Braid group actions from categorical symmetric Howe duality on deformed Webster algebras

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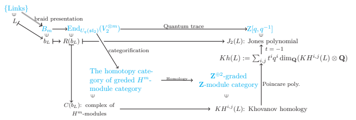
1. Jones polynomial and Khovanov homology

M. Khovanov constructed a homological link invariant whose graded Euler characteristic is Jones polynomial.



[Reference: M. Khovanov, A functor-valued invariant of tangles, Algebraic & Geometric Topology **2** (2002) 665–741.]

(A) From this fact, I have been working on a natural question:

Can we construct homological invariants which refine other (quantum) invariants?

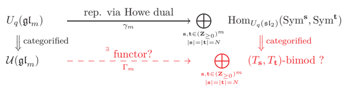
Yes, to some quantum link invariants.

(B) The following question is also interesting.

Can we construct homological invariants using other categories?

Yes, to some homological link invariants. For instance, matrix factorizations, Soergel bimodule, zigzag algebra, affine Grassmannian, quiver varieties, etc! It makes this field so interesting!!

2. Our approach



Question Can we construct a 2-category categorifying $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(\mathrm{Sym}^{\mathbf{s}},\mathrm{Sym}^{\mathbf{t}})$ and a functor from $\mathcal{U}(\mathfrak{gl}_m)$ to the 2-category?

Fact Webster defined an algebra T_λ categorifying a tensor product of irreducible representations of $U_q(\mathfrak{g})$. That is, we have an algebra $T_{\mathbf{s}}$ categorifying the representation $\mathrm{Sym}^{\mathbf{s}}$.
[Reference: B. Webster, Knot invariants and higher representation theory. Mem. Amer. Math. Soc., 250, 141.]

However, there are extra complications using the original Webster algebra. We need a *deformed Webster algebra* $W(\mathbf{s},k)$ getting a functor, where k is a non-negative integer.

Remark: (1) Khovanov–Sussan define deformed Webster algebra $W(m,k)$ of type A_1 which is related to categorification of $\mathrm{Sym}^{\overbrace{(1,\ldots,1)}^{m}}$.
(2) Cautis-Kamnitzer constuct the representation in the case $\mathbf{s}=(1,\ldots,1)$ using affine Grassmannian.

[Reference: M. Khovanov and J. Sussan, The Soergel category and the redotted Webster algebra, Journal of Pure and Applied Algebra **222** (2018) 1957-2000.]
[Reference: S. Cautis and J. Kamnitzer, Categorical geometric symmetric Howe duality, Selecta Mathematica **24** (2018), 1593–1631.]

3. Main theorem

Theorem 1 (Khovanov-Lauda-Sussan-Y)
There exists a functor $\Gamma_{m,k}:\mathcal{U}(\mathfrak{gl}_m)\to \mathrm{Bim}(m,k)$, where

$$\mathrm{Bim}(m,k)=\bigoplus_{\substack{\mathbf{s},\mathbf{t}\in(\mathbf{Z}_{\geq 0})^m\\ |\mathbf{s}|=|\mathbf{t}|=N}} (W(\mathbf{t},k),W(\mathbf{s},k))\text{-bimod}$$

Remark The algebra $W(\mathbf{s},k)$ is non-cyclotomic. So $(W(\mathbf{t},k),W(\mathbf{s},k))$-bimod is not a categorification of $\mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(\mathrm{Sym}^{\mathbf{s}},\mathrm{Sym}^{\mathbf{t}})$.

4. Categorical braid group action

A categorical braid group action on derived categories of coherent sheaves on affine Grassmannian is defined by Cautis-Kamnitzer. It's obtained generalizing the Chuang-Rouquier construction in $\mathfrak{sl}_2$ categorification. Let $\mathbf{s}=(s_1,\ldots,s_m)$. Consider the formal unbounded complexes of $\mathcal{U}(\mathfrak{gl}_m)$

$$\mathrm{T}_i 1_{\mathbf{s}} = \mathcal{E}_i^{(s_{i+1}-s_i)}1_{\mathbf{s}} \to \mathcal{E}_i^{(s_{i+1}-s_i+1)}\mathcal{F}_i^{(1)}1_{\mathbf{s}}\langle 1\rangle \to \cdots \to \mathcal{E}_i^{(s_{i+1}-s_i+j)}\mathcal{F}_i^{(j)}1_{\mathbf{s}}\langle j\rangle \to \cdots$$

for $s_{i+1}\geq s_i$ (omission for $s_i > s_{i+1}$)

