Integral region choice problems on link diagrams
絡み目射影図上の整数値領域選択問題

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Region crossing change （領域交差交換） (RCC)

[Shimizu 2014]proposed by Kishimoto

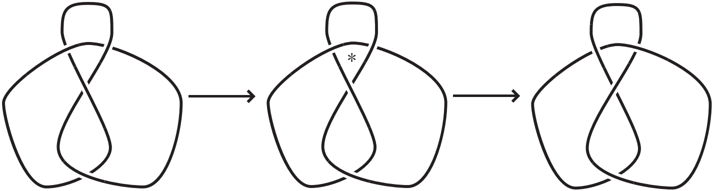


Figure: An example of a RCC.

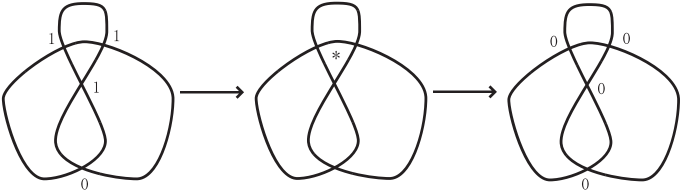


Figure: An example of an interpreted RCC.

Integral region choice problem （整数値領域選択問題）

[Ahara-Suzuki 2012]

(Definite integral region choice problem（定値整数値領域選択問題）(D$\mathbb{Z}$RCP))

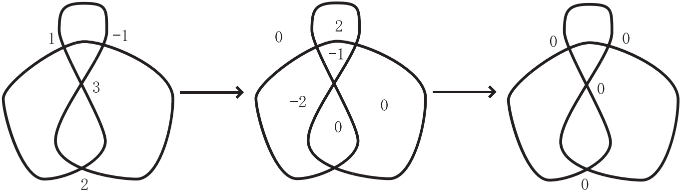


Figure: An example of a D$\mathbb{Z}$RCP.

$$\begin{aligned} 1 \mapsto \quad & 1 + 0 + 2 + (-1) + (-2) &= 0, \\ -1 \mapsto \quad & -1 + 0 + 0 + (-1) + 2 &= 0, \\ 3 \mapsto \quad & 3 + 0 + 0 + (-2) + (-1) &= 0, \\ 2 \mapsto \quad & 2 + 0 + 0 + 0 + (-2) &= 0. \end{aligned}$$

Alternating integral region choice problem
（交代的整数値領域選択問題）(A$\mathbb{Z}$RCP)

[Harada 2018] suggested by Yonezawa

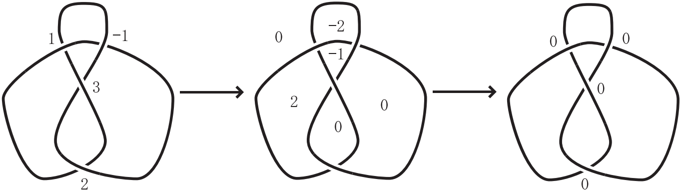
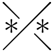
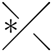
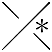
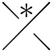
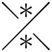


Figure: An example of an A$\mathbb{Z}$RCP.

$$
\begin{aligned}
1 &\mapsto & 1+0-(-2)+(-1)-2 &= 0,\\
-1 &\mapsto & -1+0-0+(-1)-(-2) &= 0,\\
3 &\mapsto & 3-0+0-2+(-1) &= 0,\\
2 &\mapsto & 2+0-0+0-2 &= 0.
\end{aligned}
$$

Region choice matrices

D : a link diagram with d connected components
$x_1, \cdots, x_n$: crossings of D, $n \geq 1$
$R_1, \cdots, R_{n+d+1}$: regions of D

x_i and Rj		 or 	 or 		otherwise
$a_{ij}^{(d1)}$	1	1	1	1	0
$a_{ij}^{(d2)}$	2	1	1	2	0
$a_{ij}^{(a1)}$	1	1	-1	-1	0
$a_{ij}^{(a2)}$	2	1	-1	-2	0

$A_{d1}(D) := (a_{ij}^{(d1)})$, $A_{d2}(D) := (a_{ij}^{(d2)})$: D$\mathbb{Z}$RC matrices
$A_{a1}(D) := (a_{ij}^{(a1)})$, $A_{a2}(D) := (a_{ij}^{(a2)})$: A$\mathbb{Z}$RC matrices

D$\mathbb{Z}$RCP and A$\mathbb{Z}$RCP on knot diagrams

Theorem 1 ([Ahara-Suzuki 2012])

