ルジャンドル結び目とラック彩色数

木村　直記（早稲田大学）

結び目の数理 IV

2021 年 12 月 24 日

Plan of talk

1. Legendrian knots
2. Bi-Legendrian rack
3. Distinguishing Legendrian knots by bL-rack coloring numbers

Definition

M : a 3-manifold.
A contact structure ξ on M is a plane field on M satisfying, when $\xi = \ker\alpha$ for a local 1-form α on M,
$\alpha \wedge d\alpha$ is nowhere zero.
Then (M, ξ) is called a contact 3-manifold.

Example

(x, y, z) : a global coordinate on $\mathbb{R}^3$.
$\alpha_{std} := dz + xdy$.
$\xi_{std} := \ker\alpha_{std} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \rangle_{\mathbb{R}}$.
Then ξ_{std} is a contact structure on $\mathbb{R}^3$.
ξ_{std} is called the standard contact structure on $\mathbb{R}^3$.

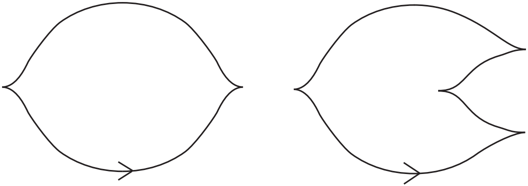
Definition

(M, ξ) : a contact 3-manifold.
A smooth knot K in (M, ξ) is called Legendrian if $T_pK \subset \xi_p$ for any $p \in K$.

Definition

K_0, K_1 : Legendrian knots in a contact 3-manifold (M, ξ). K_0 is said to be Legendrian isotopic to K_1 if there exists an isotopy φ_t of M $(t \in [0, 1])$ such that φ_0 is id_M, $\varphi_1(K_0)$ is K_1 and $\varphi_t(K_0)$ is Legendrian for any $t \in [0, 1]$.

Legendrian isotopic is more strict equivalence relation than ambient isotopic. Actually, two Legendrian unknots below are not Legendrian isotopic.



We consider the classification of Legendrian knots up to Legendrian isotopy.

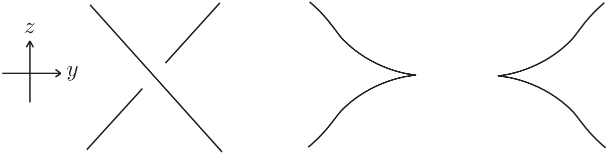
We only consider Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.

$$\mathbb{R}^3 \ni (x, y, z) \mapsto (y, z) \in \mathbb{R}^2 : \text{the front projection.}$$

Front diagrams of Legendrian knots have the following features.
$\gamma(t) = (x(t), y(t), z(t))$: a parametrization of a Legendrian knot K.
Since $\alpha_{std}(\gamma'(t)) = 0$,

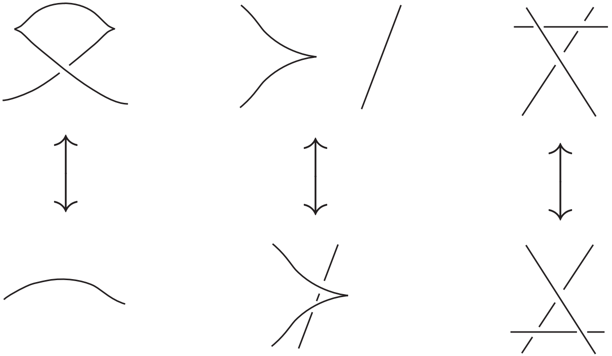
$$z'(t) + x(t)y'(t) = 0.$$

(1) Each point $(y(t), z(t))$ in $\mathbb{R}^2$ with $y'(t) = 0$ is a singular point called a cusp.
(2) Due to $x(t) = -\frac{dz}{dy}(t)$, at each crossing the slope of the over crossing is smaller than that of the under crossing.



Theorem

K_0, K_1 : Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.
D_i : the front diagram of K_i $(i = 1, 2)$.
Then K_0 and K_1 are Legendrian isotopic if and only if D_0 and D_1 are related by a finite sequence of the following three types of local moves.



The moves are called the Legendrian Reidemeister moves.

The classical invariants of Legendrian knots

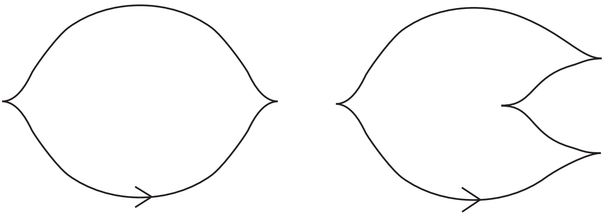
(1) the Thurston-Bennequin number $tb(K) \in \mathbb{Z}$
(an invariant of unoriented Legendrian knots)
(2) the rotation number $rot(K) \in \mathbb{Z}$
(an invariant of oriented Legendrian knots)

$$tb(K) = w(D) - \frac{1}{2}c(D),$$
$$rot(K) = \frac{1}{2}(dc(D) - uc(D)),$$

where $w(D)$: the writhe of D,
$c(D)$: the number of the cusps of D,
$dc(D)$: the number of the downward cusps of D,
$uc(D)$: the number of the upward cusps of D.

