曲面の写像類群に含まれる純ブレイド群の最高次数について

The maximum sizes of pure braid groups in the mapping class groups of surfaces

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December 23rd, 2021

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Main Theorem

S: an orientable surface of finite type

A necessary and sufficient condition for $PB_n \hookrightarrow \mathrm{Mod}(S)$.

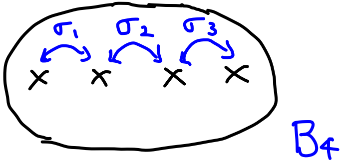
Definition of the mapping class groups of surfaces

$S = S^b_{g,p}$: an orientable surface of genus g with p punctures and b boundary components

Definition

The mapping class group $\mathrm{Mod}(S)$ is $\mathrm{Homeo}^+(S)/($isotopy rel. $\partial S)$.
$\mathrm{PMod}(S) := \mathrm{Ker}(\mathrm{Mod}(S) \to \mathfrak{S}_p)$ is called the pure mapping class group of S.
In particular,
$B_n := \mathrm{Mod}(S^1_{0,n})$ is called the braid group on n-strands.
$PB_n := \mathrm{PMod}(S^1_{0,n})$ is called the pure braid group.

$B_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \rangle$,
where $i - j \geq 2$ in the second relations.
PB_n is a finite index subgroup of B_n (index $n!$).



Rigidity of mapping class groups

Theorem (Behrstock–Kleiner–Minsky–Mosher, 2010)

S: an orientable surface with complexity $3g - 3 + p + b \geq 3$
G: finitely generated group
If $G \underset{q.i.}{\sim} \mathrm{Mod}(S)$, then $G \underset{\exists}{\to} \mathrm{Mod}(S)$ with finite kernel and finite index image.

Theorem (Bowditch, 2015)

S, S': orientable surfaces with the same complexity ≥ 4
If $\mathrm{Mod}(S)$ is quasi-isometrically embedded in $\mathrm{Mod}(S')$, then $S = S'$.

Similar result holds for injective homomorphisms: Ivanov–McCarthy (1999).
For virtual embeddings: Shackleton (2007).
Here, H is virtually embedded in G if $\exists L \leq H$: a f.i. subgp s.t. $L \hookrightarrow G$.

Question. $\mathrm{Mod}(S) \underset{\text{virtual}}{\hookrightarrow} \mathrm{Mod}(S')$?

Note: Since MCGs are virtually torsion free, $\exists$ v.e. $\Leftrightarrow$ $\exists$ virtual hom with finite kernel.

topology of surfaces $\leftrightsquigarrow$ v.e. of MCGs?

relation

Main Result

Case where S is a surface without boundary:

Main Theorem 1

$PB_n \hookrightarrow \mathrm{Mod}(S_{g,p})$ if and only if

$$n \leq \begin{cases} 1 & ((g,p) \in \{(0,0),(0,1),(0,2),(0,3)\}) \\ 2 & ((g,p) \in \{(0,4),(1,0),(1,1)\}) \\ -\chi & (g=0,\ p \geq 5) \\ 1-\chi & (g=1,\ p \geq 2) \\ 2-\chi & (g \geq 2,\ p=0) \\ 1-\chi & (g \geq 2,\ p \geq 1). \end{cases}$$

$-\chi = -\chi(S^b_{g,p}) = 2g+p+b-2.$

Remark: Similar holds for virtual embeddings of B_n.

