The Long-Moody construction and twisted Alexander invariants

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Introduction

B_n : braid group, R : commutative ring

Question : Can we (re)construct a link invariant from a given representation $\rho : B_n \longrightarrow GL_k(R)$?

[Burau, 1935] reduced Burau representation
$\rightsquigarrow$ Alexander polynomial

[Birman, 1974] reduced Gassner representation
$\rightsquigarrow$ multivariable Alexander polynomial

[Conway, 2018] reduced twisted Burau map
$\rightsquigarrow$ twisted Alexander invariant

Today

[T., 2021] reduced Long-Moody construction

$\rho : B_n \ltimes F_n \longrightarrow GL_k(R)$: representation

$\rightsquigarrow \widetilde{LM}(\rho) : B_n \longrightarrow GL_{(n-1)k}(R)$

$\rightsquigarrow$ twisted Alexander invariant $\Delta_{K,\rho}(t)$

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1. Twisted Alexander invariants
2. The Long-Moody construction
3. A relation between $\Delta_{K,\rho}(t)$ and $LM(\rho)$
4. Examples

1. Twisted Alexander invariants [Wada, 1994]

K : knot (or link) in S^3

$G(K) := \pi_1(S^3 \setminus K) \cong \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle$: fixed

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$
$$\frac{\partial (g g')}{\partial x_j} = \frac{\partial g}{\partial x_j} + g \frac{\partial g'}{\partial x_j}$$

$\alpha : G(K) \twoheadrightarrow \mathbb{Z} \cong \langle t \rangle$; ${}^\forall$meridian $\longmapsto t$

$\rho : G(K) \longrightarrow GL_k(R)$: representation (R : UFD)

$\phi : F_n = \langle x_1, \cdots, x_n \rangle \twoheadrightarrow G(K)$: quotient map

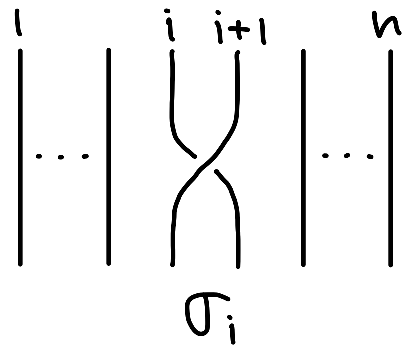
$$M := \left((\rho \otimes \alpha) \circ \phi \left(\frac{\partial r_i}{\partial x_j} \right) \right)_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n}} : \text{Alexander matrix}$$

M_j : $(n-1) \times (n-1)$ matrix obtained from M by removing the j-th column.

$$\Delta_{K,\rho}(t) := \frac{\det M_j}{\det \left((\rho \otimes \alpha) \circ \phi (x_j - 1) \right)} : \text{twisted Alexander invariant of } K \text{ associated with } \rho$$

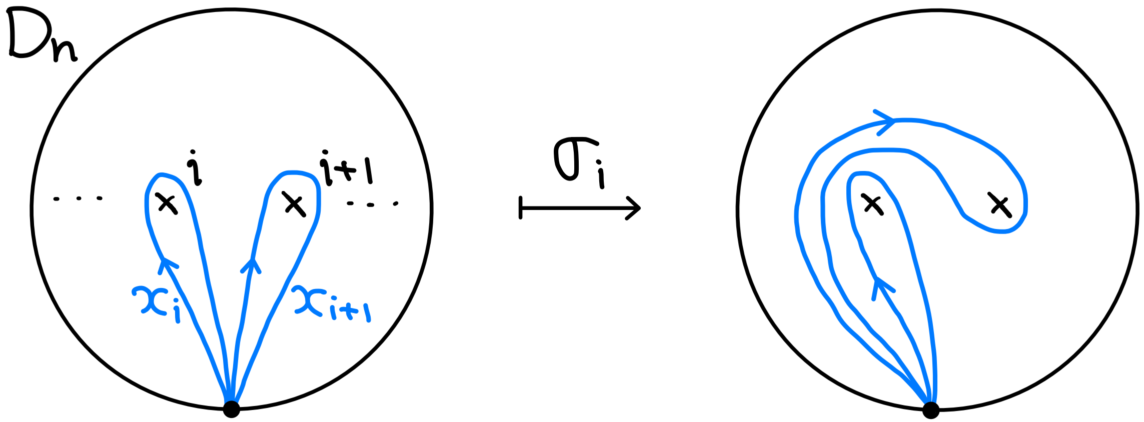
2 The Long-Moody construction

$$B_n \cong \left\langle \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (1 \leq i \leq n-2) \\ \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \geq 2) \end{array} \right\rangle$$



