Gap between the alternation number and the dealternating number

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研究集会「結び目の数理」

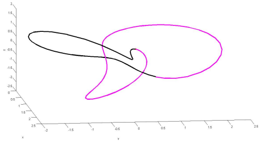
Waseda University, Japan

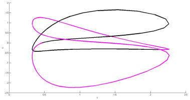
December 25, 2018

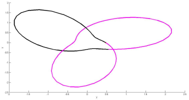
Introduction

Definition

A link is a disjoint union of circles embedded in S^3, a knot is a link with one component.



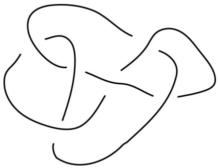




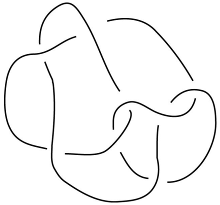
Introduction

Definition

A knot that possesses an alternating diagram is called an **alternating knot**, *otherwise it is called a non-alternating knot.*



Alternating diagram



Non-alternating diagram

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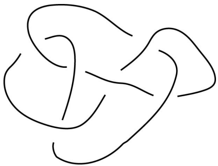
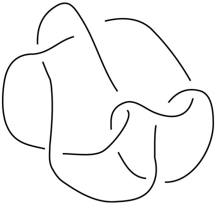


Alternating diagram



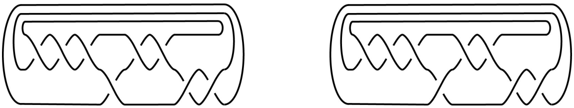
Non-alternating diagram

In 2015 Greene and Howie,independently, gave a characterization of alternating links.

Definition (Adams et al., 1992)

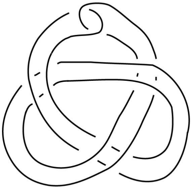
The dealternating number of a link diagram D is the minimum number of crossing changes necessary to transform D into an alternating diagram. The dealternating number of a link L, denoted dalt(L), is the minimum dealternating number of any diagram of L.

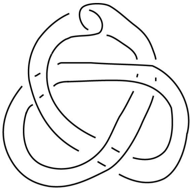
A link with dealternating number k is also called *k-almost alternating*. We say that a link is *almost alternating* if it is 1-almost alternating.



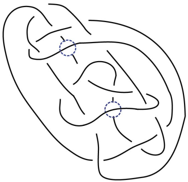
Definition (Kawauchi, 2010)

The alternation number of a link diagram D is the minimum number of crossing changes necessary to transform D into some (possibly non-alternating) diagram of an alternating link.
The alternation number of a link L, denoted alt(L), is the minimum alternation number of any diagram of L.



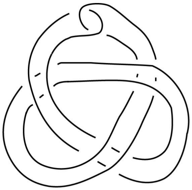


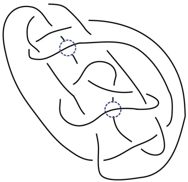
$alt(L) = 1$



$dalt(L) = 2$

$alt(L) \leq dalt(L)$





$alt(L) = 1$ $\qquad$ $dalt(L) = 2$

$$alt(L) \leq dalt(L)$$

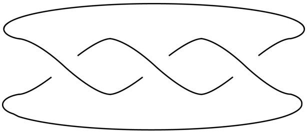
Adams et al. showed that an almost alternating knot is either a torus knot or a hyperbolic knot.

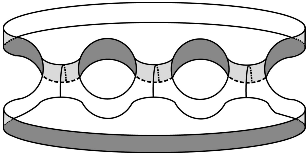
Turaev genus

To a link diagram D, Turaev associated a closed orientable surface embedded in S^3, called the *Turaev surface*.

Definition (Turaev, 1987)

The Turaev genus, $g_T(L)$, of a link L is the minimal number of the genera of the Turaev surfaces of diagrams of L.





[Dasbach et al., 2008] $g_T(L) = 0$ if and only if L is alternating.

Khovanov homology [Khovanov, 2000]

Let $L \in S^3$ be an oriented link. The Khovanov homology of L, denoted $Kh(L)$, is a bigraded $\mathbb{Z}$-module with homological grading i and polynomial (or Jones) grading j so that $Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L)$.