Case where S is a surface with boundary:

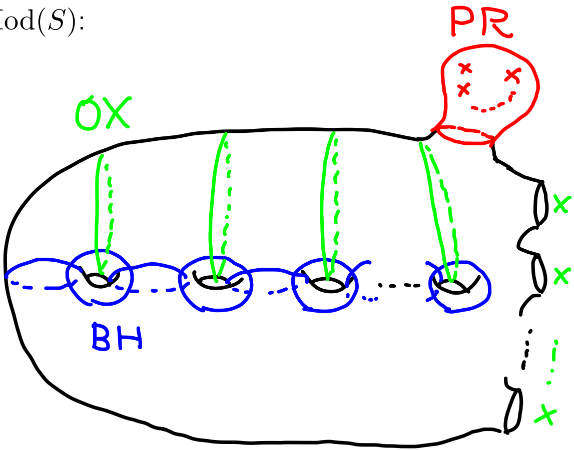
Main Theorem 2

Suppose $b \geq 1$.
Then $PB_n \hookrightarrow \mathrm{Mod}(S_{g,p}^b)$ if and only if

$$n \leq \begin{cases} 2-\chi & (g \geq 1,\ p+b \leq 2) \\ 1-\chi & (\text{otherwise}). \end{cases}$$

Remark: Similar holds for virtual embeddings of B_n.

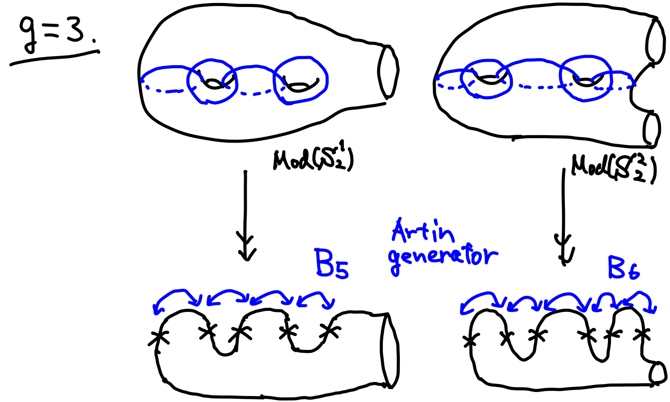
Methods for $PB_n \hookrightarrow \mathrm{Mod}(S)$:



"Geometric monodromy" obtained from Birman–Hilden theory

Theorem (Birman–Hilden, '72)

$B_{2g-1} \hookrightarrow \mathrm{Mod}(S^1_{g-1})$.
Moreover, $B_{2g} \hookrightarrow \mathrm{Mod}(S^2_{g-1})$.

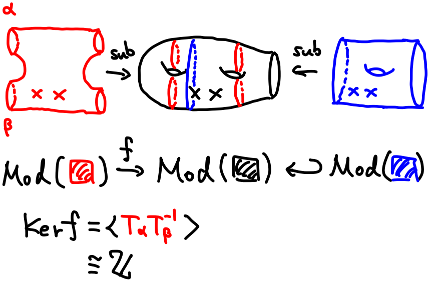


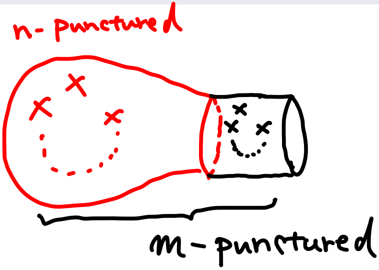
Paris–Rolfsen's work

Let $S' \subset S$ be a surface inclusion preserving punctures.
From this we have a natural homomorphism $\iota\colon \mathrm{Mod}(S') \to \mathrm{Mod}(S)$.

Theorem (Paris–Rolfsen, '00)

Suppose that $\forall$component of $S - \mathrm{Int}N(S')$ is either hyperbolic or an annulus.
Then $\mathrm{Ker}(\iota) \cong \bigoplus \mathbb{Z}[T_\alpha T_\beta^{-1}]$, where α and β are closed curves isotopic thorough an outer annulus, and the direct sum runs over the set of outer annular components.





Corollary (van der Lek, '83): If $n \leq m$, then $B_n \hookrightarrow B_m$ $(PB_n \hookrightarrow PB_m)$.

$(1-\chi)$-embedding theorem of pure braid groups

Theorem

Suppose $g \geq 1$ or $b \geq 1$.
Then $PB_{2g+p+b-1} \hookrightarrow \mathrm{Mod}(S^b_{g,p})$.