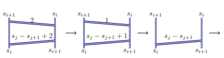
$$1_{\mathbf{s}}\mathrm{T}'_i = \cdots \to 1_{\mathbf{s}}\mathcal{E}_i^{(j)}\mathcal{F}_i^{(s_{i+1}-s_i+j)}\langle -j\rangle \to \cdots \to 1_{\mathbf{s}}\mathcal{E}_i^{(1)}\mathcal{F}_i^{(s_{i+1}-s_i+1)}\langle -1\rangle \to 1_{\mathbf{s}}\mathcal{F}_i^{(s_{i+1}-s_i)}$$

for $s_{i+1}\geq s_i$ (omission for $s_i > s_{i+1}$)

Applying the above functor from $\mathcal{U}(\mathfrak{gl}_m)$ to a 2-category to the unbounded complexes, we obtain the bounded complexes.

Corollary The bounded complexes $\Gamma_{m,k}(\mathrm{T}_i),\Gamma_{m,k}(\mathrm{T}'_i)$ satisfy the braid group relations as a functor from $\mathrm{K}^b(W(\mathbf{s},k)\text{-mod})$ to $\mathrm{K}^b(W(s_i\mathbf{s},k)\text{-mod})$.

In the case of $k=0$,

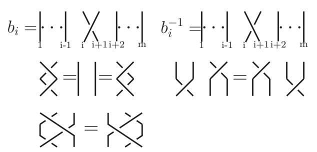
$\Gamma_{m,0}(\mathrm{T}'_i) \quad = \quad \cdots \to$ 

$\to 0$

5. Diagram description of braid group

Braid group B_m is generated by elements b_i and b_i^{-1} $(i=1,\ldots,m-1)$ subject to the following relations

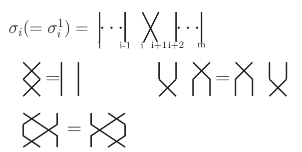
$$\begin{array}{ll} b_ib_j=b_jb_i & \text{if } |i-j|>1\\ b_ib_{i+1}b_i=b_{i+1}b_ib_{i+1} & i=1,\ldots,m-2\end{array}$$



6. Diagram description of symmetric group

Symmetric group S_m is generated by elements σ_i $(i=1,\ldots,m-1)$ subject to the following relations

$$\begin{array}{ll} \sigma_i^2=1 & i=1,\ldots,m-1\\ \sigma_i\sigma_j=\sigma_j\sigma_i & \text{if } |i-j|>1\\ \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} & i=1,\ldots,m-2\end{array}$$

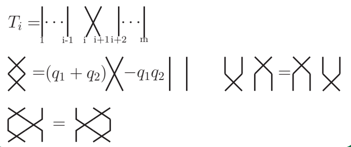


7. Diagram description of Hecke algebra

Hecke algebra $H_m(q_1,q_2)$ is a q-deformation of the group ring of symmetric group $\mathbb{C}[S_m]$ whose generators are T_i $(i=1,\ldots,m-1)$ and relations are

$$\begin{array}{ll} T_i^2=(q_1+q_2)T_i-q_1q_2 & i=1,\ldots,m-1\\ T_iT_j=T_jT_i & \text{if } |i-j|>1\\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} & i=1,\ldots,m-2\end{array}$$

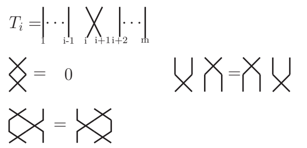
where $(q_1,q_2)\in\mathbb{C}^2$.
Remark: (1) When $(q_1,q_2)=(1,-1)$ or $(-1,1)$, $H_m(q_1,q_2)$ is $\mathbb{C}[S_m]$.
(2) When $(q_1,q_2)\in\mathbb{C}^2-\{(0,0)\}$, $H_m(q_1,q_2)$ is well known.
$(T'_i=q_1T_i$ and $q=-\frac{q_2}{q_1}$ if $q_1\neq 0)$



8. Diagram description of nil Hecke algebra

Nil Hecke algebra H_m^0 is $H_m(0,0)$. Write $T_i \xrightarrow{(q_1,q_2)\to(0,0)} \partial_i$.

