Multiple conjugation quandle coloring quivers of handlebody-links

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Today's contents

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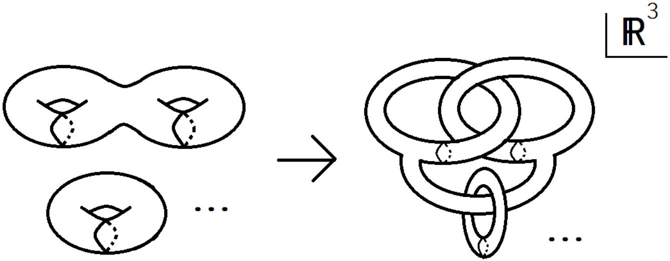
Handlebody-links

Definition (Ishii, 2008)

Handlebody-link $\Leftrightarrow$ $\mathbb{R}^3$ に埋め込まれたいくつかの handlebody.

Definition

H, H' : handlebody-links.
$H \sim H' \Leftrightarrow {}^{\exists}f : \mathbb{R}^3 \to \mathbb{R}^3$: ori-pres. homeo $\mathrm{s.t.}$ $f(H) = H'$.



Handlebody-link diagrams

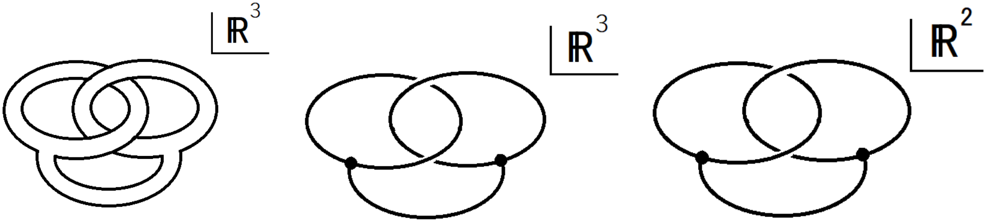
H : handlebody-link.

K : spatial trivalent graph $\Leftrightarrow$ $\mathbb{R}^3$ に埋め込まれた trivalent graph.

Definition

K は H を表す $\Leftrightarrow$ K の正則近傍が H であること.

H の *diagram* $\Leftrightarrow$ H を表す K の diagram.



H K H の diagram

Property of handlebody-links

H, H' : handlebody-links.
D, D' : H, H' の diagram.

Theorem (Ishii, 2008)

$H \sim H' \Leftrightarrow D$ と D' が有限回の $\mathrm{R1} - \mathrm{R6}$ moves で移り合う.

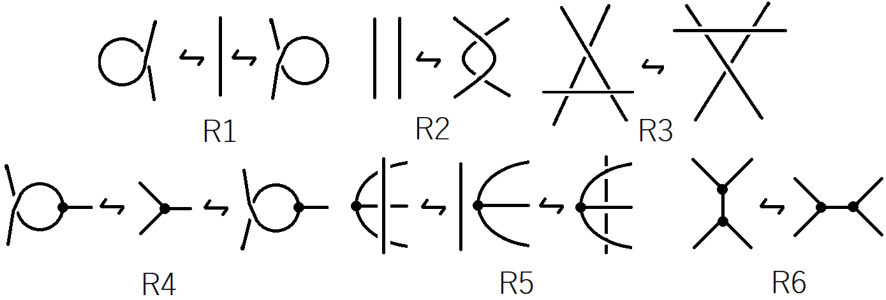


Figure: $\mathrm{R1} - \mathrm{R6}$ moves.

Quandles

Definition (Joyce, Matveev, 1982)

集合 Q と 2 項演算 $*$ の組 $(Q,*)$ が *quandle* とは次を満たすもの.

1. $^{\forall}x \in Q,\ x * x = x.$
2. $^{\forall}x, y \in Q,\ ^{\exists!}z \in Q$ s.t. $z * y = x.$
3. $^{\forall}x, y, z \in Q,\ (x * y) * z = (x * z) * (y * z).$

以後, quandle $(Q,*)$ を Q と書き, 有限なものとする.

Example

n : 自然数, Q を巡回群 $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, 演算 $*$ を $x * y = 2y - x$ とすれば quandle になる.
この quandle を位数 n の *dihedral quandle* と呼び, R_n と書く.

Quandles

Q : quandle.

$$x, y \in Q,\ n \in \mathbb{N},\ x *^n y := (\cdots(\overbrace{x * y) * y \cdots) * y}^{n}.$$

Definition

Q の *Type* $\Leftrightarrow \mathrm{Type}(Q) = \min\{n \in \mathbb{N} \mid x *^n y = x\ ({}^\forall x, y \in Q)\}$.