$j \diagdown i$	-4	-3	-2	-1	0	1	2
7							1
5							
3					1	1	
1				1	1		
-1				1	1		
-3		1	1				
-5							
-7	1						

The coefficients of the monomials $t^i q^j$ are shown.

$j - 2i = s + 1$ or $j - 2i = s - 1$, where $s = 2$ is the signature of 9_{42}.

$w_{Kh}(K)$

$\delta = j - 2i$ so that $Kh(L) = \bigoplus_{\delta} Kh^{\delta}(L)$.

Let δ_{min} be the minimum δ-grading where $Kh(L)$ is nontrivial and δ_{max} be the maximum δ-grading where $Kh(L)$ is nontrivial.

$Kh(L)$ is said to be $[\delta_{min}, \delta_{max}]$-thick, and the Khovanov width of L is defined as

$$w_{Kh}(L) = \frac{1}{2}(\delta_{max} - \delta_{min}) + 1.$$

$$alt(K) \leq dalt(K). \tag{1}$$

$$g_T(K) \leq dalt(K). \tag{2}$$

$$w_{Kh}(K) - 2 \leq g_T(K). \tag{3}$$

$$\widehat{w_{HF}(K)} - 1 \leq g_T(K). \tag{4}$$

(2) [Abe and Kishimoto, 2010];
(3)[Champanerkar et al., 2007] and [Champanerkar and Kofman, 2009];
(4)[M. Lowrance, 2008].

$$\frac{|\sigma(K) - s(K)|}{2} \leq alt(K). \tag{5}$$

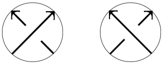
$$\frac{|\sigma(K) - s(K)|}{2} \leq g_T(K). \tag{6}$$

Skein relation

$$0 \leq \sigma(K_+) - \sigma(K_-) \leq 2. \tag{7}$$

$$0 \leq s(K_+) - s(K_-) \leq 2. \tag{8}$$

where $\sigma(K)$ and $s(K)$ are the signature and Rasmussen s-invariant of a knot K, respectively, and both invariants are equal to 2 for the positive trefoil knot.



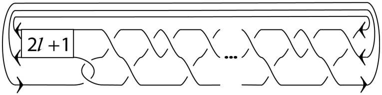
(5) [Abe, 2009];
(6)[Dasbach and Lowrance, 2011];
(7) [Cochran and Lickorish, 1986];
(8) [Rasmussen, 2010].

$alt(K)$ and $dalt(K)$

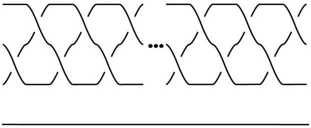
[Abe and Kishimoto, 2010] Examples where the alternation number equals the dealternating number.

[Lowrance, 2015] For all $n \in \mathbb{N}$ there exists a knot K, which is the iteration of Whitehead doubles of eight figure-eight knot, such that $alt(K) = 1$ and $n \leq dalt(K)$.

[Guevara-Hernández, 2017] For all $n \in \mathbb{N}$ there exist a knot family $\mathcal{DS}_n$ such that if $K \in \mathcal{DS}_n$ then $alt(K) = 1$ and $dalt(K) = n$.



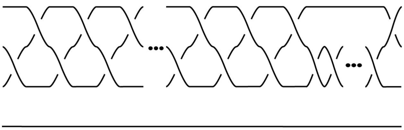
Families of knots



$$N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^l\sigma_3^{-1}(\sigma_1\sigma_2)^{3n}\cdot c)$$

where $l, m, n \in \mathbb{N}$.