D : a knot diagram with crossings $x_1, \cdots, x_n$, $n \geq 1$
$R_1, \cdots, R_{n+2}$: regions of D
$\forall \mathbf{c} \in \mathbb{Z}^n$,
$\exists \mathbf{u} \in \mathbb{Z}^{n+2};\ A_{d1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}, \quad \exists \mathbf{w} \in \mathbb{Z}^{n+2};\ A_{d2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}.$

Theorem 2 ([Harada 2018])

D : a knot diagram with crossings $x_1, \cdots, x_n$, $n \geq 1$
$R_1, \cdots, R_{n+2}$: regions of D
$\forall \mathbf{c} \in \mathbb{Z}^n$,
$\exists \mathbf{u} \in \mathbb{Z}^{n+2};\ A_{a1}(D)\mathbf{u} + \mathbf{c} = \mathbf{0}, \quad \exists \mathbf{w} \in \mathbb{Z}^{n+2};\ A_{a2}(D)\mathbf{w} + \mathbf{c} = \mathbf{0}.$

Alexander index

D: an oriented link diagram

Alexander index [Alexander 1928]:

an integer index to each region of D s.t. $\forall \gamma \subset D$ an oriented arc,

$$\begin{aligned}&(\text{an index of the left region adjacent to } \gamma)\\ =\ &(\text{an index of the right region adjacent to } \gamma) + 1\end{aligned}$$

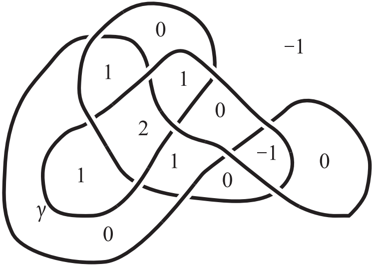


Figure: An example of an Alexander index

Lemma 3

D: an oriented link diagram with n crossings, $n \geq 1$
An Alexander index on D gives a solution for $A_{a2}(D)\mathbf{w} = \mathbf{0}$.

Remark 1

By the projection $\mathbb{Z} \to \mathbb{Z}_2$,
an Alexander index gives a 'checkerboard coloring'.

Remark 2

$A_{a2}(D)$ *coincides with the Alexander matrix defined in [Alexander 1928] if we substitute* 1 *for the variable.*

Outline of an alternative proof of Theorem 2

x: a crossing of a link diagram D in same link component

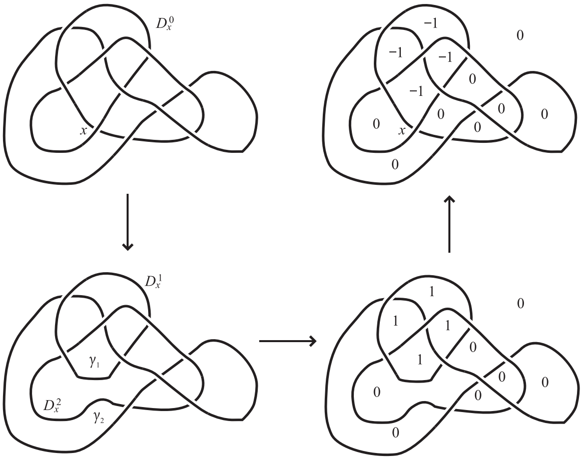


Figure: $\exists \mathbf{v}_x$ s.t. the component of $A_{a2}(D)\mathbf{v}_x$ to x is 1 and the others are 0.

Outline of an alternative proof of Theorem 1

x: a crossing of a link diagram D in same link component
$\mathbf{v}_x$ s.t. the component of $A_{a2}(D)\mathbf{v}_x$ to x is 1 and the others are 0.

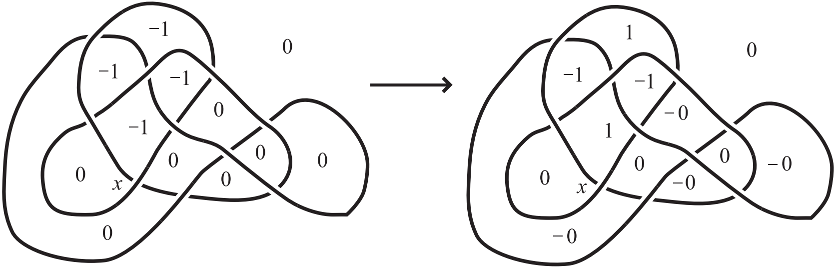


Figure: From $\mathbf{v}_x$ to $\exists\bar{\mathbf{v}}_x$ s.t. the component of $A_{d2}(D)\bar{\mathbf{v}}_x$ to x is 1 and the others are 0.