Q は有限なので, $\{n \in \mathbb{N} \mid x *^n y = x\ ({}^\forall x, y \in Q)\} \neq \phi$ である.

Example

R_n : dihedral quandle of order n.
$\mathrm{Type}(R_n) = 2$.
${}^\forall x, y \in R_n,\ x *^2 y = (x * y) * y = (2y - x) * y = \{2y - (2y - x)\} = x.$

Multiple conjugation quandles (MCQs)

Definition (Ishii, 2015)

$X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ は群 $G_\lambda (\lambda \in \Lambda)$ の非交和で, 2 項演算 $*$ との組 $(X, *)$ が MCQ とは次を満たすもの.

1. ${}^\forall a, b \in G_\lambda,\ a * b = b^{-1}ab.$
2. ${}^\forall x \in X,\ {}^\forall a, b \in G_\lambda,\ x * e_\lambda = x$ かつ $x * (ab) = (x * a) * b.$ ここで, e_λ は G_λ の単位元.
3. ${}^\forall x, y, z \in X,\ (x * y) * z = (x * z) * (y * z).$
4. ${}^\forall x \in X,\ {}^\forall a, b \in G_\lambda,\ (ab) * x = (a * x)(b * x).$ ここで, ある $\mu \in \Lambda$ が存在して, $a * x, b * x \in G_\mu$ を満たす.

以後, MCQ $(X, *)$ を単に X と書き, 有限なものとする.

$$f : X \to X \text{ が MCQ 自己準同型写像} \Leftrightarrow \begin{cases} {}^\forall x, y \in X,\ f(x * y) = f(x) * f(y), \\ {}^\forall \lambda \in \Lambda,\ {}^\forall a, b \in G_\lambda,\ f(ab) = f(a)f(b). \end{cases}$$

$\mathrm{End}(X) = \{X \text{ の MCQ 自己準同型写像}\}.$

Associated MCQs

Q : quandle.

Definition

$Q \times \mathbb{Z}_{\mathrm{Type(Q)}} = \bigsqcup_{x \in Q}(\{x\} \times \mathbb{Z}_{\mathrm{Type(Q)}})$: Q の *associated MCQ*
$\Leftrightarrow$
$(x, a) * (y, b) = (x *^b y, a)$ $(x, y \in Q,\ a, b \in \mathbb{Z}_{\mathrm{Type(Q)}})$,
$(x, a)(x, b) = (x, a + b)$ $(x \in Q,\ a, b \in \mathbb{Z}_{\mathrm{Type(Q)}})$.

以後, $n \in \mathbb{N}$ に対して, $X_n := R_n \times \mathbb{Z}_2 = \bigsqcup_{x \in R_n}(\{x\} \times \mathbb{Z}_2)$ とする.

Proposition

p : 奇素数,

$$\begin{aligned} \mathrm{End}(X_p) &= \{f : X_p \to X_p,\ f(x, i) = (ax + b, i) \mid a, b \in \mathbb{Z}_p\} \\ &\cup \{f : X_p \to X_p,\ f(x, i) = (c, 0) \mid c \in \mathbb{Z}_p\}. \end{aligned}$$

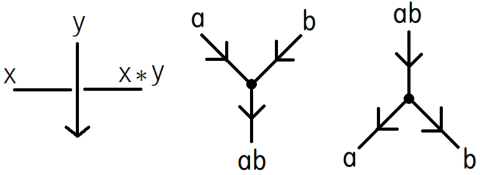
Today's contents

MCQ colorings

H : handlebody-link.
D : H の Y-oriented diagram ⇔ 各 vertex で下図のように向き付けされた diagram.
D の *arc* ⇔ D の辺を下方交叉でさらに区切ったもの. $\mathcal{A}(D) := \{D$ の arc 全体 $\}$.

Definition

$X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$: MCQ.
$c : \mathcal{A}(D) \to X$ が *X-coloring* ⇔ D の各 crossing, 各 vertex で以下の条件を満たす.
$\mathrm{Col}_X(D) = \{D$ の X-coloring $\}$.



$x, y \in X,\ a, b \in G_\lambda.$

Example of MCQ colorings

D : Y-oriented handlebody-knot diagram, X_3 : R_3 の associated MCQ. このとき, MCQ coloring は次の 18 通りである. ただし, a, b は R_3 の任意の元である.