Families of knots



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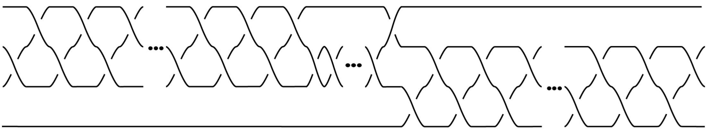

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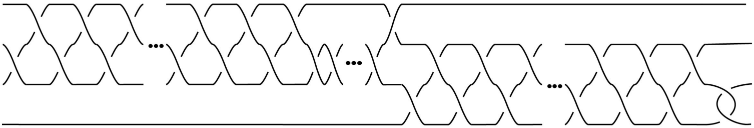

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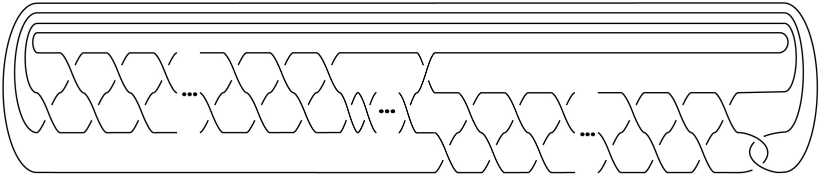

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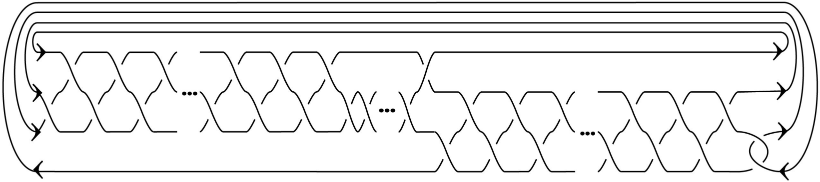


$$N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^l\sigma_3^{-1}(\sigma_1\sigma_2)^{3n}\cdot c)$$

where $l, m, n \in \mathbb{N}$.

$w_{Kh}(K) - 2 \leq dalt(K)$

Theorem (Khovanov, 2010)

There are long exact sequences

$$\cdots Kh^{i-e-1,j-3e-2}(D_h) \to Kh^{i,j}(D_+) \to Kh^{i,j-1}(D_v) \to Kh^{i-3,j-3e-2}(D_h) \to \cdots$$

and

$$\cdots Kh^{i,j+1}(D_v) \to Kh^{i,j}(D_-) \to Kh^{i-e+1,j-3e+2}(D_h) \to Kh^{i+1,j+1}(D_v) \to \cdots$$

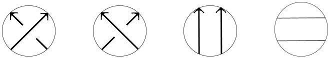
When only the $\delta = j - 2i$ grading is considered, the long exact sequence become

$$\cdots Kh^{\delta-e}(D_h) \xrightarrow{f_+^{\delta-e}} Kh^{\delta}(D_+) \xrightarrow{g_+^{\delta}} Kh^{\delta-1}(D_v) \xrightarrow{f_+^{\delta-1}} Kh^{\delta-e-2}(D_h) \to \cdots$$

and

$$\cdots Kh^{\delta+1}(D_v) \xrightarrow{f_-^{\delta+1}} Kh^{\delta}(D_-) \xrightarrow{g_-^{\delta}} Kh^{\delta-e}(D_h) \xrightarrow{h_-^{\delta-e}} Kh^{\delta-1}(D_v) \to \cdots$$

$e = neg(D_h) - neg(D_+)$



The crossings D_+, D_-, D_v, D_h, respectively.

Corollary

Let D_+, D_-, D_v and D_h be as above. Suppose $Kh(D_v)$ is $[v_{min}, v_{max}]$-thick and $Kh(D_h)$ is $[h_{min}, h_{max}]$-thick. Then $Kh(D_+)$ is $[\delta^+_{min}, \delta^+_{max}]$-thick, and $Kh(D_-)$ is $[\delta^-_{min}, \delta^-_{max}]$-thick, where

$$\delta^+_{min} = \begin{cases} min\{v_{min}+1, h_{min}+e\} & \text{if } v_{min} \neq h_{min}+e+1 \\ v_{min}+1 & \text{if } v_{min} = h_{min}+e+1 \text{ and } h_+^{v_{min}} \text{ is surjective} \\ v_{min}-1 & \text{if } v_{min} = h_{min}+e+1 \text{ and } h_+^{v_{min}} \text{ is not surjective,} \end{cases}$$