Main result 1 : ranks of region choice matrices

Theorem 4

D: an l-component link diagram with d connected components
$x_1, \cdots, x_n$: crossings of D, $n \geq 1$
$R_1, \cdots, R_{n+d+1}$: regions of D

$$\mathrm{rank} A_{d2}(D) = \mathrm{rank} A_{a2}(D) = n + d - l,$$
$$\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{d2}(D)\mathbf{u} = \mathbf{0}\} \cong \mathbb{Z}^{l+1},$$
$$\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{a2}(D)\mathbf{u} = \mathbf{0}\} \cong \mathbb{Z}^{l+1}$$

Remark 3

The above theorem modulo 2 implies some results in [Cheng-Gao 2012] and [Hashizume 2013].

Lemma 5 ([Ahara-Suzuki 2012, Harada 2018])

Changing a crossing on a link diagram does not change A_{d1} *and* A_{d2}*, and makes the row concerning the crossing multiplied by* -1 *in* A_{a1} *and* A_{a2}*.*

Lemma 6 (cf. [Ahara-Suzuki 2012, Harada 2018])

Reidemeister moves preserve the equalities in Theorem 4.

Lemma 5 and 6 $\Rightarrow$ Theorem 4.

Reidemeister move I

Figure: Reidemeister moves I among D_+, D, and D_-.

$$A_{d2}(D) = \begin{pmatrix} \mathbf{a}_d & \mathbf{b}_d & P_d \end{pmatrix}, \quad A_{a2}(D) = \begin{pmatrix} \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}.$$

$$A_{d2}(D_+) = A_{d2}(D_-) = \begin{pmatrix} 1 & 2 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & P_d \end{pmatrix},$$

$$A_{a2}(D_+) = \begin{pmatrix} 1 & -2 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}, \quad A_{a2}(D_-) = \begin{pmatrix} -1 & 2 & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & P_a \end{pmatrix}.$$

Reidemeister move II

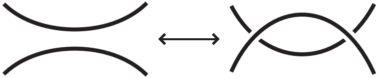


Figure: A Reidemeister move II between D and D'.

If the regions appearing around the move are different each other,

$$A_{d2}(D) = \begin{pmatrix} \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & P_d \end{pmatrix}, \quad A_{a2}(D) = \begin{pmatrix} \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & P_a \end{pmatrix}.$$

$$A_{d2}(D') = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & O \\ 1 & 1 & 0 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}'_d & \mathbf{b}''_d & \mathbf{c}_d & P_d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ 1 & 0 & 0 & 1 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{b}''_d & \mathbf{c}_d & P_d \end{pmatrix}$$

$$A_{a2}(D') = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & O \\ -1 & 1 & 0 & -1 & 1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}'_a & \mathbf{b}''_a & \mathbf{c}_a & P_a \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & O \\ -1 & 0 & 0 & -1 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{b}''_a & \mathbf{c}_a & P_a \end{pmatrix}$$

where $\mathbf{b}'_d + \mathbf{b}''_d = \mathbf{b}_d$, $\mathbf{b}'_a + \mathbf{b}''_a = \mathbf{b}_a$. We omit the other cases here.

Reidemeister move III and A_{d2}

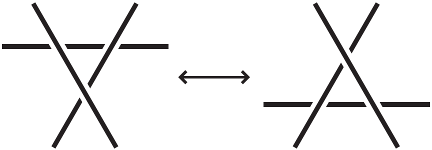


Figure: A Reidemeister move III between D_∇ and D_Δ.

If regions appearing around the move are different each other,

$$A_{d2}(D_\nabla) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & O \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & \mathbf{d}_d & \mathbf{e}_d & \mathbf{f}_d & P_d \end{pmatrix}$$

$$\sim A_{d2}(D_\Delta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & O \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_d & \mathbf{b}_d & \mathbf{c}_d & \mathbf{d}_d & \mathbf{e}_d & \mathbf{f}_d & P_d \end{pmatrix}.$$

We omit the other cases here.

Reidemeister move III and A_{a2}

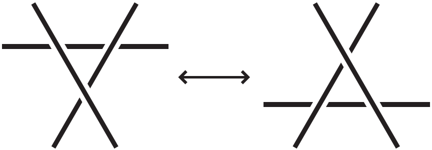


Figure: A Reidemeister move III between D_∇ and D_Δ.