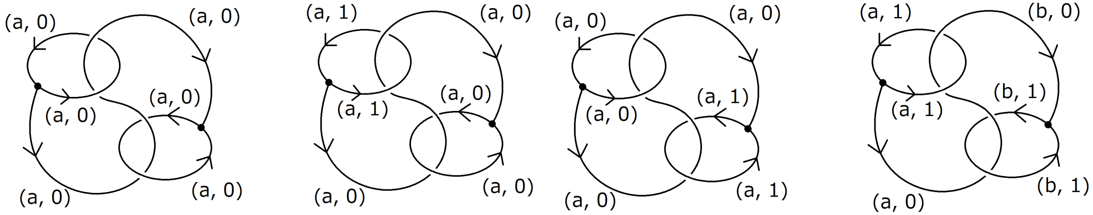


Figure: $|\mathrm{Col}_{X_3}(D)| = 18$.

Property of MCQ colorings

H, H' : handlebody-links.
D, D' : H, H' の Y-oriented diagrams.
$X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$: MCQ.

Theorem (Ishii, 2015)

$H \sim H' \Rightarrow |\mathrm{Col}_X(D)| = |\mathrm{Col}_X(D')|$.
つまり, $|\mathrm{Col}_X(D)|$ は handlebody-link の不変量である.

MCQ coloring quivers

H : handlebody-link.
D : H の Y-oriented diagram.
$X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$: MCQ.

Definition (U.)

$S \subset \mathrm{End}(X)$, $Q^{\mathrm{MCQ}}_{X,S}(D)$: *MCQ coloring quiver*
$\Leftrightarrow Q^{\mathrm{MCQ}}_{X,S}(D) = (V, E, s, t)$: quiver.

1. $V = \mathrm{Col}_X(D)$.
2. $E = \{(v, w, f) \in V \times V \times S \mid w = f \circ v\}$.
3. $s : E \to V$; $s((v, w, f)) = v$.
4. $t : E \to V$; $t((v, w, f)) = w$.

Example of MCQ coloring quivers (1)

D : Y-oriented handlebody-knot diagram, X_3 : R_3 の associated MCQ, $f(x,i) = (0,i)$,
$S = \{f\} \subset \mathrm{End}(X_3)$.

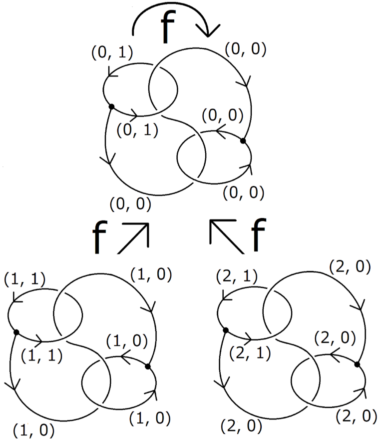


Figure: $\mathrm{Col}_{X_3}(D)$ の一部.

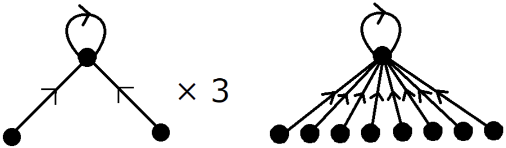


Figure: $Q^{\mathrm{MCQ}}_{X_3,S}(D)$.

Today's contents

Main results

$H,\ H'$: handlebody-links.
$D,\ D'$: $H,\ H'$ の Y-oriented diagrams.
$X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$: MCQ.

Main theorem 1 (U.)

$H \sim H' \Rightarrow {}^\forall S \subset \mathrm{End}(X),\ Q^{\mathrm{MCQ}}_{X,S}(D) \cong Q^{\mathrm{MCQ}}_{X,S}(D')$.
つまり, $Q^{\mathrm{MCQ}}_{X,S}(D)$ は handlebody-link の不変量である.

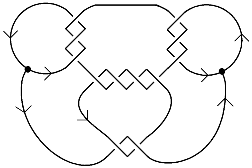
Remark

定義より, $Q^{\mathrm{MCQ}}_{X,S}(D) \cong Q^{\mathrm{MCQ}}_{X,S}(D') \Rightarrow |\mathrm{Col}_X(D)| = |\mathrm{Col}_X(D')|$.

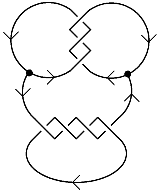
逆は真とは限らない.
つまり, $|\mathrm{Col}_X(D)| = |\mathrm{Col}_X(D')|$ であるが $Q^{\mathrm{MCQ}}_{X,S}(D) \not\cong Q^{\mathrm{MCQ}}_{X,S}(D')$ となる例が存在する.

