Generalization of the Conway-Gordon theorems and intrinsic knotting and linking on complete graphs

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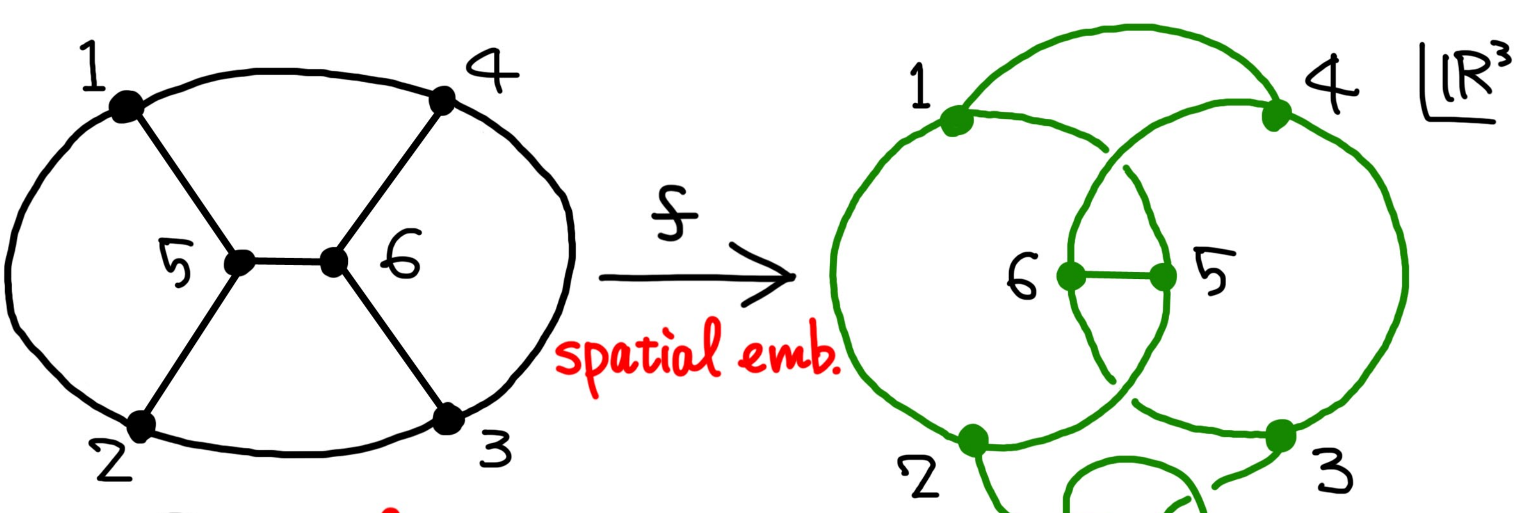
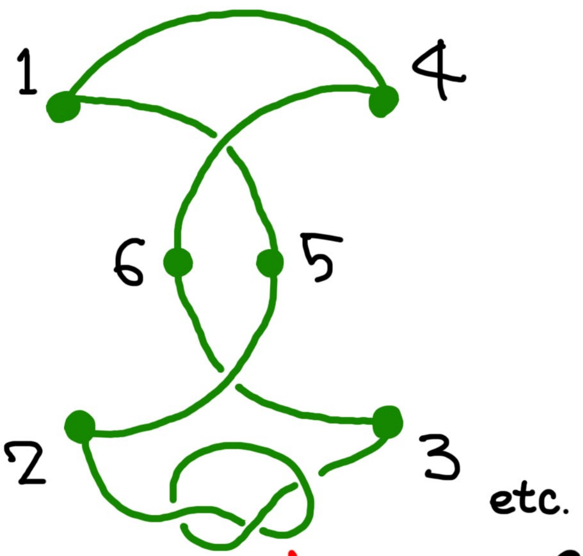
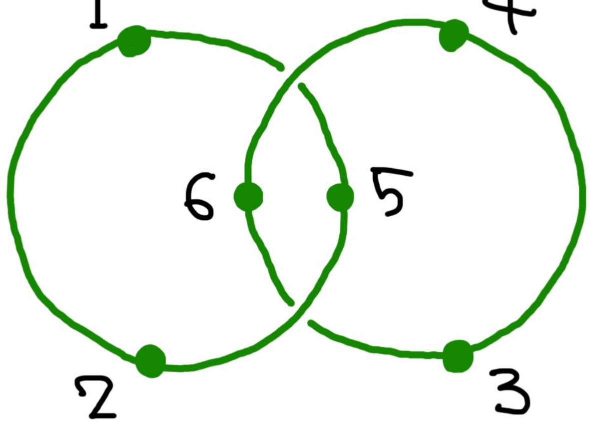