If regions appearing around the move are different each other,

$$A_{a2}(D_\nabla) = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 & O \\ 1 & 0 & 0 & 0 & -1 & 1 & -1 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & \mathbf{d}_a & \mathbf{e}_a & \mathbf{f}_a & P_a \end{pmatrix}$$

$$\sim A_{a2}(D_\Delta) = \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 1 & 0 & \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 & O \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & \\ \mathbf{0} & \mathbf{a}_a & \mathbf{b}_a & \mathbf{c}_a & \mathbf{d}_a & \mathbf{e}_a & \mathbf{f}_a & P_a \end{pmatrix}.$$

We omit the other cases here.

Componentwise Alexander index

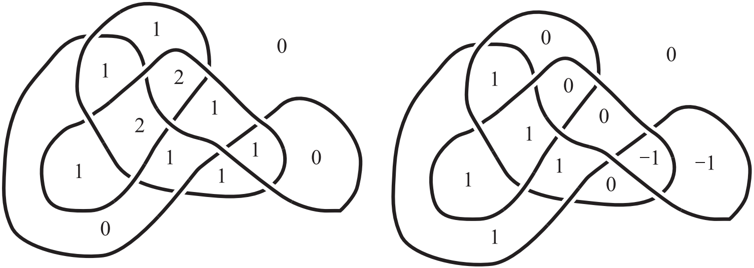


Figure: Examples of componentwise Alexander indexes.

Lemma 7

$D = D_1 \cup \cdots \cup D_l$: an oriented link diagram with n crossings, $n \geq 1$
A componentwise Alexander index associated with D_i gives a solution $\mathbf{w} = \mathbf{u}_i$ for $A_{a2}(D)\mathbf{w} = \mathbf{0}$, $i = 1, \cdots, l$.

Assigning a same integer to all regions

Lemma 8

On any link diagram D, assigning a same integer to all regions gives a solution for $A_{a2}(D)\mathbf{w} = \mathbf{0}$.

$\mathbf{u}_\infty$: assigning 1 to all regions.

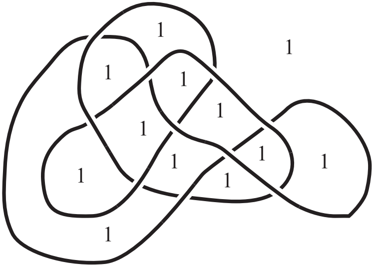


Figure: An example of $\mathbf{u}_\infty$.

Main result 2 : bases of kernels for matrices

Theorem 9

$D = D_1 \cup \cdots \cup D_l$: an oriented link diagram with n crossings, $n \geq 1$.
$\mathbf{u}_1, \cdots, \mathbf{u}_l$: componentwise Alexander indexes,
the unbounded region assigned 0 for each $\mathbf{u}_i$.
$\mathbf{u}_\infty$: assigning 1 to all regions.
The set of $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$ is a basis of $\{\mathbf{w} \mid A_{a2}(D)\mathbf{w} = \mathbf{0}\}$.

Lemma 7, 8, Theorem 4, the linearly independence of $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$
$\Rightarrow$ Theorem 9.

Remark 4

The above theorem modulo 2 implies a result in [Hashizume 2013].

Fix a checkerboard coloring.
From the basis $\mathbf{u}_1, \cdots, \mathbf{u}_l, \mathbf{u}_\infty$ of $\{\mathbf{w} \mid A_{a2}(D)\mathbf{w} = \mathbf{0}\}$,
the basis $\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_l, \bar{\mathbf{u}}_\infty$ of $\{\mathbf{w} \mid A_{d2}(D)\mathbf{w} = \mathbf{0}\}$ is obtained.

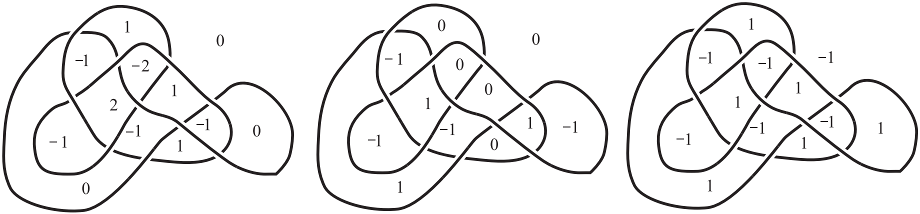


Figure: $\bar{\mathbf{u}}_1$, $\bar{\mathbf{u}}_2$, $\bar{\mathbf{u}}_\infty$.

