A virtualized skein relation for a multivariable polynomial invariant

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Introduction

The multivariable polynomial invariant is a refinement of the Jones polynomial, which is an invariant of virtual links introduced by Dye, Kauffman and Miyazawa.

On the other hand, a virtualized skein relation for a Jones polynomial was introduced by N.Kamada, Nakabo and Satoh.

Theorem [N.Kamada, Nakabo, Satoh]

(D_+, D_-, D_v) : virtual skein triple
V_D: Jones polynomial of D

D_+, D_-: checkerboard colorable virtual link diagram
$\Rightarrow A^3 V_{D_+} + A^{-3} V_{D_-} = (A^3 + A^{-3}) V_{D_v}$

In this talk, we discuss a virtualized skein relation for multivariable polynomial invariants.

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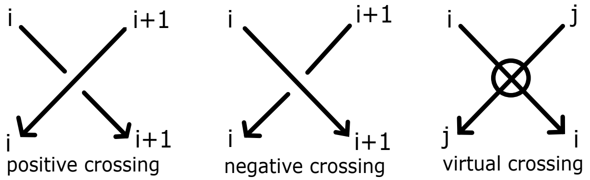
Alexander numbering and almost classical

D : virtual link diagram

A **semi-arc** of D := an arc of D between two classical crossings or loop without classical crossing of D

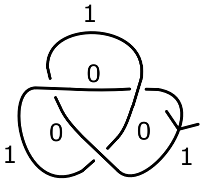
Alexander numbering of D := an assignment of a number of $\mathbb{Z}$ to each semi-arc of D depicted as in figure for crossings.

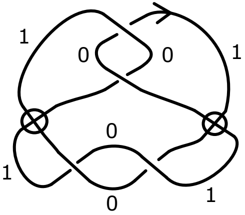


D is **almost classical** $\Leftrightarrow$ D admit an Alexander numbering

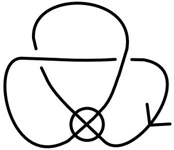
A virtual link L is **almost classical** $\Leftrightarrow$ L has an almost classical virtual link diagram

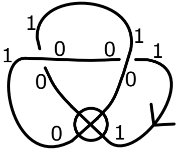
· Example of almost classical virtual link diagram





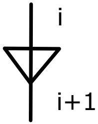
A classical link diagram is always an almost classical virtual link diagram.
Not every virtual link diagram is an almost classical virtual link diagram.



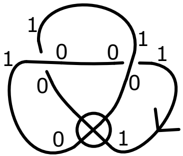


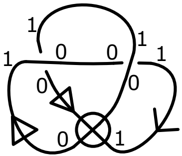
cut system (1/2)

An **oriented cut point** := a point on a semi-arc with an orientation of D and Alexander numbering increses by one on the direction.



A **cut system** C of D := a set of oriented cut points s.t. D admit Alexander numbering with C.



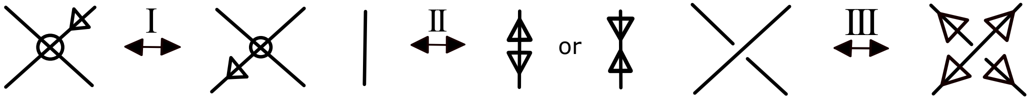


Remark

Note that an $\emptyset$ set is a cut system of almost classical virtual link diagram.

cut system (2/2)

The local transformations of oriented cut points as below are called **oriented cut point moves**.



Theorem [N.Kamada]

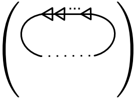
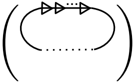
Two cut systems of the same virtual limk diagram are related by a finite sequence of oriented cut point moves.

multivariable polynomial invariant (1/3)

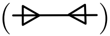
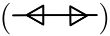
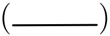
(D, C) : a pair of a virtual link diagram and a cut system

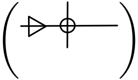
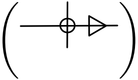
A **cut point state diagram** of (D, C), σ^c of D
$:=$ a virtual link diagram with oriented cut points obtained by splicing all classical crossings of D.

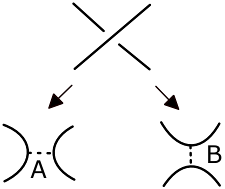
A map ι : { loops of a cut point state diagram } $\to \mathbb{Z}$.

(i) ι  $= \iota$  $=$ r ,

where 2r oriented cut points in the same direction appear.