D_n : n-punctured disk, $\pi_1(D_n) \cong F_n$

$$F_n \curvearrowleft B_n : x_j \cdot \sigma_i = \begin{cases} x_i x_{i+1} x_i^{-1} & (j = i) \\ x_i & (j = i+1) \\ x_j & (j \neq i, i+1) \end{cases}$$



Fact $b \in B_n$, $\hat{b}$: its closure

$$\pi_1(S^3 \setminus \hat{b}) \cong \langle x_1, \cdots, x_n \mid x_1 = x_1 \cdot b, \cdots, x_n = x_n \cdot b \rangle$$

$\rho : B_n \ltimes F_n \longrightarrow GL_k(R)$: representation

Notation : $v = (v_1, \cdots, v_k) \in R^{\oplus k}$

$\rightsquigarrow$ For $^{\forall} x \in B_n \ltimes F_n$, $v \longmapsto v\rho(x)$

$$\mathcal{I}_{F_n} := \mathrm{Ker}\left(\begin{array}{ccc} R[F_n] & \longrightarrow & R \\ x & \longmapsto & 1 \end{array}\right) : \text{augmentation ideal}$$

Def [Long-Moody, 1994]

The Long-Moody construction of ρ is a representation $LM(\rho) : B_n \longrightarrow \mathrm{Aut}_R(R^{\oplus k} \otimes_{R[F_n]} \mathcal{I}_{F_n})$ given by

$$LM(\rho)(b)(v \otimes h) := v\rho(b) \otimes h \cdot b$$

for any $b \in B_n$, $v \in R^{\oplus k}$, $h \in \mathcal{I}_{F_n}$.

Remark Adding one variable t:

Consider the 1-dim. rep. $t^{\ell} : B_n \ltimes F_n \longrightarrow GL_1(R[t^{\pm 1}])$ $(\ell \in \mathbb{Z})$

$$\sigma_i,\ x_j \longmapsto \cdot\, t^{\ell}$$

and define $t^{\ell}\rho := t^{\ell} \otimes \rho : B_n \ltimes F_n \longrightarrow GL_k(R[t^{\pm 1}])$.

$$\rightsquigarrow\ t^{-1}LM(t\rho) := t^{-1} \otimes LM(t\rho) : B_n \longrightarrow \mathrm{Aut}_{R[t^{\pm 1}]}\left(R[t^{\pm 1}]^{\oplus k} \otimes_{R[F_n]} \mathcal{I}_{F_n}\right)$$

$t\rho$

$$\mathcal{I}_{F_n} \cong R[F_n]^{\oplus n} \cong \langle x_1 - 1, \cdots, x_n - 1 \rangle_{R[F_n]}$$

$$\therefore\ LM(\rho) : B_n \longrightarrow \mathrm{Aut}_R\left(R^{\oplus k} \otimes_{R[F_n]} \mathcal{I}_{F_n}\right) \cong GL_{nk}(R)$$

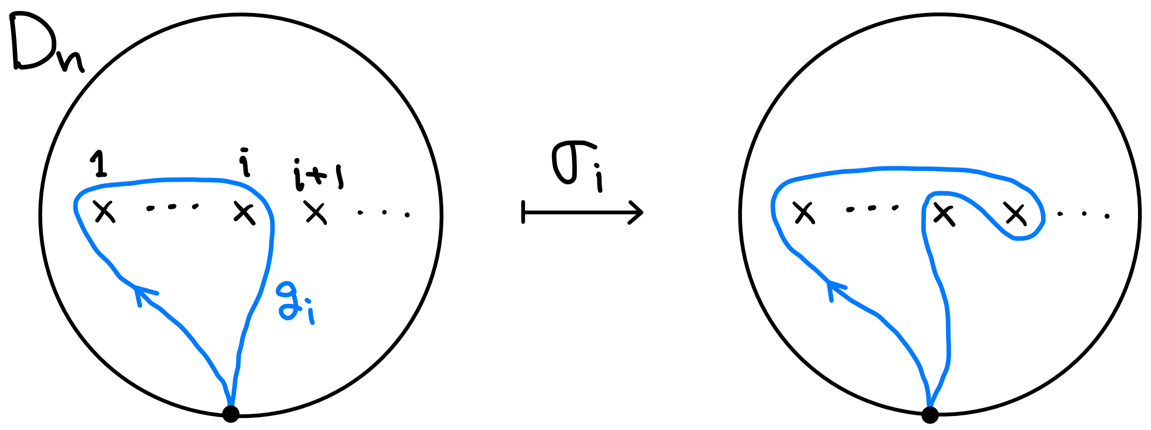
ρ

Example Let $\rho : B_n \ltimes F_n \longrightarrow GL_1(\mathbb{Z})$ be the trivial rep.
$t^{-1}LM(t\rho) : B_n \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}])$ is equivalent to the unreduced Burau representation.

