gamma,\omega) := \frac{\Delta(L\cup\Gamma)}{2^{\underbrace{r(L)}_{\text{comp}\,\#\,\text{of}\,L}}\,(-1)^{\underbrace{\sigma_+(L)}_{\#\text{ of + eigenvalues of lk matrix}}}}$$

is a topol. invariant of (M,Γ,ω)

Each component of $K\subset L$ has $\underbrace{\text{Kirby color}}_{\text{CGP のまね.}}$

$\Omega(\omega([m_K]),1)$

Kirby color :

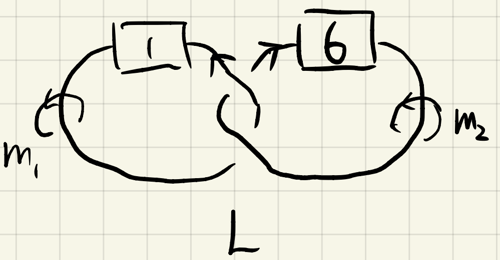
$$\Delta\Big(\underset{\Omega(t,N)}{\uparrow}\Big) = \underbrace{d(t)}_{\frac{1}{t^2-t^{-2}}}\,\Delta\Big(\underset{(t,N)}{\uparrow}\Big) - d(t)\,\Delta\Big(\underset{(t^{-1},2-N)}{\downarrow}\Big)$$

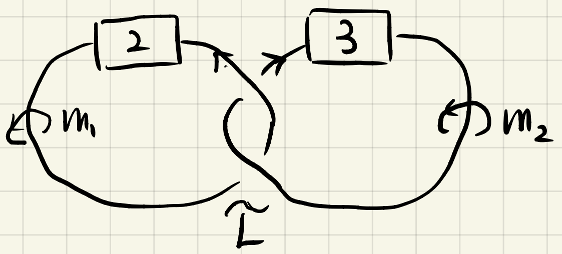
§ Example

$B=\mathbb{Q}(\pi,\xi)$ $\qquad \xi=\exp\left(\frac{2\pi\sqrt{-1}}{5}\right)$

$G=\mathbb{Z}\langle\pi,\xi\rangle\cong\mathbb{Z}\oplus\mathbb{Z}_5$ $\qquad H_1(M,\mathbb{Z})\cong H_1(\tilde{M},\mathbb{Z})\cong\mathbb{Z}/5\mathbb{Z}$

$M=L(5,1)$ $\qquad \tilde{M}=L(5,2)$ $\qquad \Gamma=\phi$





$$\omega : H_1(M, \mathbb{Z}) \to G$$

$$\begin{pmatrix} [m_1] \\ [m_2] \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$$

$\exists$ 4 non-trivial cohomology class given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underset{\omega_1}{\begin{pmatrix} \xi \\ \xi \end{pmatrix}}, \underset{\omega_2}{\begin{pmatrix} \xi^2 \\ \xi^2 \end{pmatrix}} \underset{\omega_3}{\begin{pmatrix} \xi^3 \\ \xi^3 \end{pmatrix}} \underset{\omega_4}{\begin{pmatrix} \xi^4 \\ \xi^4 \end{pmatrix}}$$

$$\Delta_i := \Delta(M, \phi, \omega_i) = -d(\xi^i)^2 \qquad d(t) = \frac{1}{t^2 - t^{-2}}$$

$$\tilde{\omega} : H_1(\tilde{M}, \mathbb{Z}) \longrightarrow G$$

$$\begin{pmatrix} [m_1] \\ [m_2] \end{pmatrix} \mapsto \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

$\exists$ 4 non-trivial cohomology classes given by

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \underset{\tilde{\omega}_1}{\begin{pmatrix} \xi \\ \xi^2 \end{pmatrix}}, \underset{\tilde{\omega}_2}{\begin{pmatrix} \xi^2 \\ \xi^4 \end{pmatrix}}, \underset{\tilde{\omega}_3}{\begin{pmatrix} \xi^3 \\ \xi \end{pmatrix}} \underset{\tilde{\omega}_4}{\begin{pmatrix} \xi^4 \\ \xi^3 \end{pmatrix}}$$

$$\tilde{\Delta}_i := \Delta(\tilde{M}, \phi, \tilde{\omega}_i) = -d(\xi^i)d(\xi^{2i})$$

$$M \cong \tilde{M} \Rightarrow \{\Delta_i\}_{i=1}^4 = \{\tilde{\Delta}_i\}_{i=1}^4$$

$$\tilde{\Delta}_i \neq \Delta_i \quad 1 \leq i \leq 4 \qquad \Rightarrow \quad M \not\cong \tilde{M}$$

