Non-existence of epimorphisms between certain handlebody-knot groups

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§1 A preorder of handlebody-knots

$\left[\begin{array}{l} H: \text{genus } g \text{ handlebody-knot} \\ G(H) := \pi_1(S^3 - H): \text{(handlebody-)knot group of } H \end{array}\right.$

$$H_1 \geq H_2 \overset{\text{def}}{\Longleftrightarrow} {}^{\exists}\sigma : G(H_1) \longrightarrow G(H_2) : \text{epimorphism}$$

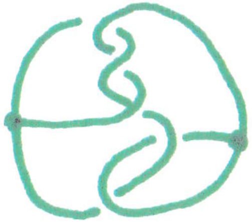
Prop. 1 $\geq$ is a preorder.
i.e. (i) $H_1 \geq H_1$, (ii) $H_1 \geq H_2, H_2 \geq H_3 \Rightarrow H_1 \geq H_3$

We also have: $H_1 \geq H_2, H_2 \geq H_1 \Rightarrow G(H_1) \cong G(H_2)$

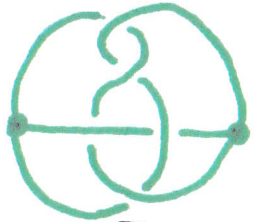
- In particular for prime knots ($g=1$),
  $\geq$ is a partial order.
- This partial order is determined up to 11 crossings.
  [Kitano-S '05 '11] [Horie-Kitano-Matsumoto-S '11]

∃ Table of irreducible genus 2 handlebody-knots up to 6 crossings. [Ishii-Kishimoto-Moriuchi-S '12]


4_1


5_1


5_2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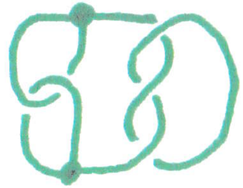
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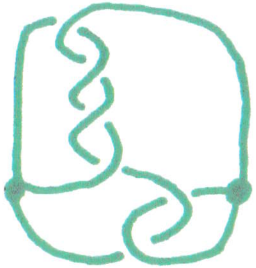
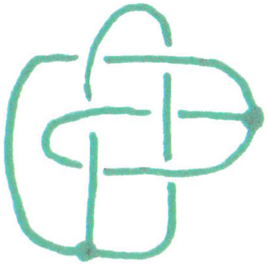

6_1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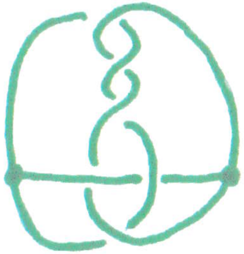
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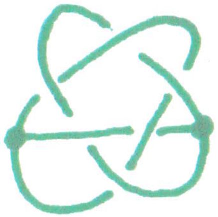
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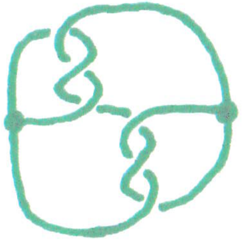
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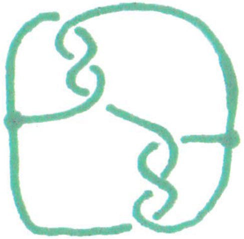
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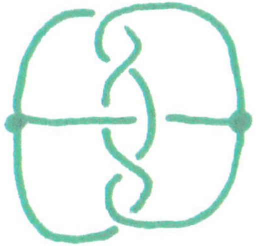
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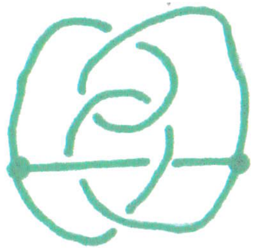
6_8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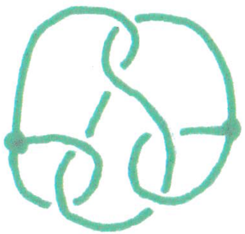
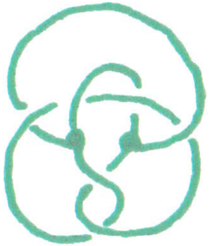
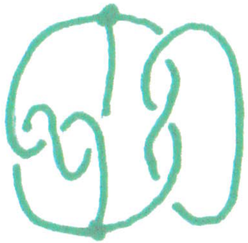
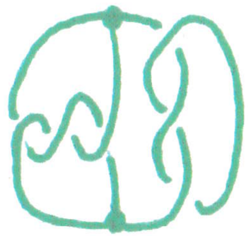