G: graph

f(G): spatial graph of G



constituent knot in f(G)



constituent link in f(G)

Notation

- $SE(G) \overset{\text{def}}{=} \{ f : G \to \mathbb{R}^3 \text{ sp. emb.} \}$

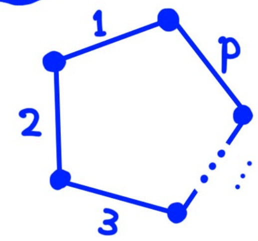
- $\Gamma(G) \overset{\text{def}}{=} \{ \text{cycle of } G \}$

  $\cup$

- $\Gamma_p(G) \overset{\text{def}}{=} \{ p\text{- cycle of } G \}$

(cycle $\approx$ ○

p-cycle =



)

- $\Gamma_{p.q}(G) \overset{\text{def}}{=} \{ \lambda = \gamma \cup \gamma' \mid \gamma \in \Gamma_p(G), \gamma' \in \Gamma_q(G), \gamma \cap \gamma' = \phi \}$

- $a_2(\cdot)$: 2nd coefficient of the Conway polynomial $\nabla(z)$

- $lk(\cdot)$: linking number in $\mathbb{R}^3$

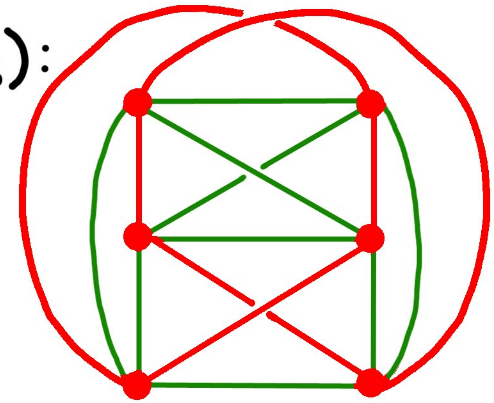
- K_n : complete graph on n vertices

Thm. 1 [Conway-Gordon '83]

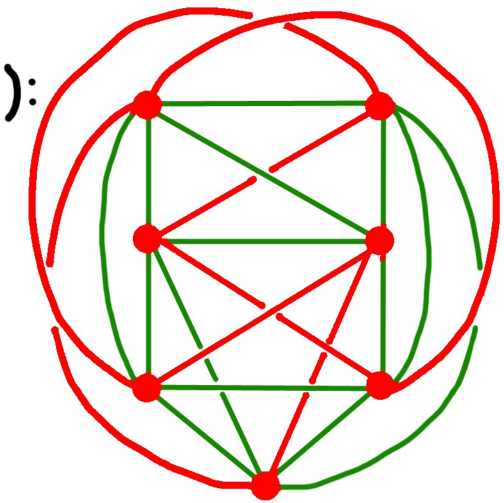
(1) $\forall f \in SE(K_6)$, $\displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_6)} lk(f(\lambda)) \equiv 1 \pmod{2}$

(2) $\forall f \in SE(K_7)$, $\displaystyle\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) \equiv 1 \pmod{2}$

$f(K_6)$:



$f(K_7)$:



Our result:

We succeeded in generalizing the Conway-Gordon theorems to complete graphs with arbitrary number of vertices ≥ 6.

Integral lifts of the Conway-Gordon theorem:

Thm.3 [Nikkuni '09]

(1) $\forall f \in SE(K_6)$, $\displaystyle\sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) = \frac{1}{2} \sum_{\lambda \in \Gamma_{3,3}(K_6)} lk(f(\lambda))^2 - \frac{1}{2}$

(2) $\forall f \in SE(K_7)$,

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 2\sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma)) = \frac{1}{7}\left(2\sum_{\lambda \in \Gamma_{3,4}(K_7)} lk(f(\lambda))^2 + 3\sum_{\lambda \in \Gamma_{3,3}(K_7)} lk(f(\lambda))^2\right) - 6$$

☆ We generalize Thm.3(1) to K_n : (MAIN Thm.1)

Thm.4 [MN] $n \geq 6$, $\forall f \in SE(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2(f(\gamma))$$

$$= \frac{(n-5)!}{2}\left\{\sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f(\lambda))^2 - \binom{n-1}{5}\right\}$$