Note that $rot(-K) = -rot(K)$,
where $-K$ is the same knot as K with the reverse orientation.

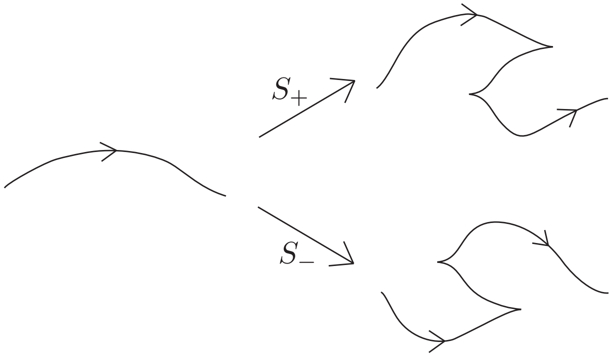
U_0



$tb = -1$, $rot = 0$ $\qquad$ $tb = -2$, $rot = -1$

These two Legendrian unknots are not Legendrian isotopic.

The classical invariants are rather strong.
In fact, the pair of the classical invariants completely classifies the Legendrian isotopy classes of the unknot, those of each torus knot and those of the figure eight knot.



A positive (or negative) stabilization S_+ (or S_-) is an operation represented by adding two downward (or upward) cusps to a trivial arc for the front diagram. $S_\pm$ changes the Legendrian isotopy class and does not change the knot type. Note that $S_+S_- = S_-S_+$.

$$tb(S_\pm(K)) = tb(K) - 1,$$
$$rot(S_\pm(K)) = rot(K) \pm 1.$$

Recall the definitions of a rack and a quandle.

Definition

$(X, *)$ is called a rack if X is a set with a binary operation $*$ satisfying the following conditions for all $x, y, z \in X$:

$$*x \text{ is a bijection on } X,$$
$$(x * y) * z = (x * z) * (y * z).$$

A rack which satisfies $x * x = x$ for all $x \in X$ is called a quandle.

We denote the inverse map of $*x$ by $\bar{*}x$.

Since the axioms of a quandle correspond to the Reidemeister moves, quandle coloring numbers of diagrams are invariants of topological knots.

Definition (K)

$(X, *, f, g)$ is called a bi-Legendrian rack if $(X, *)$ is a rack and f and g are maps on X satisfying the following conditions for all $x, y \in X$:

$$\begin{aligned} f \circ g &= g \circ f, \\ fg(x * x) &= x, \\ f(x * y) &= f(x) * y, \\ g(x * y) &= g(x) * y, \\ x * f(y) &= x * y, \\ x * g(y) &= x * y. \end{aligned}$$

Remark

$(X, *, f, g)$: a bi-Legendrian rack.
Then $(X, *, f, g)$ is a bi-Legendrian quandle if and only if g is the inverse map of f.

Example

$(G, *)$: a conjugation quandle,
i.e. G is a group and $x * y = y^{-1}xy$ for $x, y \in G$.
Take $z \in Z(G)$ and define $f(x) := zx$.
Then $(G, *, f, f^{-1})$ is a bi-Legendrian quandle.

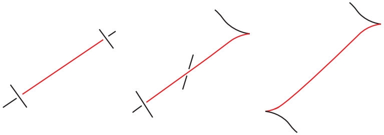
Example

X : a set.
f, g : bijections on X such that they are commutative.
Define

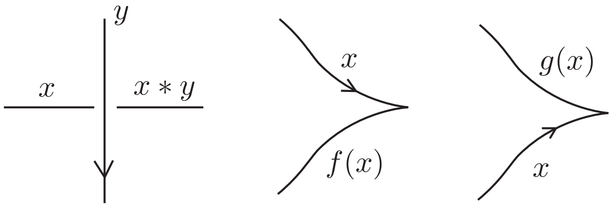
$$x * y := (f \circ g)^{-1}(x).$$

Then $(X, *, f, g)$ is a constant bi-Legendrian rack.

In this talk, an arc means a part of a diagram which does not have cusps in the interior of it.



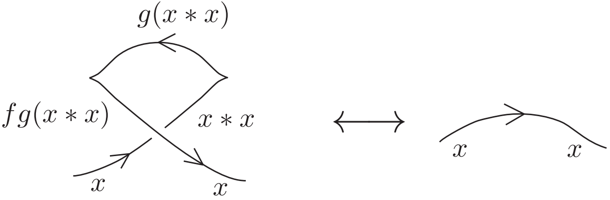
Relations at crossings and cusps are settled as below.