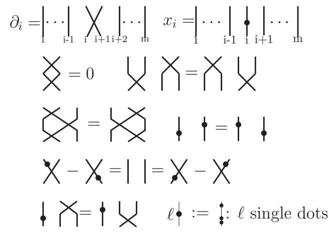
$$\begin{array}{ll} \partial_i^2=0 & i=1,\ldots,m-1\\ \partial_i\partial_j=\partial_j\partial_i & \text{if } |i-j|>1\\ \partial_i\partial_{i+1}\partial_i=\partial_{i+1}\partial_i\partial_{i+1} & i=1,\ldots,m-2\end{array}$$



9. Diagram description of nil affine Hecke algebra

Nil Hecke algebra H_m^0 is generated by ∂_i $(i=1,\ldots,m-1)$ and x_j $(j=1,\ldots,m)$ subject to the following relations.

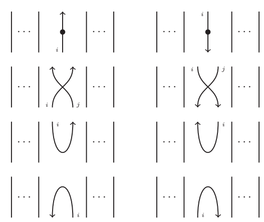
$$\begin{array}{ll} \partial_i^2=0 & i=1,\ldots,m-1\\ \partial_i\partial_j=\partial_j\partial_i & \text{if } |i-j|>1\\ \partial_i\partial_{i+1}\partial_i=\partial_{i+1}\partial_i\partial_{i+1} & i=1,\ldots,m-2\\ x_ix_j=x_jx_i & \\ x_i\partial_i=\partial_ix_{i+1}+1 & \\ \partial_ix_i=x_{i+1}\partial_i+1 & \\ \partial_ix_j=x_j\partial_i & \text{unless } j=i,i+1\end{array}$$



Remark
(1) This is a graded algebra with $\deg\partial_i=-2$ and $\deg x_i=2$.
(2) Actually, This is the (half) A_1 quiver Hecke algebra!!
(3) The projective module category of this algebra is a categorification of $U_q^-(\mathfrak{sl}_2)$.

10. Quiver Hecke algebra $\mathfrak{gl}_m$

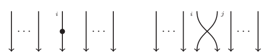
Quiver Hecke algebra $\mathcal{U}(\mathfrak{g})$ (KLR algebra, categorified quantum group) is introduced by Khovanov-Lauda and Rouquier.
Quiver Hecke algebra $\mathcal{U}(\mathfrak{g})$ is generated by the following colored oriented diagrams subject to some relations (omit).



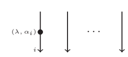
where colorings are corresponding to Dynkin verteces of $\mathfrak{g}$.
Theorem (Khovanov-Lauda, Rouquier) The projective module category is a categorification of $U_q(\mathfrak{g})$.
[Reference: M. Khovanov and A. Lauda, A categorification of quantum sl(n), Quantum Topology **1** (2010) 1–92.]
[Reference: R. Rouquier, 2-Kac-Moody algebras, arXiv:0812.5023.]

11. Cyclotomic quotient of half quiver Hecke alg

Let $\mathcal{U}^-(\mathfrak{g})$ be the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by



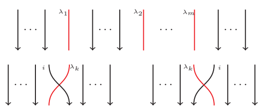
Cyclotomic quotient of $\mathcal{U}^-(\mathfrak{g})$ is introduced for categorifying representations of quantum group.
Cyclotomic quotient $\mathcal{U}^\lambda(\mathfrak{g})$ of weight λ is the quotient algebra $\mathcal{U}^-(\mathfrak{g})/I_\lambda$, where I_λ is the cyclotomic ideal of λ (2-sided ideal generated by elements associated with the weight λ)



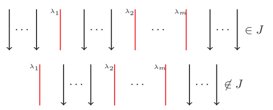
Theorem (Kang-Kashiwara) The projective module category of $\mathcal{U}^\lambda(\mathfrak{g})$ is a categorification of V_λ.