Cor.5 $n \geq 6$, ${}^{\forall} f \in SE(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2(f(\gamma)) \geq \frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!}$$

$$\left(\because \sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f(\lambda))^2 \geq \binom{n}{6} \right)$$

Remark [Otsuki '96]

f_b: "canonical book presentation" [Endo-Otsuki '94]

${}^{\exists} f_b(K_n) \in SE(K_n)$ $(n \geq 6)$

s.t. $f_b(K_n) \supset {}^{\exists}$exactly $\binom{n}{6}$ $(3,3)$-Hopf links

as nonsplittable $(3,3)$-links.

$\therefore$ Cor.5 is sharp.

Cor. 6 $n \geq 7$, ${}^\forall f \in SE(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv \begin{cases} -\dfrac{(n-5)!}{2}\dbinom{n-1}{5} & (n \equiv 0 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8) \\ \dfrac{(n-5)!}{2}\dbinom{n}{6} & (n \equiv 7 \pmod 8) \end{cases} \pmod{(n-5)!}$$

Remark

$(n=7)$ • $\displaystyle\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) \equiv \frac{(7-5)!}{2}\binom{7}{6} \equiv 1 \pmod 2$ [CG'83]

$(n=8)$ • $\displaystyle\sum_{\gamma \in \Gamma_8(K_8)} a_2(f(\gamma)) \equiv -\frac{(8-5)!}{2}\binom{7}{5} \equiv -63 \equiv 3 \pmod 6$

[Foisy'07]+[Hirano'10]

$(n \geq 9)$ • $\displaystyle\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv 0 \pmod 2$ [Hirano'10]

Thm. 7 [Kazakov-Korablev '14]
$n \geq 6$, ${}^\forall f \in SE(K_n)$,

$$\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} lk(f(\lambda)) \equiv \begin{cases} 1 & (n=6) \\ 0 & (n \geq 7) \end{cases} \pmod 2$$

☆ We refine Thm. 7 : (MAIN Thm. 2)

Thm. 8 [MN] $n \geq 6$, ${}^\forall f \in SE(K_n)$, $n = p+q$,

$$\sum_{\lambda \in \Gamma_{p,q}(K_n)} lk(f(\lambda))^2 = \begin{cases} (n-6)! \sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f(\lambda))^2 & (p=q) \\ 2(n-6)! \sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f(\lambda))^2 & (p \neq q) \end{cases}$$

Cor. 9 $n \geq 6$, ${}^\forall f \in SE(K_n)$, $n = p + q$,

$$\sum_{\lambda \in \Gamma_{p,q}(K_n)} lk(f(\lambda))^2 \geq \begin{cases} \dfrac{n!}{6!} & (p = q) \\ 2 \times \dfrac{n!}{6!} & (p \neq q) \end{cases}$$

In particular, $\displaystyle\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} lk(f(\lambda))^2 \geq (n-5)\frac{n!}{6!}$

Cor. 10 $n \geq 6$, ${}^\forall f \in SE(K_n)$,

$$\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} lk(f(\lambda))^2 = (n-5)! \sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f(\lambda))^2$$