Note: $\chi(S^b_{g,p}) = 2 - 2g - p - b$ and so $1 - \chi = 2g + p + b - 1$.

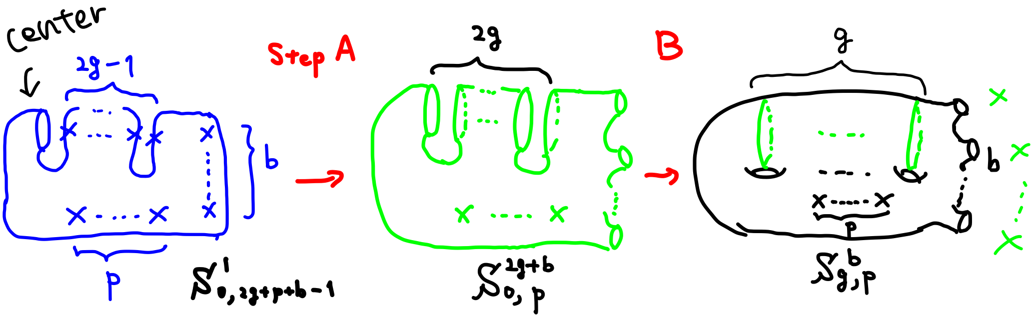
Step A. Show that $PB_{2g+p+b-1} \overset{\exists}{\hookrightarrow} \mathrm{Mod}(S^{2g+b}_{0,p})$.
Direct decomposition $\mathrm{PMod}(S^{2g+b}_{0,p}) \cong \mathbb{Z}^{2g+b-1} \times PB_{2g+p+b-1}$ (Clay–Leininger–Margalit).

Step B. Show that the restriction of the hom $\mathrm{Mod}(S^{2g+b}_{0,p}) \to \mathrm{Mod}(S^b_{g,p})$ to $PB_{2g+p+b-1}$ is injective.
Use Paris–Rolfsen' result.

g copies of an annulus

Step A + Step B $\rightsquigarrow$ the desired embedding.

Picture for (OX):



Difference between $(1-\chi)$ and others

We have the following methods for embedding pure braid groups into MCGs:

(BH) Birman–Hilden theory

(PR) Paris–Rolfsen's result

(OX) $(1-\chi)$-embedding theorem

From (BH) and (PR), we obtain embeddings of braid groups into MCGs.
On the other hand, according to Castel's result, (OX) cannot be extended to braid groups.

Theorem (Castel, '16)

Suppose $b \geq 2$.
$B_n \hookrightarrow \mathrm{Mod}(S_g^b)$ if and only if $n \leq 2g+2$.

From our result, we can deduce $PB_{2g+b-1} \hookrightarrow \mathrm{Mod}(S_g^b)$.
If $b \geq 4$, (OX) provides us with new embeddings of pure braid groups.

Finding the “largest” pure braid groups

Emphasize that all of (BH), (PR) and (OX) are effective methods.

The list of the largest pure braid groups when $g = 0$ or $g \geq 2$

Genus 0 case

$b = 0$ and $p \geq 3$; $PB_{p-2} \hookrightarrow \mathrm{Mod}(S_{0,p})$ (PR).

$b \geq 1$; $PB_{p+b-1} \hookrightarrow \mathrm{Mod}(S_{0,p}^b)$ (OX).

Genus ≥ 2 case

$p + b = 0$; $PB_{2g} \hookrightarrow \mathrm{Mod}(S_g)$ (BH)+(PR).

$1 \leq p + b \leq 2$; $PB_{2g} \hookrightarrow \mathrm{Mod}(S_{g,1})$ (BH) or (OX).

$PB_{2g+1} \hookrightarrow \mathrm{Mod}(S_g^1)$ (BH).