(ii) ι  $= \iota$  $= \iota$ 

(iii) ι  $= \iota$ 



multivariable polynomial invariant (2/3)

S : A set of all cut point state diagrams of (D, C)

$$\langle\langle D, C\rangle\rangle := \sum_{\sigma^c \in S} A^{\natural\sigma^c}(-A^2 - A^{-2})^{\sharp\sigma^c - 1} d_1^{\tau_1(\sigma^c)} d_2^{\tau_2(\sigma^c)} \cdots \in \mathbb{Z}[A^{\pm 1}, d_1, d_2 \cdots]$$

$\natural\sigma^c := \sharp$(A splice obtaining σ^c) $- \sharp$(B splice obtaining σ^c)
$\sharp\sigma^c := \sharp$(loops in σ^c)
$\tau_i(\sigma^c) := \sharp$($\sigma^c$ whose indices by ι are i)

Proposition [N.Kamada]

$\langle\langle D, C\rangle\rangle$ does not depend on a choice of C

$\langle\langle D\rangle\rangle := \langle\langle D, C\rangle\rangle$

・multivariable polynomial invariant (3/3)

$$X_D := (-A^3)^{-w(D)} \langle\langle D \rangle\rangle$$

$w(D) = \sharp$(positive crossings of D) $- \sharp$(negative crossings of D)

Theorem [N.Kamada]

X_D coincides with the multivariable polynomial invariant defined by Dye, Kauffman and Miyazawa.

Proposition [Nakamura, Nakanishi, Satoh]

D : an almost classical virtual link diagram
$\Rightarrow X_D \in \mathbb{Z}[A^{\pm 1}]$

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Main result

A **virtual skein triple** (D_+, D_-, D_v)
: a triple of virtual link diagrams
s.t. D_- is obtained from D_+ by crossing change at a positive crossing p,
and D_v is obtained from D_+ by replacing p with a virtual crossing.

Main result

(D_+, D_-, D_v) : virtual skein triple

D_+, D_- : almost classical virtual link diagrams

$\Rightarrow (A^6 - d_1)X_{D_+} + (-A^{-6} + d_1)X_{D_-} = (A^6 - A^{-6})X_{D_v}$

$S' \subset S$: A subset of cut point state diagrams S of (D, C)

$$\langle\langle D|S'\rangle\rangle := \sum_{\sigma^c \in S'} A^{\natural\sigma^c}(-A^2 - A^{-2})^{\sharp\sigma^c - 1} d_1^{\tau_1(\sigma^c)} d_2^{\tau_2(\sigma^c)} \cdots \in \mathbb{Z}[A^{\pm 1}, d_1, d_2 \cdots]$$

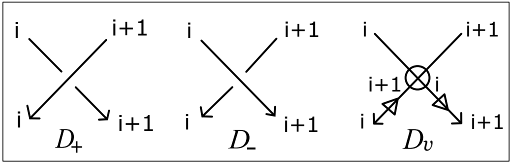
Outline of proof (1/2)

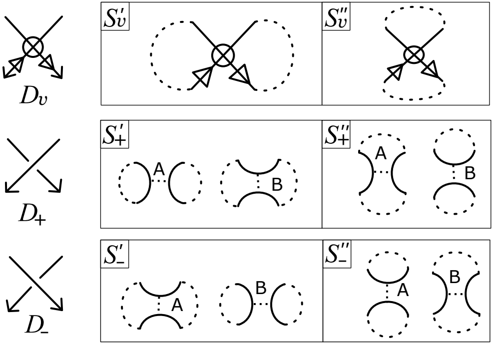
D_+, D_- : almost classical diagram
The cut system C of D_+, D_- : $\emptyset$ set
S_+, S_- and S_v: the sets of all states of D_+, D_- and D_v

Remark

D : an almost classical diagram
$\Rightarrow D$: checkerboard colorable diagram

$S_v = S'_v \sqcup S''_v$
S'_ε and S''_ε : the subsets of S_ε as in the figure, where $\varepsilon = \pm$.