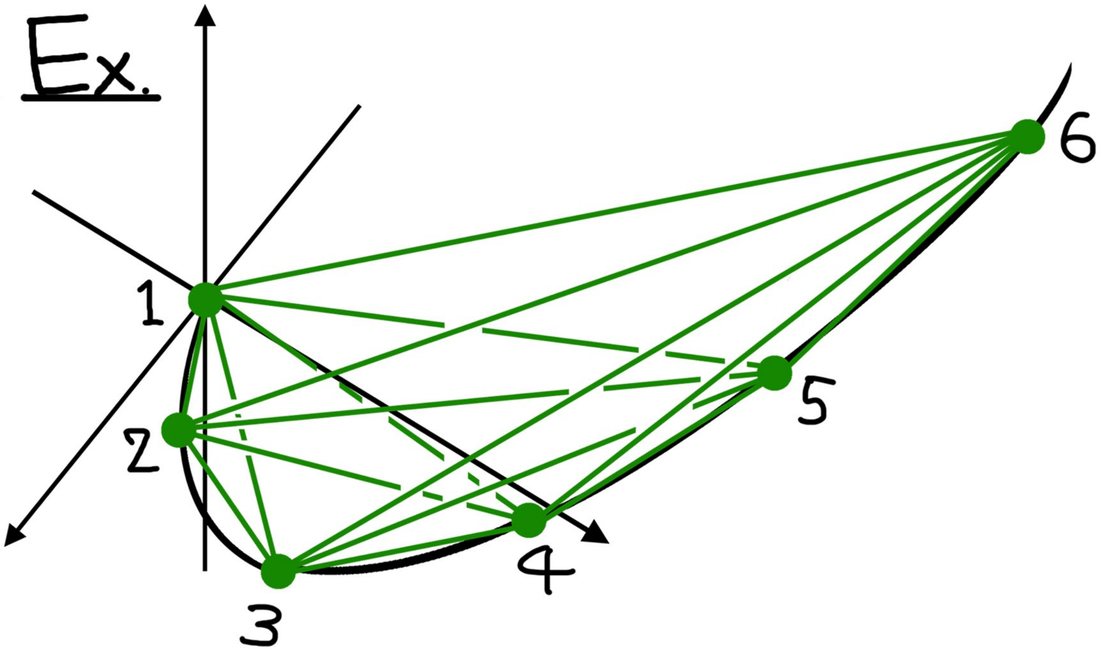
In particular, $\displaystyle\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} lk(f(\lambda))^2 \equiv 0 \pmod{(n-5)!}$

☆ Application for rectilinear spatial graphs:

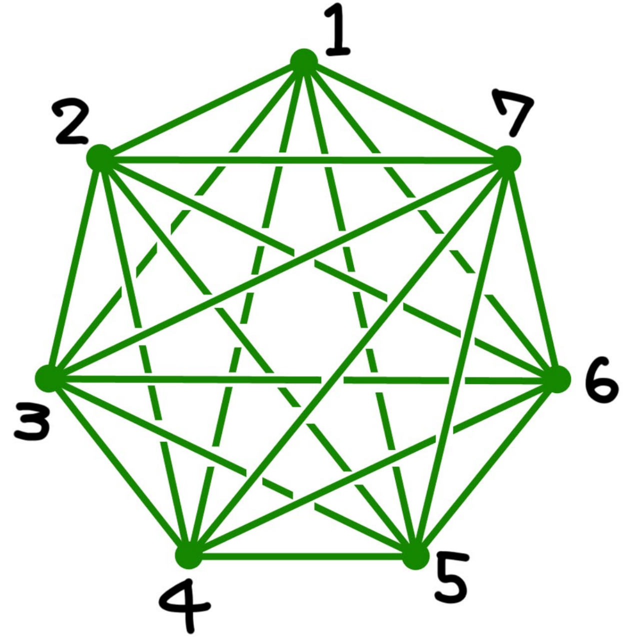
Def. $f_r \in SE(K_n)$ is rectilinear

$\overset{\text{def}}{\iff}$ ${}^\forall$edge in $f_r(K_n)$ is a straight line segment in $\mathbb{R}^3$.

Ex.



$f_r(K_6)$



$f_r(K_7)$

$RSE(K_n) \overset{\text{def}}{=} \{$rectilinear spatial embedding of $K_n\}$

Cor. 11 $n \geq 6$, $^\forall f_r \in RSE(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_r(\gamma)) = \frac{(n-5)!}{2}\left\{ \sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f_r(\lambda))^2 - \binom{n-1}{5} \right\}$$

(∵ Every five stick knot is trivial.)

Cor. 12 $n \geq 6$, $^\forall f_r \in RSE(K_n)$,

$$\frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!} \leq \sum_{\gamma \in \Gamma_n(K_n)} a_2(f_r(\gamma)) \leq \frac{3(n-2)(n-5)(n-6)!}{2 \cdot 6!}$$

(∵ $^\forall f_r \in RSE(K_n)$, $f_r(K_6) \supset$ at most 3 Hopf links.