12. Webster algebra

Webster introduced red strands in $\mathcal{U}^-(\mathfrak{g})$ for categorifying tensor preduct representations $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_m}$ of quantum group $U_q(\mathfrak{g})$.
• (Non-cyclotomic) Webster algebra $\tilde{T}^{\underline{\lambda}}(\mathfrak{g})$ is generated by colored oriented diagrams of $\mathcal{U}^-(\mathfrak{g})$ with red strands and the red-black crossings subject to some relations (omit).
Ex.



• Webster algebra $T^{\underline{\lambda}}(\mathfrak{g})$ of multiweight $\underline{\lambda}=(\lambda_1,\ldots,\lambda_m)$ is the quotient algebra $\tilde{T}^{\underline{\lambda}}(\mathfrak{g})/J$, where J is the 2-sided ideal generated by colored oriented strands whose the left-most strand is black.



Theorem (Webster) The projective module category of $T^{\underline{\lambda}}(\mathfrak{g})$ is a categorification of $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_m}$ of $U_q(\mathfrak{g})$.

Remark
When $m=1$, that is $\underline{\lambda}=\lambda$, $T^\lambda(\mathfrak{g})$ is isomorphic to $\mathcal{U}^\lambda(\mathfrak{g})$.

13. Deformed Webster algebra $W(\mathrm{s},k)$ of type A_1

It's enough to consider Webster algebra for getting a categorification of tensor product representation but we need a deformation for getting a categorification of intertwining operators (R-matices).
Definition (Khovanov-Lauda-Sussan-Y)
$m,k\in\mathbf{Z}_{\geq 0}$, $\mathbf{s}=(s_1,\ldots,s_m)\in\mathbf{Z}^m_{\geq 0}$. $\mathrm{Seq}(\mathbf{s},k)$ is the set of all sequences $\mathbf{i}=(i_1,\ldots,i_{m+k})$ in which

- k entries in sequence are $\mathfrak{b}$,
- s_i appears exactly once and in the order of $\mathbf{s}$,

where $\mathbf{i}_j$: the j-th entry of $\mathbf{i}$ and $I_{\mathbf{s}}$: the set $\{s_i|1\leq i\leq m\}$.
Let $W(\mathbf{s},k)$ be the algebra over $\Bbbk$ generated by

- $e(\mathbf{i})$, where $\mathbf{i}\in\mathrm{Seq}(\mathbf{s},k)$
- x_j, $E(d)_j$, where $1\leq j\leq m+k$, $d\geq 1$
- ψ_j, where $1\leq j\leq m+k-1$