$$\delta^+_{max} = \begin{cases} max\{v_{max}+1, h_{max}+e\} & \text{if } v_{max} \neq h_{max}+e+1 \\ v_{max}-1 & \text{if } v_{max} = h_{max}+e+1 \text{ and } h_+^{v_{max}} \text{ is injective} \\ v_{max}+1 & \text{if } v_{max} = h_{max}+e+1 \text{ and } h_+^{v_{max}} \text{ is not injective,} \end{cases}$$

$$\delta^-_{min} = \begin{cases} min\{v_{min}-1, h_{min}+e\} & \text{if } v_{min} \neq h_{min}+e-1 \\ v_{min}+1 & \text{if } v_{min} = h_{min}+e-1 \text{ and } h_-^{v_{min}} \text{ is surjective} \\ v_{min}-1 & \text{if } v_{min} = h_{min}+e-1 \text{ and } h_-^{v_{min}} \text{ is not surjective,} \end{cases}$$

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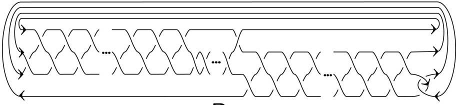
Lemma (G.)

If $D = N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^l\sigma_3^{-1}(\sigma_1\sigma_2)^{3n}\cdot c)$, *then* $Kh(D)$ *is* $[4m+l+2, 6m+2n+l+4]$*-thick. Hence,* $w_{Kh}(D) = m+n+2$.

Proof. (outline)



D_+

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If $D = N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^l\sigma_3^{-1}(\sigma_1\sigma_2)^{3n} \cdot c)$*, then* $Kh(D)$ *is* $[4m + l + 2, 6m + 2n + l + 4]$*-thick. Hence,* $w_{Kh}(D) = m + n + 2$*.*

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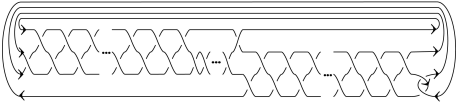

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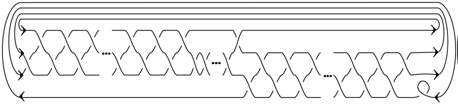
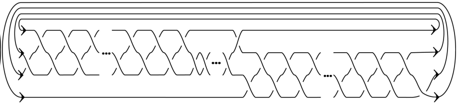


D_v



D_h

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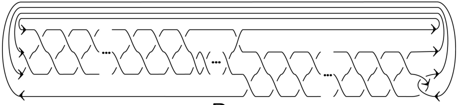

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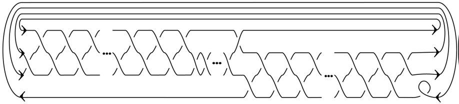
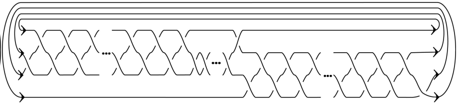
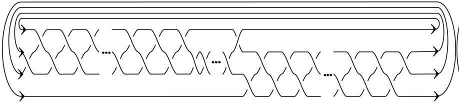


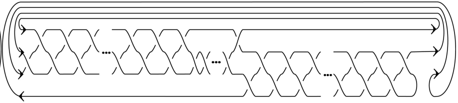
D_v



D_h



D_{h_v}

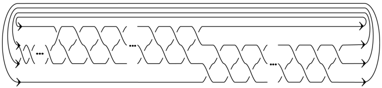


D_{h_h}

Lemma (G.)

Let D be the closure of the 3-braid $(\sigma_2\sigma_3)^{3k}\sigma_2^r\sigma_3^{-1}(\sigma_1\sigma_2)^{3n}$ with $k, r, m \in \mathbb{N}$ and $k \geq 2$, then $Kh(D)$ is $[4(k+n)+r-3, 6(k+n)+r-3]$-thick.

Proof. Induction over n by using the braid $\sigma_2^r\sigma_3^{-1}(\sigma_2\sigma_3)^{3k}(\sigma_1\sigma_2)^{3n}$. □



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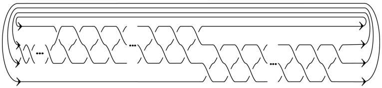
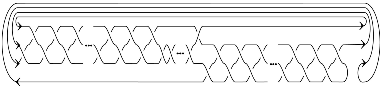
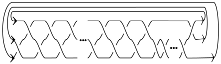


Proposition (Lowrance, 2009)

Let D be the closure of the braid $(\sigma_1\sigma_2)^{3k}\sigma_1^a\sigma_2^{-1}$ where a and k are positive integers. Then $Kh(D)$ is $[4k+a-2, 6k+a-2]$-thick.