[Hughes '06]

∴ $\binom{n}{6} \leq \sum_{\lambda \in \Gamma_{3,3}(K_n)} lk(f_r(\lambda))^2 \leq 3\binom{n}{6}$

[Huh-Jeon '07]

[Nikkuni '09])

Cor. 13 $n \geq 7$, ${}^\forall f_r \in RSE(K_n)$,
$f_r(K_n) \supset$ at least

$$r_n = \left\lceil \frac{(n-5)(n-6)(n-1)! \,/\, 2 \cdot 6!}{\lfloor (n-3)^2(n-4)^2 \,/\, 32 \rfloor} \right\rceil$$

non trivial Hamiltonian knots with $a_2 > 0$.
(n-cycle knot)

$\because$ $c(K)$: crossing number, $s(K)$: stick number

[Calvo '01]
$$c(K) \leq \frac{(s(K)-3)(s(K)-4)}{2}$$

[Polyak-Viro '01]
$$a_2(K) \leq \frac{c(k)^2}{8}$$

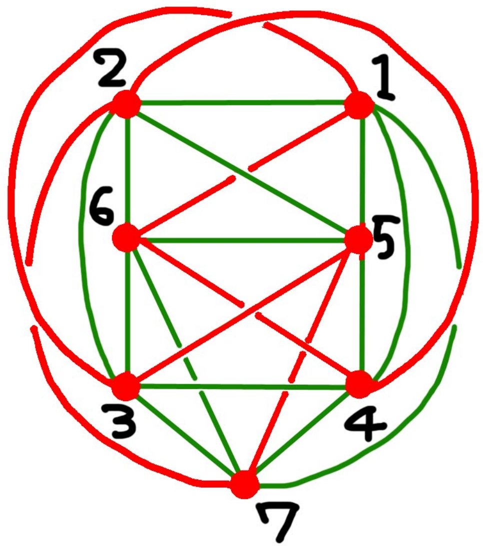
$\Rightarrow$ K: n stick knot
$$a_2(K) \leq \left\lfloor \frac{(n-3)^2(n-4)^2}{32} \right\rfloor$$

n	7	8	9	10	11	12	13	14	15	$\cdots$
r_n	1	2	12	92	772	7187	73628	823680	10015889	$\cdots$

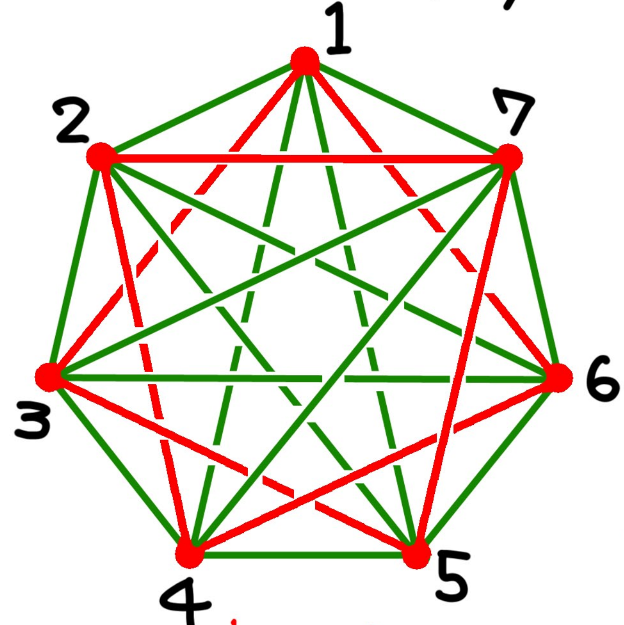
Examples $f \in SE(K_n)$, $f_r \in RSE(K_n)$.

(n=7)
$$\begin{cases} 1 \le \sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) \ (\le 15 \text{ if } f = f_r) \\ 14 \le \sum_{\lambda \in \Gamma_{3,4}(K_7)} lk(f(\lambda))^2 \ (\le 42 \text{ if } f = f_r) \end{cases}$$

[CG '83]