6_{10}


6_{11}


6_{12}


6_{13}

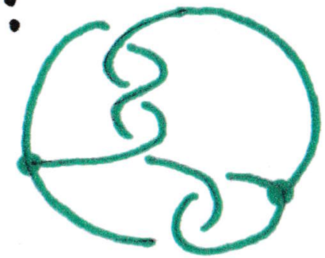
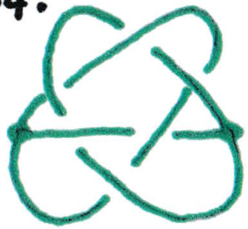
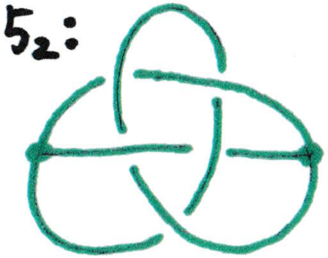
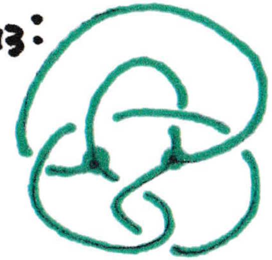

6_{14}


6_{15}

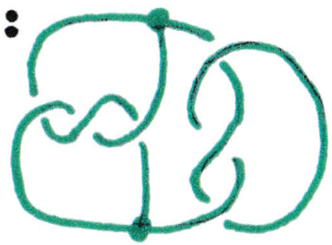

6_{16}

Remark

$\geq$ is **NOT** a partial order.

5_1:  $\not\cong$ 6_4:  , 5_2:  $\not\cong$ 6_{13}: 

[J.H. Lee – S. Lee '12]

6_{14}:  $\not\cong$ 6_{15}: 

[Ishii, Kishimoto, M. Ozawa '15]

Each of the pairs has the same knot group. $\square$

§2 (Twisted) Alexander invariants

$F_s = \langle x_1, x_2, \cdots, x_s \mid \phi \rangle$

$G = \langle x_1, x_2, \cdots, x_s \mid r_1, r_2, \cdots, r_t \rangle$: finitely pr. group

C : commutative group

$$\begin{tikzcd} F_s \arrow[r, "\varphi"] \arrow[dr] & G \arrow[d, "\psi"] \\ & C \end{tikzcd} \qquad \varphi : \text{cano. epi.}, \quad \psi : \text{homo.}$$

$$\Longrightarrow \quad \begin{tikzcd} \mathbb{Z}F_s \arrow[r, "\tilde{\varphi}"] \arrow[dr] & \mathbb{Z}G \arrow[d, "\tilde{\psi}"] \\ & \mathbb{Z}C \end{tikzcd}$$

$\frac{\partial r_i}{\partial x_j} \in \mathbb{Z}F_s$: Fox derivative

$\frac{\partial r_i}{\partial x_j} \mapsto \left(\frac{\partial r_i}{\partial x_j}\right)^{\tilde{\varphi}\tilde{\psi}} \in \mathbb{Z}C$

- $A(G, \tilde{\psi}) \overset{\text{def}}{=} \left(\left(\frac{\partial r_i}{\partial x_j}\right)^{\tilde{\varphi}\tilde{\psi}} \right)$: Alexander matrix of G asso. with ψ

- $E_d(A(G,\tilde{\psi})) \overset{\text{def}}{=} \begin{cases} (0) & (d < s-t) \\ ((s-d)\text{-minors}) & (s-t \leq d < s) \\ (1) & (d \geq s) \end{cases}$: d-th Alexander ideal of $A(G, \tilde{\psi})$

R : ring

$\rho : G \to GL(n;R)$: group rep. $\Rightarrow \tilde{\rho} : \mathbb{Z}G \to M_n(R)$

$$\begin{array}{ccc} \tilde{\rho}\otimes\tilde{\psi} : \mathbb{Z}G & \longrightarrow & M_n(RC) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sum m_i g_i & \longmapsto & \sum m_i \psi(g_i)\rho(g_i) \end{array} \quad (m_i \in \mathbb{Z},\ g_i \in G)$$

: tensor product homo. of $\tilde{\rho}$ and $\tilde{\psi}$

- $A(G, \tilde{\rho}\otimes\tilde{\psi}) \overset{\text{def}}{=} \left((\tilde{\rho}\otimes\tilde{\psi}) \circ \tilde{\varphi} \left(\frac{\partial r_i}{\partial x_j} \right) \right)$