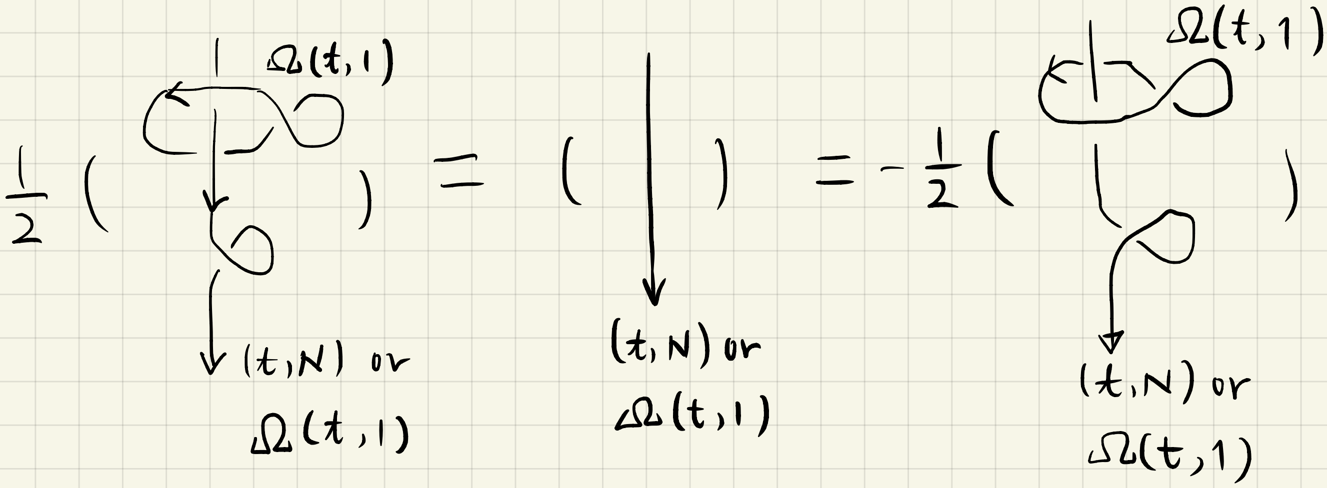
§ Sketch of proof

(M, Γ, ω) : compatible triple.

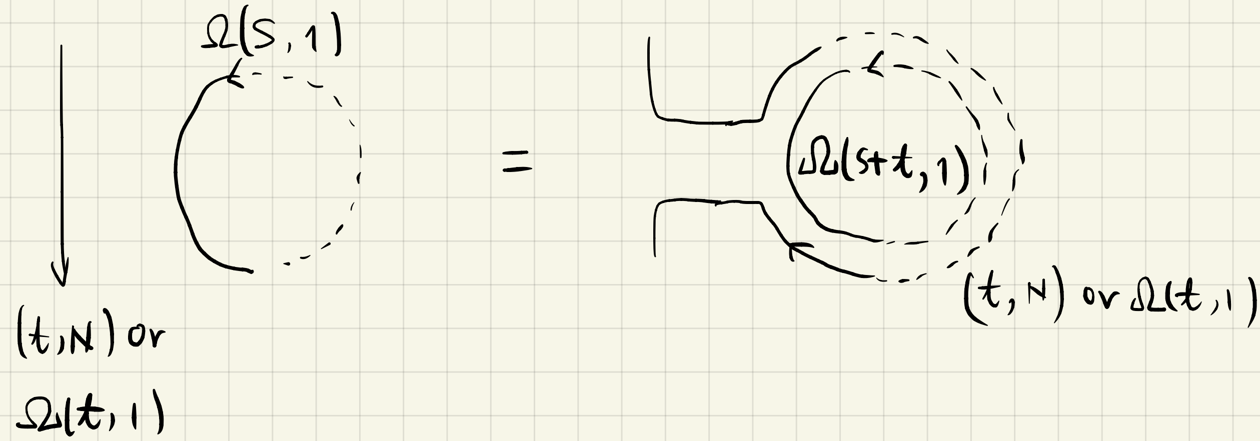
L_1, L_2 : two computable surgery presentations.

Then $L_1 \cup \Gamma$ and $L_2 \cup \Gamma$ can be connected by

1) blow up / down move



2) handle-slide move



$$\Delta(M, \Gamma, \omega) := \frac{\Delta(L \cup \Gamma)}{2^{r(L)} (-1)^{\sigma_+(L)}}$$