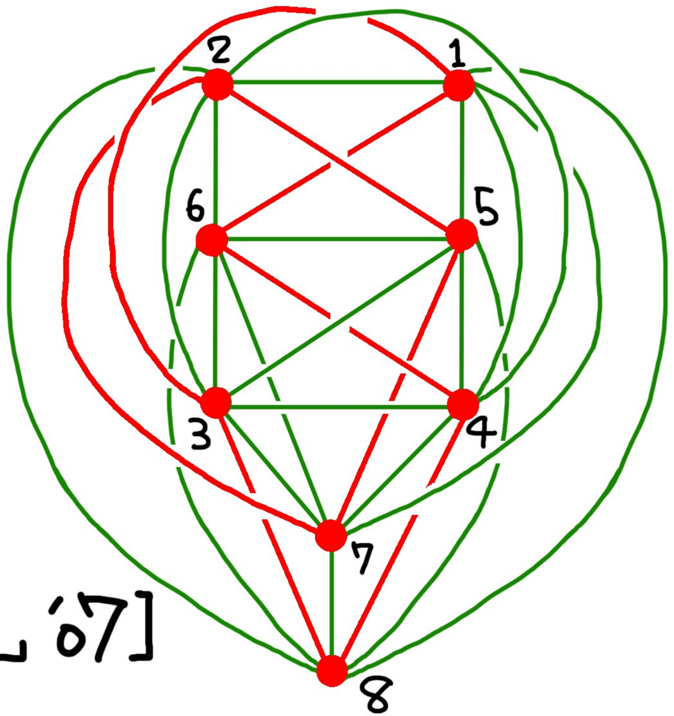
$\cong$



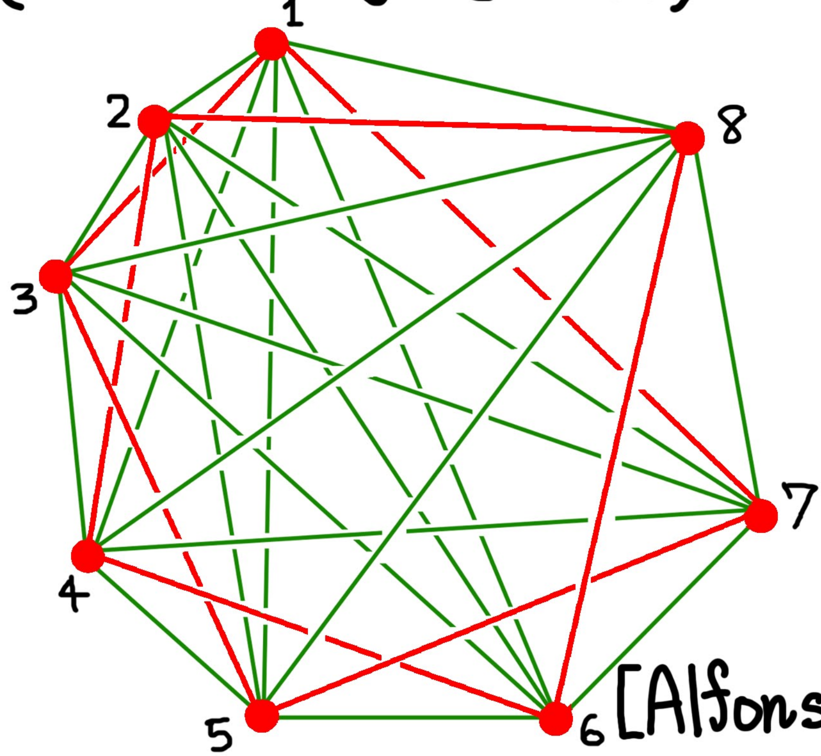
[Brown '77]
[Alfonsín '99]

- $\sum_{\gamma \in \Gamma_7(K_7)} a_2(f_r(\gamma)) \ge 1 \Rightarrow {}^{\exists}\gamma \in \Gamma_7(K_7)$ s.t. $a_2(f_r(\gamma)) > 0$ $\overset{\text{stick number}}{\Rightarrow}$ $f_r(\gamma) \cong$ trefoil
- [Jeon, Jin, et al. '10] $\sum_{\gamma \in \Gamma_7(K_7)} a_2(f_r(\gamma)) \le 11$?

(n=8)
$$\left[\begin{array}{l} 21 \leq \sum_{\gamma \in \Gamma_8(K_8)} a_2(f(\gamma)) \ (\leq 189 \text{ if } f = f_r) \\ 56 \leq \sum_{\lambda \in \Gamma_{4,4}(K_8)} lk(f(\lambda))^2 \ (\leq 168 \text{ if } f = f_r) \\ 112 \leq \sum_{\lambda \in \Gamma_{3,5}(K_8)} lk(f(\lambda))^2 \ (\leq 336 \text{ if } f = f_r) \end{array}\right.$$



[BBFHL '07]



[Alfonsin '08]

$\exists f \in SE(K_8)$ s.t. $f(K_8) \supset \exists$ exactly 21 trefoil knots as non-triv. Ham. knots

Problems

(1) $n \geq 7$, $f_r \in RSE(K_n)$,

Determine the sharp upper bound of $\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_r(\gamma))$.

(2) $n \geq 8$, $f \in SE(K_n)$ $(f_r \in RSE(K_n))$

Determine the minimum number of nontrivial Hamiltonian knots.