3. A relation between $\Delta_{K,\rho}(t)$ and $LM(\rho)$

Regard F_n as being generated by $g_1, \cdots, g_n$,
where $g_i := x_1 \cdots x_i$.

$$F_n \curvearrowleft B_n : g_j \cdot \sigma_i = \begin{cases} g_j & (j \neq i) \\ g_{i+1} g_i g_{i-1}^{-1} & (j = i \neq 1) \\ g_2 g_1^{-1} & (j = i = 1) \end{cases}$$



In particular, the action on g_n is trivial.
I_{F_n} is also generated by $g_1 - 1, \cdots, g_n - 1$.

Matrix presentation of $LM(\rho)$

By the fundamental formula of the Fox derivation,

$$w-1=\sum_{j=1}^{n}\frac{\partial w}{\partial g_j}(g_j-1)$$

for any $w\in\mathbb{Z}[F_n]$. Hence

$$\begin{aligned}LM(\rho)(b)(v\otimes(g_i-1)) &= v\rho(b)\otimes(g_i-1)\cdot b\\ &= v\rho(b)\otimes\sum_{j=1}^{n}\frac{\partial(g_i\cdot b)}{\partial g_j}(g_j-1)=\sum_{j=1}^{n}v\rho(b)\rho\left(\frac{\partial(g_i\cdot b)}{\partial g_j}\right)\otimes(g_j-1).\end{aligned}$$

Thus we obtain a matrix presentation with respect to the basis $\{g_i-1\mid 1\le i\le n\}$:

$$LM(\rho)(b)=\mathrm{Diag}(\underbrace{\rho(b),\cdots,\rho(b)}_{n})\cdot\begin{pmatrix}\left(\rho\left(\frac{\partial(g_i\cdot b)}{\partial g_j}\right)\right)_{1\le i,j\le n-1} & * \\ 0 & I_k\end{pmatrix}$$

Def $F_{n-1} := \langle g_1, \cdots, g_{n-1} \rangle < F_n$

The reduced Long-Moody construction of ρ is a representation $\widetilde{LM}(\rho) : B_n \longrightarrow \mathrm{Aut}_R(R^{\oplus k} \otimes_{R[F_{n-1}]} \mathcal{I}_{F_{n-1}})$ given by

$$\widetilde{LM}(\rho)(b)(v \otimes h) := v\rho(b) \otimes h \cdot b$$

for any $b \in B_n$, $v \in R^{\oplus k}$, $h \in \mathcal{I}_{F_{n-1}}$.

Its matrix presentation with respect to the basis $\{g_i - 1 \mid 1 \leq i \leq n-1\}$ is

$$\widetilde{LM}(\rho)(b) = \mathrm{Diag}(\underbrace{\rho(b), \cdots, \rho(b)}_{n-1}) \cdot \left(\rho\left(\frac{\partial(g_i \cdot b)}{\partial g_j}\right)\right)_{1 \leq i,j \leq n-1}$$

Remark t-added version:

$$t^{-1}\widetilde{LM}(t\rho)(b) = \mathrm{Diag}(\rho(b), \cdots, \rho(b)) \cdot \left(t\rho\left(\frac{\partial(g_i \cdot b)}{\partial g_j}\right)\right)_{1 \leq i,j \leq n-1}$$

<u>Thm</u> [T., 2021]

$b \in B_n$: fixed, $\hat{b}$: its closure

Suppose that $\rho|_{F_n}$ factors through ϕ and a representation $G(\hat{b}) \longrightarrow GL_k(R)$:

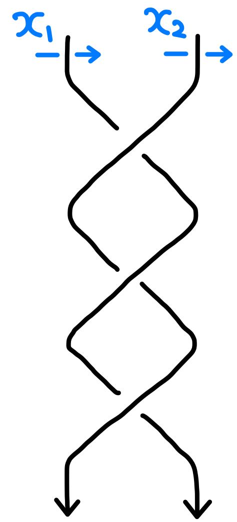
$$\begin{tikzcd} F_n \arrow[r, "\rho|_{F_n}"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & GL_k(R) \\ G(\hat{b}) \arrow[ur] & \end{tikzcd}$$

Then

$$\Delta_{\hat{b},\rho}(t) \det(\rho(x_1 \cdots x_n)t^n - I_k)$$
$$\doteq \det\left(t^{-1} \widetilde{LM}(t\rho)(b) - \mathrm{Diag}(\rho(b), \cdots, \rho(b))\right)$$