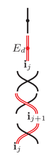
satisfying the following relations

- $e(\mathbf{i})e(\mathbf{j})=\delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i})$
- $E(d)_je(\mathbf{i})=e(\mathbf{i})E(d)_j$
- $E(d)_je(\mathbf{i})=0$ if $\mathbf{i}_j=\mathfrak{b}$
- $E(d)_je(\mathbf{i})=0$ if $d>\mathbf{i}_j\in I_{\mathbf{s}}$
- $x_je(\mathbf{i})=e(\mathbf{i})x_j$
- $x_je(\mathbf{i})=0$ if $\mathbf{i}_j\in I_{\mathbf{s}}$
- $x_jx_\ell=x_\ell x_j$
- $E(d)_jx_\ell=x_\ell E(d)_j$
- $E(d)_jE(d')_\ell=E(d')_\ell E(d)_j$
- $\psi_je(\mathbf{i})=e(\sigma_j(\mathbf{i}))\psi_j$
- $\psi_je(\mathbf{i})=0$ if $\mathbf{i}_j,\mathbf{i}_{j+1}\in I_{\mathbf{s}}$
- $\psi_jx_\ell=x_\ell\psi_j$ if $\ell\neq j$ and $\ell\neq j+1$
- $\psi_jE(d)_\ell=E(d)_\ell\psi_j$ if $\ell\neq j$ and $\ell\neq j+1$
- $\psi_jE(d)_j=E(d)_{j+1}\psi_j$
- $E(d)_j\psi_j=\psi_jE(d)_{j+1}$
- $x_j\psi_je(\mathbf{i})-\psi_jx_{j+1}e(\mathbf{i})=\delta_{\mathbf{i}_j,\mathbf{i}_{j+1}}e(\mathbf{i})$ unless $\mathbf{i}_j\in I_{\mathbf{s}}$ and $\mathbf{i}_{j+1}\in I_{\mathbf{s}}$
- $\psi_jx_je(\mathbf{i})-x_{j+1}\psi_je(\mathbf{i})=\delta_{\mathbf{i}_j,\mathbf{i}_{j+1}}e(\mathbf{i})$ unless $\mathbf{i}_j\in I_{\mathbf{s}}$ and $\mathbf{i}_{j+1}\in I_{\mathbf{s}}$
- $\psi_j\psi_\ell=\psi_\ell\psi_j$ if $|j-\ell|>1$
- $\psi_j^2e(\mathbf{i})=\begin{cases}0 & \text{if } \mathbf{i}_j=\mathbf{i}_{j+1}=\mathfrak{b},\\ \sum_{d=0}^{\mathbf{i}_j}(-1)^dE(d)_jx_{j+1}^{\mathbf{i}_j-d}e(\mathbf{i}) & \text{if } \mathbf{i}_j\in I_{\mathbf{s}},\mathbf{i}_{j+1}=\mathfrak{b},\\ \sum_{d=0}^{\mathbf{i}_{j+1}}(-1)^dx_j^{\mathbf{i}_{j+1}-d}E(d)_{j+1}e(\mathbf{i}) & \text{if } \mathbf{i}_j=\mathfrak{b},\mathbf{i}_{j+1}\in I_{\mathbf{s}},\end{cases}$
- $(\psi_j\psi_{j+1}\psi_j-\psi_{j+1}\psi_j\psi_{j+1})e(\mathbf{i})$
$=\begin{cases}\sum_{d_1+d_2+d_3=\mathbf{i}_{j+1}-1}(-1)^{d_3}x_j^{d_1}E(d_3)_{j+1}x_{j+2}^{d_2}e(\mathbf{i}) & \text{if } \mathbf{i}_{j+1}\in I_s,\mathbf{i}_j=\mathbf{i}_{j+2}=\mathfrak{b},\\ 0 & \text{otherwise,}\end{cases}$

14. Diagram description of $W(\mathrm{s},k)$

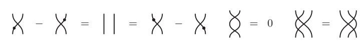
1. Generators $e(\mathbf{i})$ for $\mathbf{i}\in\mathrm{Seq}(\mathbf{s},k)$ represent $(m+k)$ vertical strands s.t.
   - j-th strand is black if $\mathbf{i}_j$ is $\mathfrak{b}$
   - j-th strand is red with $\mathbf{i}_j$-labelling if $\mathbf{i}_j$ is in $I_{\mathbf{s}}$
2. Dots on black j-strand correspond to generators x_j
3. E_d-dot on red j-strand corresponds to generators $E(d)_j$
4. Crossing of $(j,j+1)$-strands corresponds to a generator ψ_j

Generators:

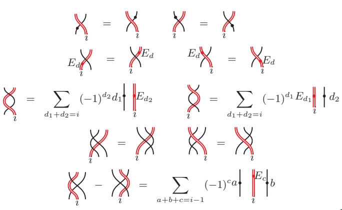


$x_je(\mathbf{i}) =$ if $\mathbf{i}_j=\mathfrak{b}$,
$E(d)_je(\mathbf{i}) =$ if $\mathbf{i}_j\in I_{\mathbf{s}}$ and $1\leq d\leq\mathbf{i}_j$,
$\psi_je(\mathbf{i}) =$ if $\mathbf{i}_j=\mathfrak{b}$, $\mathbf{i}_{j+1}=\mathfrak{b}$,
$\psi_je(\mathbf{i}) =$ if $\mathbf{i}_j=\mathfrak{b}$, $\mathbf{i}_{j+1}\in I_{\mathbf{s}}$,
$\psi_je(\mathbf{i}) =$ if $\mathbf{i}_j\in I_s$, $\mathbf{i}_{j+1}=\mathfrak{b}$.

Relations:
• Diagram isotopy relation
• Diagram relations of categorified quantum $\mathfrak{sl}_2$



• The following diagram relations:



15. Conclusion