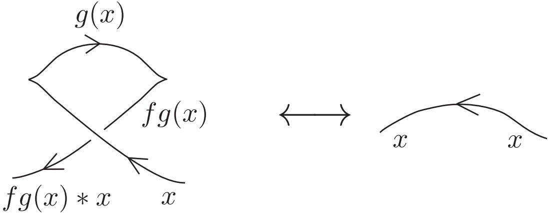
The axioms of a bL-rack correspond to the Legendrian Reidemeister moves.

Proposition (K)

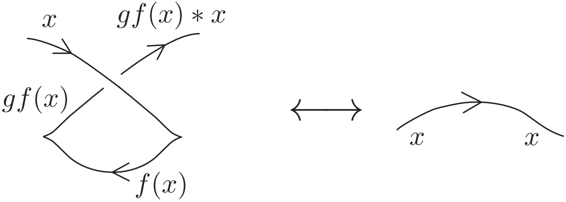
K : a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$.
D : the front diagram of K.
$(X, *, f, g)$: a bi-Legendrian rack.
Then the number of the $(X, *, f, g)$-colorings of D is invariant under the Legendrian Reidemeister moves. Namely, the number of the $(X, *, f, g)$-colorings of D is an invariant of a Legendrian knot K, denoted by $\#\mathrm{Col}(K, X)$.



$g(x * x)$
$fg(x * x)$
$x * x$
x
x
x
x



$g(x)$
$fg(x)$
$fg(x) * x$
x
x
x



x
$gf(x) * x$
$gf(x)$
$f(x)$
x
x

1. Legendrian knots

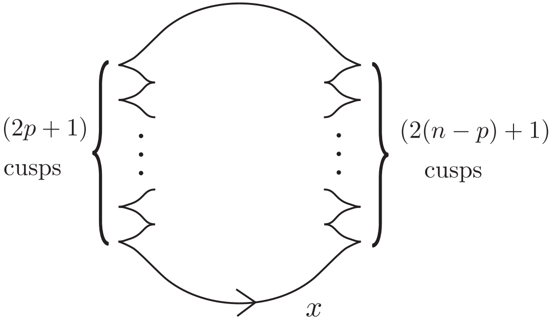
2. Bi-Legendrian rack

3. Distinguishing Legendrian knots by bL-rack coloring numbers

Theorem (K)

For any $n \in \mathbb{Z}_{\geq 0}$, there exists a bi-Legendrian rack $(X_n, *, f, g)$ such that all Legendrian unknots K with $tb(K) = -1 - n$ are distinguished by $\#\mathrm{Col}(K, X_n)$.

From Bennequin's ineq. and Eliashberg-Fraser's results, any Legendrian unknot is Legendrian isotopic to $K_{p,n}$ for some $p, n \in \mathbb{Z}_{\geq 0}$ $(0 \leq p \leq n)$ below.



$K_{p,n} = S_+^p S_-^{n-p}(U_0)$, $tb(K_{p,n}) = -1-n$, $rot(K_{p,n}) = 2p-n$.

The previous theorem states that bL-rack coloring numbers can distinguish some Legendrian knots with the same tb.

A natural question to consider next is whether or not bL-rack coloring numbers can distinguish pairs of Legendrian knots with the same tb and the same rot.

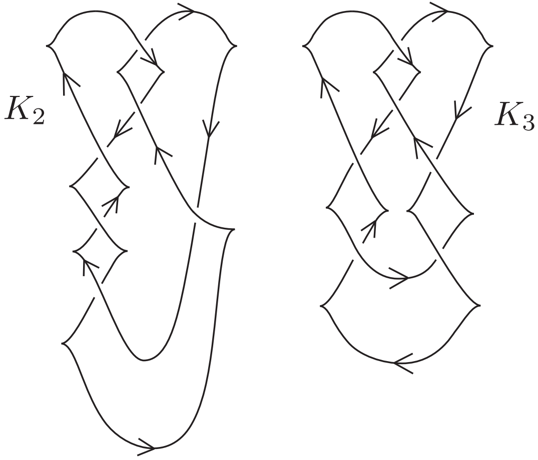
Although I do not have a complete answer, the following theorems give a partial answer to the question.

Theorem (K)

K_0, K_1 : Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.
If K_0 and K_1 are of the same knot type, $tb(K_0) = tb(K_1)$ and $rot(K_0) = rot(K_1)$,
then $\#\mathrm{Col}(K_0, X) = \#\mathrm{Col}(K_1, X)$ for any bi-Legendrian quandle $(X, *, f, g)$.

Theorem (K)

K_2, K_3 : the Chekanov knots.
Then $\#\mathrm{Col}(K_2, X) = \#\mathrm{Col}(K_3, X)$ for any finite bi-Legendrian rack $(X, *, f, g)$.



K_2 and K_3 are of the same knot type 5_2,
$tb(K_2) = tb(K_3) = 1$ and $rot(K_2) = rot(K_3) = 0$.
However, Chekanov proved K_2 and K_3 are not Legendrian isotopic. K_2 and K_3 are called the Chekanov knots.