$PB_{2g+1} \hookrightarrow \mathrm{Mod}(S_{g,2})$ (BH)+(PR) or (OX).

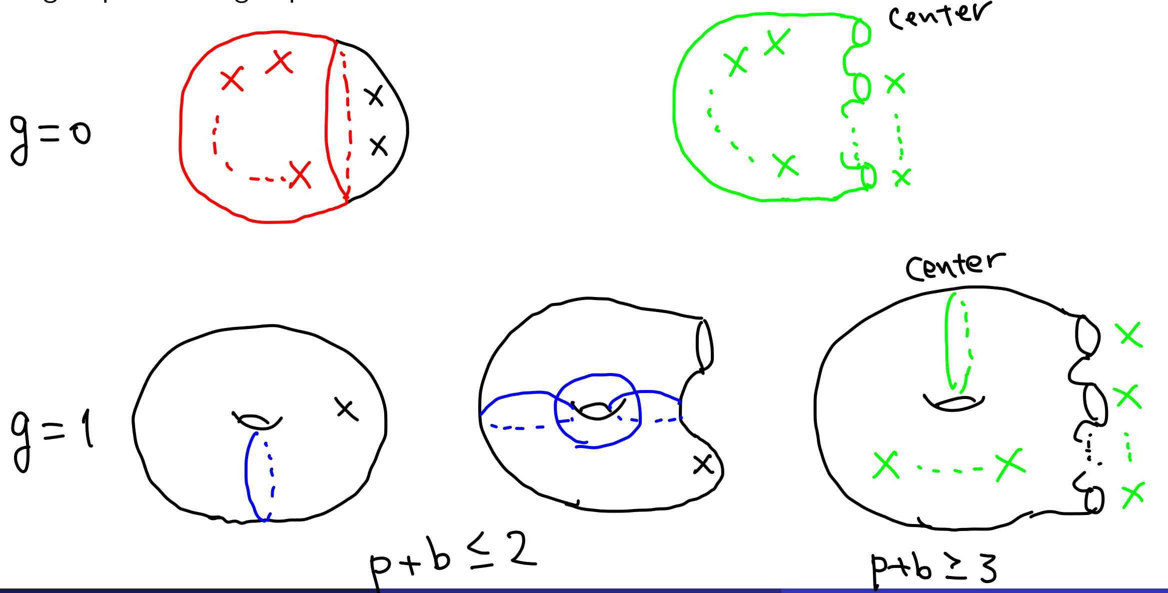
$PB_{2g+2} \hookrightarrow \mathrm{Mod}(S_g^2)$ (BH).

$PB_{2g+2} \hookrightarrow \mathrm{Mod}(S_{g,1}^1)$ (BH)+(PR).

$p + b = 3$; $PB_{2g+2} \hookrightarrow \mathrm{Mod}(S_{g,p}^b)$ (BH)+(PR) or (OX).

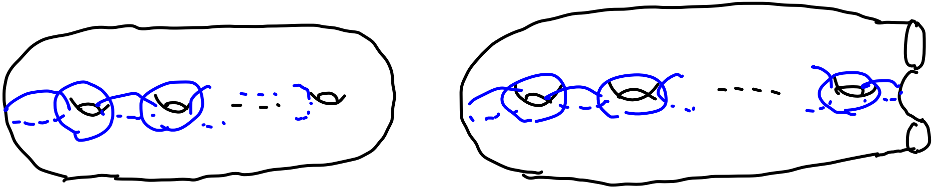
$p + b \geq 4$; $PB_{2g+p+b-1} \hookrightarrow \mathrm{Mod}(S_{g,p}^b)$ (OX).

Largest pure braid groups:

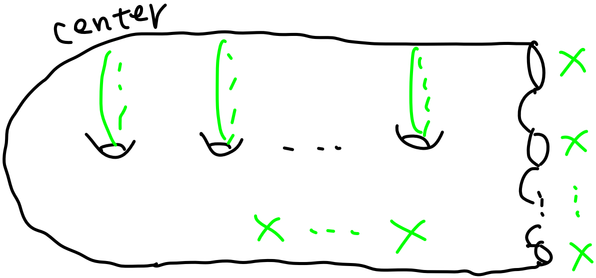


$g \geq 2$.