: twisted Alexander matrix of G asso. with ρ and ψ

- $E_d(A(G, \tilde{\rho}\otimes\tilde{\psi})) \overset{\text{def}}{=} \begin{cases} (0) & (d < nt - ns) \\ ((nt-d)\text{-minors}) & (nt - ns \leq d < nt) \\ (1) & (d \geq nt) \end{cases}$

: d-th twisted Alexander ideal of G asso. with ρ and ψ

§3 Group epimorphism and Alexander ideal

Thm. 2

$$\left[\begin{array}{l} G_i : \text{finitely pr. group } (i=1,2) \\ C : \text{commutative group} \\ \psi_i : G_i \to C : \text{homo. } (i=1,2) \\ \rho_i : G_i \to GL(n;R) : \text{group rep. } (i=1,2) \end{array}\right.$$

If $\exists\, \sigma : G_1 \to G_2$: epi.

s.t.

$$\begin{tikzcd} G_1 \arrow[r, "\psi_1"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & C \\ G_2 \arrow[ur, "\psi_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G_1 \arrow[r, "\rho_1"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & GL(n;R) \\ G_2 \arrow[ur, "\rho_2"'] & \end{tikzcd}$$

$$\Rightarrow E_d(A(G_1, \tilde{\hat{\rho}}_1 \otimes \tilde{\hat{\psi}}_1)) \subset E_d(A(G_2, \tilde{\hat{\rho}}_2 \otimes \tilde{\hat{\psi}}_2))$$

Thm. 3

G_i : finitely pr. group $(i=1,2)$

$\exists\, \gamma_2 : G_2 \longrightarrow \mathbb{C}$: homo. and $\exists\, \rho_2 : G_2 \longrightarrow GL(n;R)$: group rep.

s.t. $Ed(A(G_1, \tilde{\rho}_1 \otimes \tilde{\gamma}_1)) \not\subset Ed(A(G_2, \tilde{\rho}_2 \otimes \tilde{\gamma}_2))$

for $\forall \gamma_1 : G_1 \to \mathbb{C}$ and $\forall \rho_1 : G_1 \longrightarrow GL(n;R)$: group rep.

$\Longrightarrow \nexists\, \sigma : G_1 \longrightarrow G_2$: epi.

$\because$ If $\exists\, \sigma : G_1 \longrightarrow G_2$: epi.

Then for

$$\begin{tikzcd} G_1 \arrow[r, "\gamma_1"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \\ G_2 \arrow[ur, "\gamma_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G_1 \arrow[r, "\rho_1"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & GL(n;R) \\ G_2 \arrow[ur, "\rho_2"'] & \end{tikzcd}$$

By. Thm. 2

$\Longrightarrow Ed(A(G_1, \tilde{\rho}_1 \otimes \tilde{\gamma}_1)) \subset Ed(A(G_2, \tilde{\rho}_2 \otimes \tilde{\gamma}_2))$

contradiction. $\square$

Cor. 4

$$\begin{cases} G_i : \text{finitely pr. group} \\ \psi_i : G_i \to C : \text{homo.} \end{cases} \quad (i=1,2)$$

If $\exists\, \sigma : G_1 \to G_2$: epi. s.t.
$$\begin{tikzcd} G_1 \arrow[r, "\psi_1"] \arrow[d, "\sigma"'] & C \\ G_2 \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

$\Rightarrow E_d(A(G_1, \tilde{\psi}_1)) \subset E_d(A(G_2, \tilde{\psi}_2))$

Remark

For knots, from Cor. 4, we also have

$$K_1 \geq K_2 \Rightarrow \Delta_{K_2}(t) \mid \Delta_{K_1}(t)$$

Cor. 5

G_i : finitely pr. group $(i=1,2)$

$\exists \psi_2 : G_2 \to \mathbb{C}$: homo.

s.t. $Ed(A(G_1, \tilde{\psi}_1))$ & $Ed(A(G_2, \tilde{\psi}_2))$

for $\forall \psi_1 : G_1 \to \mathbb{C}$: homo.

$\Rightarrow \nexists \sigma : G_1 \to G_2$: epi.