4. Examples

$b = \sigma_1^3 \rightsquigarrow \hat{b}$: trefoil knot



$$G(\hat{b}) \cong \langle x_1, x_2 \mid x_1x_2x_1 = x_2x_1x_2 \rangle \cong B_3$$
$$\cong \langle g_1, g_2 \mid g_1 = g_1 \cdot b = g_2^2 g_1^{-1} g_2^{-1} \rangle$$

$\rho : G(\hat{b}) \longrightarrow GL_2(\mathbb{Z}[s^{\pm 1}])$: reduced Burau rep.

$$\rho(x_1) = \begin{pmatrix} -s & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 1 & 0 \\ s & -s \end{pmatrix}$$

$$B_2 \ltimes F_2 \cong \langle x_1, x_2, \sigma_1 \mid x_1\sigma_1 = \sigma_1 x_1 x_2 x_1^{-1}, \ x_2\sigma_1 = \sigma_1 x_1 \rangle$$

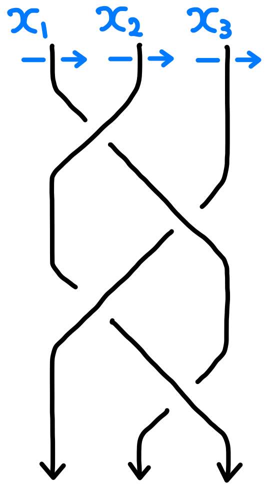
$$\rho(\sigma_1) := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \rightsquigarrow \rho : B_2 \ltimes F_2 \longrightarrow GL_2(\mathbb{Z}[t^{\pm 1}])$$

is well-defined.

$$\therefore t^{-1}\widetilde{LM}(t\rho)(b) = \begin{pmatrix} st^3 & -st^3 + s^2t^3 \\ st^3 & -st^3 \end{pmatrix}$$

$\therefore \Delta_{\hat{b},\rho}(t) = 1 - st^2$ [Wada, 1994]

$b = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \rightsquigarrow \hat{b}$: figure eight knot



$$G(\hat{b}) \cong \langle x_2, x_3 \mid x_2[x_3^{-1}, x_2] = [x_3^{-1}, x_2]x_3 \rangle$$
$$\cong \langle g_1, g_2, g_3 \mid g_1 = g_1 \cdot b,\ g_2 = g_2 \cdot b \rangle$$

Define a rep. $\rho : G(\hat{b}) \longrightarrow SL_2(\mathbb{C})$ given by

$$\rho(x_1) = \frac{1}{3}\begin{pmatrix} -s-2 & 3s \\ -2s^2 & s-1 \end{pmatrix},\quad \rho(x_2) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix},\quad \rho(x_3) = \frac{1}{3}\begin{pmatrix} -s-2 & 3 \\ -2 & s-1 \end{pmatrix}.$$

$$(s^2 + s + 1 = 0)$$

$$B_3 \ltimes F_3 \cong \left\langle \begin{matrix} x_1, x_2, x_3, \\ \sigma_1, \sigma_2 \end{matrix} \;\middle|\; \begin{matrix} x_1\sigma_1 = \sigma_1 x_1 x_2 x_1^{-1},\ x_2\sigma_1 = \sigma_1 x_1,\ x_3\sigma_1 = \sigma_1 x_3 \\ x_1\sigma_2 = \sigma_2 x_1,\ x_2\sigma_2 = \sigma_2 x_2 x_3 x_2^{-1},\ x_3\sigma_2 = \sigma_2 x_2 \\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \end{matrix} \right\rangle$$

$$\rho(\sigma_1) := \sqrt{\frac{-s}{3}}\begin{pmatrix} 1 & s+2 \\ \frac{-2(s+2)}{3} & -s^2 \end{pmatrix},\quad \rho(\sigma_2) := \sqrt{\frac{-s}{3}}\begin{pmatrix} 1 & s-1 \\ \frac{2(2s+1)}{3} & -s^2 \end{pmatrix}$$

$\rightsquigarrow$ $\rho : B_3 \ltimes F_3 \longrightarrow SL_2(\mathbb{C})$ is well-defined.

$\therefore\ \Delta_{\hat{b},\rho}(t) = (1+t)^2$

Future works

- Find other representations of $G(\hat{b})$ extended to $B_n \ltimes F_n$.

Fact $\hat{b} = 4_1, 5_2$ or 6_1.
The holonomy representation $\rho : G(\hat{b}) \longrightarrow SL_2(\mathbb{C})$ is not extended to $B_n \ltimes F_n$.

- Let $\rho : B_n \longrightarrow GL_k(R)$ be a representation which constructs a link invariant.

  Q1. Construct a link invariant from $LM(\rho)$.

  Q2. Find some relations between these two invariants.

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