$Kh(D_v^*)$ is $[4(m+n)+l+1, 6(m+n)+l+3]$-thick
$neg(D_v) = 4n+1$ and $neg(D_v^*) = 1$ $Kh^\delta(D_v) \cong Kh^{\delta+s}(D_v^*)$.
Therefore $Kh(D_v)$ is $[4m+l+1, 6m+l+3]$-thick.

Note that $D_{h_v} = D_v^*$ and $Kh(D_{h_h})$ is $[4m+l+2, 6m+l+4]$-thick.
$neg(D_{h_h}) - neg(D_h) = 4n+1-1$.
Then, $Kh(D_h)$ is $[4(m+n)+l+2, 6(m+n)+l+4]$-thick.

$e = neg(D_h) - neg(D_+) = -4n$,
since $4m+l+1 \neq (4(m+n)+l+2)+e+1$ and
$6m+l+3 \neq (6(m+n)+l+4)+e+1$
It implies that $Kh(D_+)$ is $[4m+l+2, 6m+2n+l+4]$-thick. Hence,
$w_{Kh}(N(D)) = m+n+2$.

□

Theorem (G.)

For all pair m, n of positive integers there exists a family of knots

$$\mathcal{F}^{m,n} = \{N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^l\sigma_3^{-1}(\sigma_1\sigma_2)^{3n} \cdot c) \,|\, l \in \mathbb{N}, l \text{ is odd.}\}$$

such that, if $K \in \mathcal{F}^{m,n}$ *then*

$$dalt(K) = m + n \quad \text{and} \quad m - 1 \leq alt(K) \leq m + 1.$$

Proof.

Due to the previous lemma we have that

$$w_{Kh}(K) = m + n + 2.$$

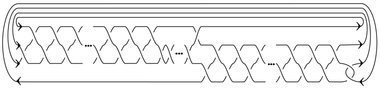
Beside,

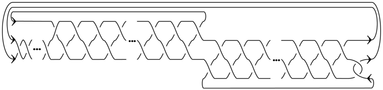
$$w_{Kh}(K) - 2 \leq g_T(K) \leq dalt(K).$$

It follows that

$$m + n \leq dalt(K).$$

$alt(K) \leq m + n$





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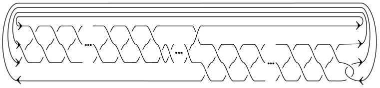

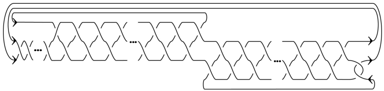

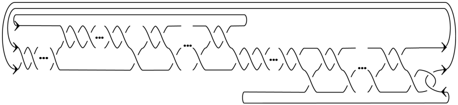

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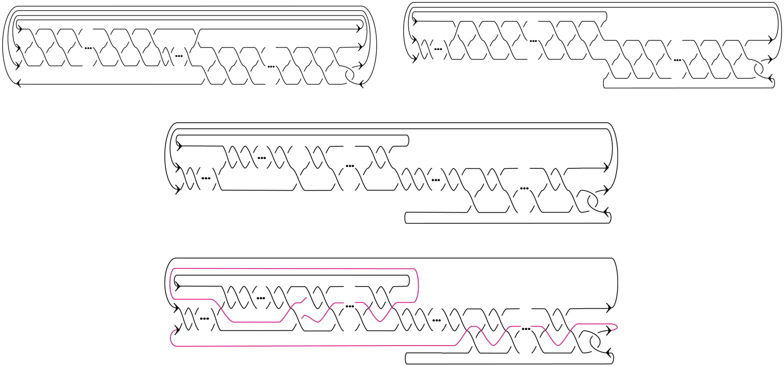