Prop. 6

H: genus g handlebody-knot

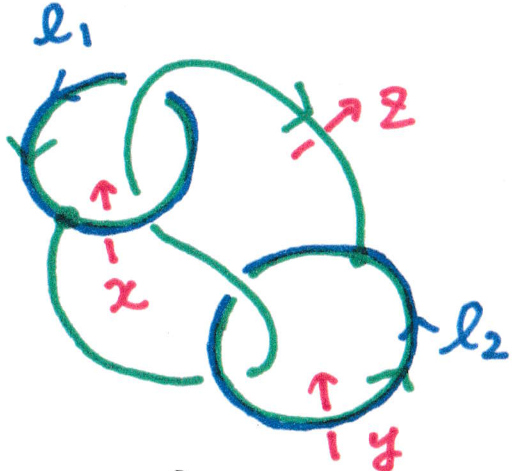
$\forall \psi : G(H) \to \langle t | \phi \rangle$, $\exists \ell \in H_1(H;\mathbb{Z})$

s.t. $\psi = \psi_\ell : G(H) \longrightarrow \langle t | \phi \rangle$

$$\begin{array}{ccc} G(H) & \longrightarrow & \langle t|\phi\rangle \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & t^{\ell k(\alpha(x), \ell)} \end{array}$$

ex

4_1 :



$$G(4_1) \cong \langle x, y, z \mid \underbrace{zyxzx^{-1}y^{-1}xz^{-1}x^{-1}}_{= r} \rangle$$

$l \overset{\text{def}}{=} c_1 l_1 + c_2 l_2$
$\in H_1(4_1)$
$(c_1, c_2 \in \mathbb{Z})$

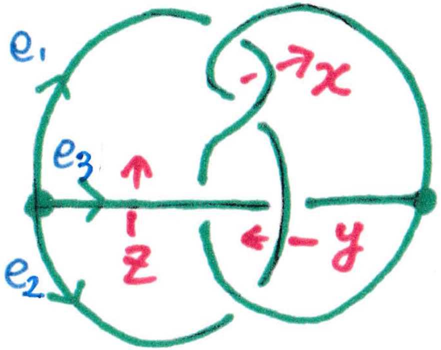
$$\psi_l : G(4_1) \longrightarrow \langle t \mid \phi \rangle$$

$$\begin{aligned} x &\longmapsto t^{c_1} \\ y &\longmapsto t^{c_2} \\ z &\longmapsto 1 \end{aligned}$$

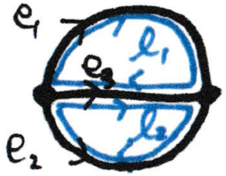
$$E_2(A(G(4_1), \psi_l)) = (1 - t^{c_1} + t^{c_1 + c_2})$$

- $c_1 = c_2 = 1$: $E_2 = (1 - t + t^2)$
- $c_1 = 1,\ c_2 = -1$: $E_2 = (2 - t)$

5_3:



$$\begin{cases} G(5_3) \cong \langle x, y, z \mid r \rangle \\ r = z x^{-1} z^{-1} x y^{-1} x^{-1} y z^{-1} y^{-1} \end{cases}$$



$l \overset{\text{def}}{=} c_1 l_1 + c_2 l_2 \in H_1(5_3)$ $(c_1, c_2 \in \mathbb{Z})$

$$\gamma_l : G(5_3) \longrightarrow \langle t \mid \phi \rangle$$
$$x \longmapsto t^{c_1}$$
$$y \longmapsto t^{c_2}$$
$$z \longmapsto t^{-c_1-c_2}$$

$$E_2(A(G(5_3), \gamma_l)) = (2, 1 - t^{c_1} + t^{c_1+c_2})$$

- If $(2, 1 - t^{c_1} + t^{c_1+c_2}) \subset (2-t)$,

$\exists f(t) \in \mathbb{Z}[t^{\pm 1}]$ s.t. $2 = f(t)(2-t)$

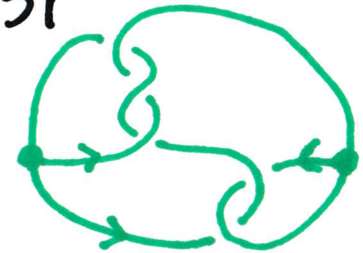
$t = 2$: $2 = 0$ $\rightarrow$ contradiction $\quad \therefore 5_3 \not\approx 4_1$

6_3

$\forall \psi : G(6_3) \to \langle t \mid \phi \rangle$,

$E_2(A(G(6_3), \tilde{\psi})) = (1)$

5_1



$\forall \psi : G(5_1) \to \langle t \mid \phi \rangle$,

$E_2(A(G(5_1), \tilde{\psi})) = (1)$

$\left[\begin{array}{l} \forall \psi : G(6_3) \to \langle t \mid t^2 \rangle \\ \forall \rho : G(6_3) \to SL(2; \mathbb{Z}/2\mathbb{Z}) \end{array}\right.$