$p+b \leq 2$



$p+b \geq 3$



Main Theorem 1

PB_n is embedded in $\mathrm{Mod}(S_{g,p})$ if and only if

$$n \leq \begin{cases} -\chi & (g = 0,\ p \geq 5)\ \text{(PR)} \\ 1 - \chi & (g = 1,\ p \geq 2)\ \text{(OX)} \\ 2 - \chi & (g \geq 2,\ p = 0)\ \text{(BH)} \\ 1 - \chi & (g \geq 2,\ p \geq 1).\ \text{(OX)} \end{cases}$$

Main Theorem 2

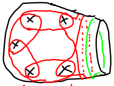
Suppose $b \geq 1$. Then PB_n is embedded in $\mathrm{Mod}(S^b_{g,p})$ if and only if

$$n \leq \begin{cases} 2 - \chi & (g \geq 1,\ p + b \leq 2)\ \text{(BH)+(PR)} \\ 1 - \chi & \text{(otherwise). (OX)} \end{cases}$$

Sketch of proof for the "only if" part

Want to show that PB_n is not embedded in $\mathrm{Mod}(S)$ if n is bigger than the bound given in the main thms.

KEY: Suppose n is bigger than the bound.
Then $S^1_{0,n}$ contains a complicated curve system which cannot appear in S.



Curve system on a surface $\xrightarrow{\text{Dehn twist}}$ right-angled Artin group (Koberda, 2012)
∃a right-angled Artin group in PB_n but not in $\mathrm{Mod}(S)$?

Lemma 1: Suppose Γ is a "nice" graph. Then $A(\Gamma) \hookrightarrow \mathrm{Mod}(S)$ (if and) only if $\Gamma \leq \mathcal{C}(S)$.
Lemma 2: If n is larger than the bound, then $\exists \Gamma \leq \mathcal{C}(S^1_{0,n})$ such that Γ is "nice" and $\Gamma \not\leq \mathcal{C}(S)$.

Right-angled Artin groups

Γ: a finite simple graph
$V(\Gamma)$: the vertex set of Γ
$E(\Gamma)$: the edge set of Γ

Definition

The right-angled Artin group (RAAG) on Γ is the group defined by the following group presentation:

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(\Gamma) \rangle .$$

E.g.
Every finitely generated free group is the RAAG on an edgeless graph.
Every finitely generated free abelian group is the RAAG on a complete graph.
RAAGs includes the direct products, free products and HNN extensions of these groups.

Curve graphs and powers of Dehn twists

Definition

For a surface S, the curve graph $\mathcal{C}(S)$ is the simple graph such that

- $V(\mathcal{C}(S))$ is the set of the isotopy classes of essential simple closed curves in S and
- $E(\mathcal{C}(S))$ consists of all non-ordered pair of essential simple closed curves which can be represented disjointly.

$\mathcal{C}(S)$ is the 1-skeleton of the Harvey's curve complex.

Theorem (Koberda, '12)

$\alpha_1, \ldots, \alpha_n$: mutually non-isotopic essential or boundary-parallel simple closed curves in S
For a sufficiently large m, $\langle T_{\alpha_1}^m, \ldots, T_{\alpha_n}^m \rangle$ is isomorphic to a RAAG.
The defining graph of the RAAG coincides with the subgraph of $\mathcal{C}(S)$, which is induced by $\alpha_1, \ldots, \alpha_n$.

Also phrased as follows: If Γ is a finite induced subgraph of $\mathcal{C}(S)$, then $A(\Gamma) \hookrightarrow \mathrm{Mod}(S)$.

Thank you for the attention!