Example of MCQ coloring quivers (2)

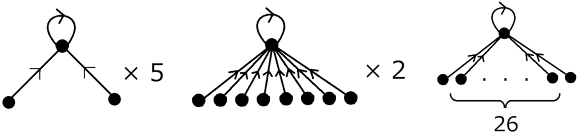
X_3 : R_3 の associated MCQ. $f(x,i) = (0,i)$, $S = \{f\} \subset \mathrm{End}(X_3)$,
$|\mathrm{Col}_{X_3}(D)| = |\mathrm{Col}_{X_3}(D')| = 60$.



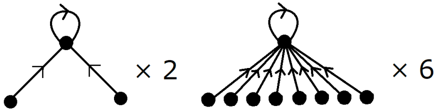
D



D'



$Q^{\mathrm{MCQ}}_{X_3,S}(D)$

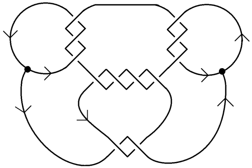


$Q^{\mathrm{MCQ}}_{X_3,S}(D')$

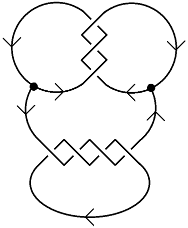
$Q^{\mathrm{MCQ}}_{X_3,S}(D) \not\cong Q^{\mathrm{MCQ}}_{X_3,S}(D')$. つまり $D \not\sim D'$. □

Example of MCQ coloring quivers (2′)

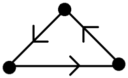
X_3 : R_3 の associated MCQ. $f'(x,i)=(x+1,i)$, $S'=\{f'\}\subset \mathrm{End}(X_3)$,
$|\mathrm{Col}_{X_3}(D)|=|\mathrm{Col}_{X_3}(D')|=60$.



D

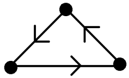


D'



× 20

$Q^{\mathrm{MCQ}}_{X_3,S'}(D)$



× 20

$Q^{\mathrm{MCQ}}_{X_3,S'}(D')$

$Q^{\mathrm{MCQ}}_{X_3,S'}(D)\cong Q^{\mathrm{MCQ}}_{X_3,S'}(D')$. S の選び方によって $D\nsim D'$ か分からない場合もある.

Main results

H, H' : 全種数が n の handlebody-links で成分数が同じもの.
D, D' : H, H' の Y-oriented diagrams, X_n : R_n の associated MCQ.

Main theorem 2 (U.)

$p \geq 2^n - 1$: 奇素数.
$|\mathrm{Col}_{X_p}(D)| = |\mathrm{Col}_{X_p}(D')|$
$\Rightarrow {}^{\forall}S \subset \mathrm{End}(X_p)$, $Q^{\mathrm{MCQ}}_{X_p,S}(D) \cong Q^{\mathrm{MCQ}}_{X_p,S}(D')$.

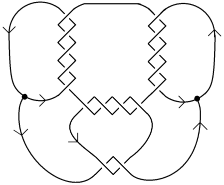
Corollary

H, H' : 種数が 2 の handlebody-knots. p : 奇素数.
$|\mathrm{Col}_{X_p}(D)| = |\mathrm{Col}_{X_p}(D')|$
$\Rightarrow {}^{\forall}S \subset \mathrm{End}(X_p)$, $Q^{\mathrm{MCQ}}_{X_p,S}(D) \cong Q^{\mathrm{MCQ}}_{X_p,S}(D')$.

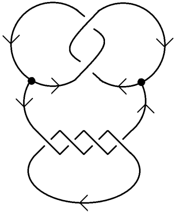
$p \geq 2^n - 1$ の仮定は本質的である. $(n = 3, p = 5)$

Example of MCQ coloring quivers (3)

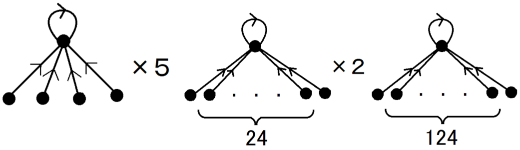
X_5 : R_5 の associated MCQ. $f(x,i)=(0,i)$, $S=\{f\}\subset \mathrm{End}(X_5)$,
$|\mathrm{Col}_{X_5}(D)| = |\mathrm{Col}_{X_5}(D')| = 200$.



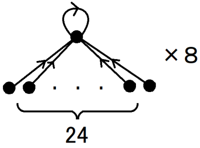
D



D'



$Q^{\mathrm{MCQ}}_{X_5,S}(D)$



$Q^{\mathrm{MCQ}}_{X_5,S}(D')$

$Q^{\mathrm{MCQ}}_{X_5,S}(D) \not\cong Q^{\mathrm{MCQ}}_{X_5,S}(D')$.

Thank you for your attention.