$\left[\begin{array}{l} \exists \psi : G(5_1) \to \langle t \mid t^2 \rangle \\ \exists \rho : G(5_1) \to SL(2; \mathbb{Z}/2\mathbb{Z}) \end{array}\right.$

$E_4(A(G(6_3), \tilde{\rho} \otimes \tilde{\psi})) = (1) \not\subset E_4(A(G(5_1), \tilde{\rho} \otimes \tilde{\psi})) = (0)$

[Ishii-N-Oshiro'18] [INO'18]

$\therefore 6_3 \not\geq 5_1$

In a similar way,

$6_3, 6_8, 6_{10}, 6_{11} \not\geq 4_1, 5_1, 5_2, 5_3, 5_4, 6_1, 6_2, 6_4, 6_5, 6_6, 6_7, 6_9, 6_{12}, 6_{13}, 6_{14}, 6_{15}, 6_{16}$

§4 Other methods

0_1 4_1 6_1 6_7 6_{10}

Prop. 7 [Jaco-McMillan '70] + [S. Suzuki '84]

H : genus 2 handlebody-knot

$^{\exists}\sigma : G(H) \to F_2$: epi. $\Rightarrow E_2(A(G(H), {}^{\forall}\psi))$ is principal.

Prop. 8

(1) $4_1, 6_1, 6_7, 6_{10} \geq 0_1$

(2) $5_2, 5_3, 6_2, 6_5, 6_6, 6_8, 6_9, 6_{12}, 6_{13} \not\geq 0_1, 4_1, 6_1, 6_7, 6_{10}$

Prop. 9 $\exists \sigma : G(H_1) \to G(H_2)$: epi.
$\Rightarrow G$: finite group
$\#\{G(H_1) \to G : \text{homo.}\} \geq \#\{G(H_2) \to G : \text{homo.}\}$

Prop. 9' $\exists G$: finite group
s.t. $\#\{G(H_1) \to G : \text{homo.}\} < \#\{G(H_2) \to G : \text{homo.}\}$
$\Rightarrow \nexists \sigma : G(H_1) \to G(H_2)$: epi.

- $SL(2; \mathbb{Z}/p\mathbb{Z})$ ($p \leq 17$, prime)
- $SL(3; \mathbb{Z}/q\mathbb{Z})$ ($q = 2, 3$)
- non-commutative finite simple groups of order ≤ 4000.

Unsolved pairs

$(4_1,5_1),(4_1,5_2),$

$(4_1,5_3),(4_1,6_2),(4_1,6_3),(4_1,6_5),(4_1,6_6),(4_1,6_8),(4_1,6_{11}),(5_2,5_1),(5_2,5_3),$

$(5_2,6_2),(5_2,6_3),(5_2,6_5),(5_2,6_6),(5_2,6_8),(5_2,6_{11}),(5_3,6_3),(5_3,6_6),(5_4,0_1),$

$(5_4,5_1),(5_4,6_3),$ $(5_4,6_{10}),(5_4,6_{11}),(6_1,5_1),(6_1,5_3),(6_1,6_2),(6_1,6_3),$

$(6_1,6_5),(6_1,6_6),(6_1,6_8),(6_1,6_{11}),(6_2,6_3),(6_2,6_5),(6_2,6_6),(6_6,6_5),(6_7,5_1),$

$(6_7,5_3),(6_7,6_1),(6_7,6_2),(6_7,6_3),(6_7,6_5),(6_7,6_6),(6_7,6_8),(6_7,6_{11}),(6_9,5_1),$

$(6_9,5_3),(6_9,6_2),(6_9,6_3),(6_9,6_5),(6_9,6_6),(6_9,6_8),(6_9,6_{11}),(6_{10},6_3),(6_{10},6_{11}),$

$(6_{12},5_1),(6_{12},5_3),(6_{12},6_2),(6_{12},6_3),(6_{12},6_5),(6_{12},6_6),(6_{12},6_8),(6_{12},6_{11}),(6_{14},0_1),$

$(6_{14},5_1),(6_{14},5_2),(6_{14},5_3),(6_{14},6_1),(6_{14},6_2),(6_{14},6_3),(6_{14},6_5),(6_{14},6_6),(6_{14},6_8),$

$(6_{14},6_9),(6_{14},6_{10}),(6_{14},6_{11}),(6_{16},0_1),(6_{16},5_1),(6_{16},6_3),$ $(6_{16},6_{10}),(6_{16},6_{11}).$