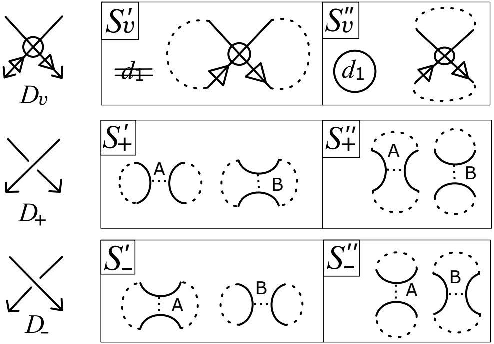


Outline of proof (2/2)

$$\langle\langle D_+|S'_+\rangle\rangle = A(-A^2-A^{-2})\langle\langle D_v|S'_v\rangle\rangle + A^{-1}\langle\langle D_v|S'_v\rangle\rangle = -A^3\langle\langle D_v|S'_v\rangle\rangle$$

$$\langle\langle D_-|S'_-\rangle\rangle = -A^{-3}\langle\langle D_v|S'_v\rangle\rangle$$

$$d_1(\langle\langle D_+|S''_+\rangle\rangle + \langle\langle D_-|S''_-\rangle\rangle) = (A + A^{-1}(-A^2-A^{-2}))\langle\langle D_v|S''_v\rangle\rangle + (A(-A^2-A^{-2})+A^{-1})\langle\langle D_v|S''_v\rangle\rangle = -(A^{-3}+A^3)\langle\langle D_v|S''_v\rangle\rangle$$



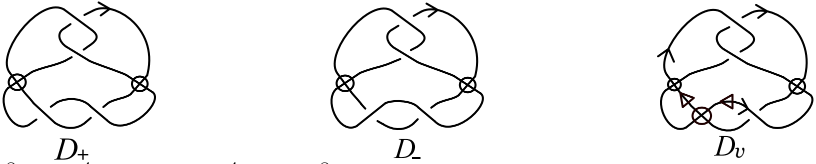
$$\Rightarrow (-A^3 + A^{-3}d_1)\langle\langle D_+\rangle\rangle + (-A^{-3}+A^3d_1)\langle\langle D_-\rangle\rangle = (A^6 - A^{-6})\langle\langle D_v\rangle\rangle$$

Note that $w(D_+)=w(D_v)+1$, $w(D_-)=w(D_v)-1$.

$$\therefore (A^6-d_1)X_{D_+} + (-A^{-6}+d_1)X_{D_-} = (A^6-A^{-6})X_{D_v}$$

Example

$$(A^6 - d_1)X_{D_+} + (-A^{-6} + d_1)X_{D_-} = (A^6 - A^{-6})X_{D_v}$$



$$X_{D_+} = A^8 - A^4 + 1 - A^{-4} + A^{-8}$$
$$X_{D_-} = 1$$
$$X_{D_v} = A^8 - A^4 + 1 + (-A^2 + A^{-2})d_1$$

$$\begin{aligned}
\text{(left side)} &= (A^6 - d_1)(A^8 - A^4 + 1 - A^{-4} + A^{-8}) + (-A^{-6} + d_1)1 \\
&= A^{14} - A^{10} + A^6 - A^2 + A^{-2} - A^{-6} + (-A^8 + A^4 + A^{-4} - A^{-8})d_1 \\
\text{(right side)} &= (A^6 - A^{-6})(A^8 - A^4 + 1 + (-A^2 + A^{-2})d_1) \\
&= A^{14} - A^{10} + A^6 - A^2 + A^{-2} - A^{-6} + (-A^8 + A^4 + A^{-4} - A^{-8})d_1
\end{aligned}$$

Corollary

Corollary 1

D_+, D_- : an almost classical diagram $\Rightarrow X_{D_v} \in \mathbb{Z}[A^{\pm 1}, d_1]$

Exp(X_D) : the set of integers appearing as exponents of A in the term without d_1 in X_D

Exp$(X_D|d_1)$: the set of integers appearing as exponent of A in the term with d_1 in X_D

Corollary 2

(D_+, D_-, D_v) : virtual skein triple

D_+, D_- : almost classical virtual link diagram with n compornent

$$\Rightarrow \text{Exp}(X_{D_v}) \subset \begin{cases} 4\mathbb{Z} & (n : odd) \\ 4\mathbb{Z}+2 & (n : even) \end{cases}$$

$$\text{Exp}(X_{D_v}|d_1) \subset \begin{cases} 4\mathbb{Z}+2 & (n : odd) \\ 4\mathbb{Z} & (n : even) \end{cases}$$

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Future works

In this talk, we concidered a multivariable polynomial invariants of almost classical virtual link diagram, but I would like to consider virtual skein relation for a multivariable polynomial invariants of other conditions.

Thank you for your attention !