Main result 3 : solvabilities of A$\mathbb{Z}$RC and D$\mathbb{Z}$RC

$D = D_1 \cup D_2$: a connected diagram of a 2-component oriented link
$x_1, \cdots, x_k$ $(k < n)$: crossings in same link component
$x_{k+1}, \cdots, x_n$: crossings of D_1 and D_2
Suppose D_2 crosses D_1 from the right to the left at x_n.
By Theorem 4, $\mathrm{rank} A_{a2}(D) = n + d - l = n + 1 - 2 = n - 1$.
Take $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1} \in \mathbb{Z}^n$ such that:

1. for $i = 1, \cdots, k$,
   all components of $\mathbf{e}_i$ are 0 but the i-th component is 1;
2. for $i = k+1, \cdots, n-1$,
   all components of $\mathbf{e}_i$ are 0 but the i-th component and the n-th component are ε_{x_i} (resp. $-\varepsilon_{x_i}$) and ε_{x_n} if D_2 crosses D_1 from the left to the right (resp. from the right to the left) at x_i, where ε_x is a sign of x.

Theorem 10

The set of $\mathbf{e}_1, \cdots, \mathbf{e}_{n-1}$ *is a basis of* $\{A_{a2}(D_1 \cup D_2)\mathbf{v}\}$.

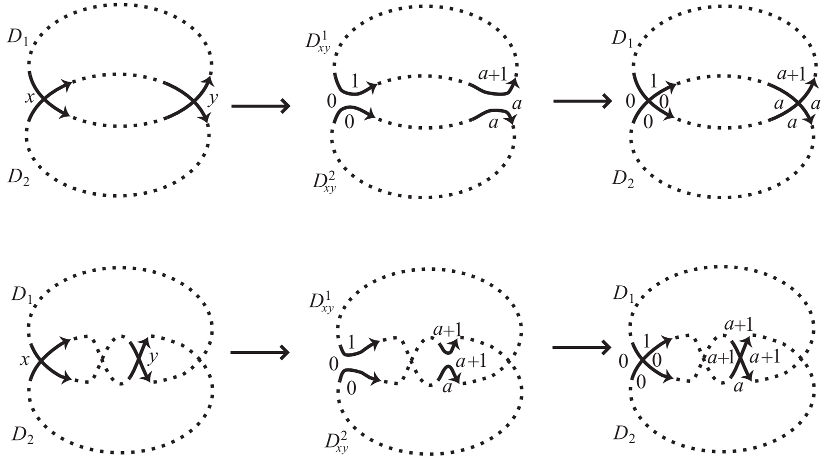


Figure: Finding $\mathbf{v}_i$ with $A_{a2}(D_1 \cup D_2)\mathbf{v}_i = \mathbf{e}_i$ for $k < i < n$, $x = x_n$, $y = x_i$.

$\forall i$, $\exists \mathbf{v}_i$ s.t. $A_{a2}(D_1 \cup D_2)\mathbf{v}_i = \mathbf{e}_i$. $\Rightarrow$ Theorem 10.

Fix a checkerboard coloring on $D = D_1 \cup D_2$.
Multiply all components of $\mathbf{v}_i$ assigned to the shaded regions by -1 to obtain $\bar{\mathbf{v}}_i$.
$\bar{\mathbf{e}}_i := A_{d2}(D)\bar{\mathbf{v}}_i$.
The set of $\mathbf{e}_1, \cdots, \mathbf{e}_k, \bar{\mathbf{e}}_{k+1}, \cdots, \bar{\mathbf{e}}_{n-1}$ is a basis of $\{A_{d2}(D_1 \cup D_2)\mathbf{v}\}$.

Remark 5

Theorem 10 and the above are partial extensions of a result in [Hashizume 2015].

Main results for A_{d1} and A_{a1}

Theorem 11

D: a link diagram with d connected components and n crossings.
$\{A_{d1}(D)\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}^{n+d+1}\} = \{A_{d2}(D)\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}^{n+d+1}\} \subset \mathbb{Z}^n$,
$\{A_{a1}(D)\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}^{n+d+1}\} = \{A_{a2}(D)\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}^{n+d+1}\} \subset \mathbb{Z}^n$.

$\because$ Same arguments as that in [Ahara-Suzuki 2012, Harada 2018].

Theorem 12

D: an l-component link diagram with d connected components and n crossings, $n \geq 1$
$\mathrm{rank} A_{d1}(D) = \mathrm{rank} A_{a1}(D) = n + d - l$,
$\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{d1}(D)\mathbf{u} = \mathbf{0}\} \cong \mathbb{Z}^{l+1}$,
$\{\mathbf{u} \in \mathbb{Z}^{n+d+1} \mid A_{a1}(D)\mathbf{u} = \mathbf{0}\} \cong \mathbb{Z}^{l+1}$

$\because$ Theorem 4 and 11

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