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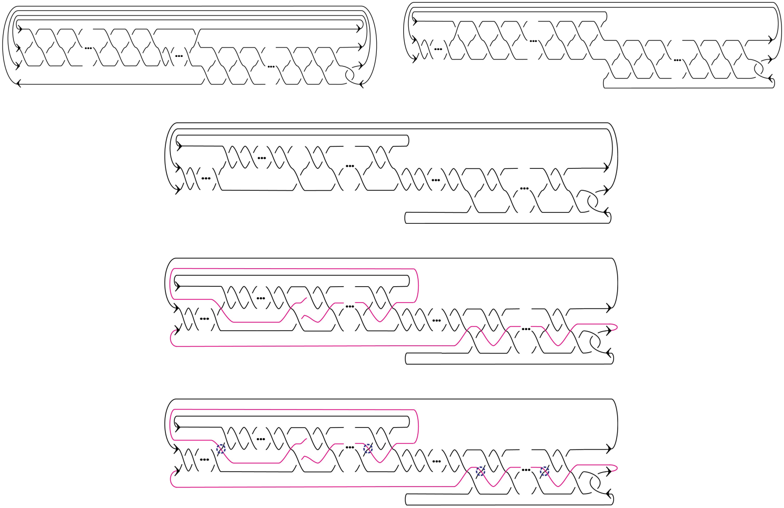
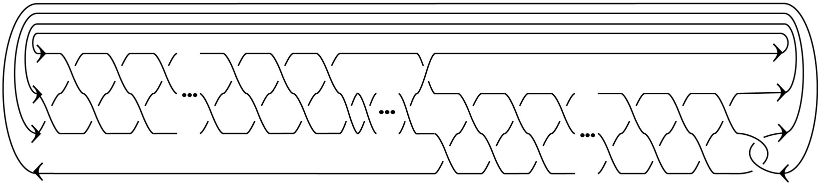


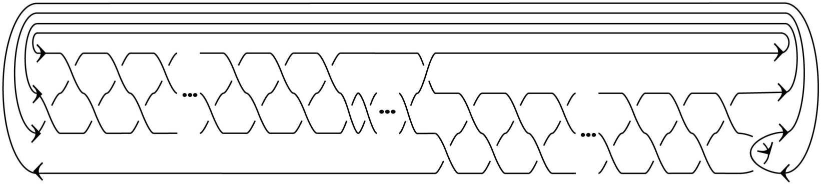
After $m + n$ crossings changes we have an alternating diagram. Therefore, $dalt(K) = m + n$.

Alternation number

One crossing change.



One crossing change.



Alternation number

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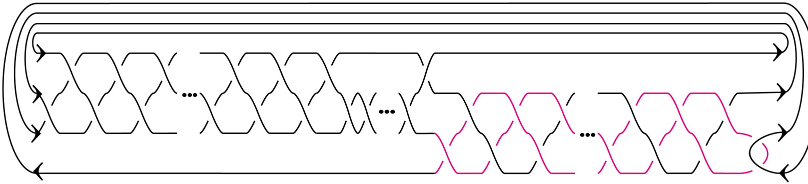
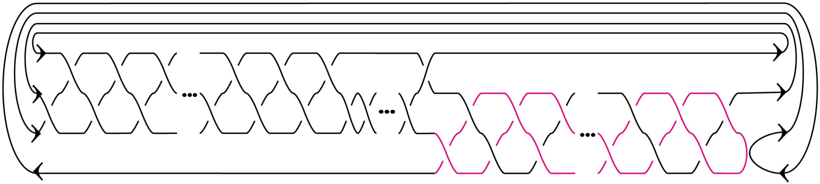
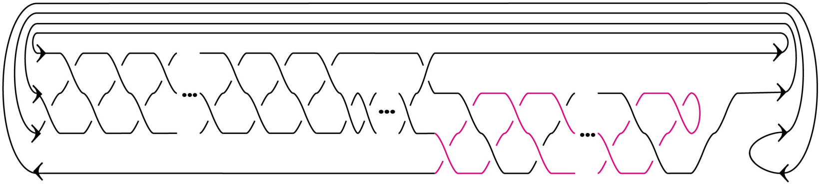


One crossing change.



Alternation number

One crossing change.



Alternation number

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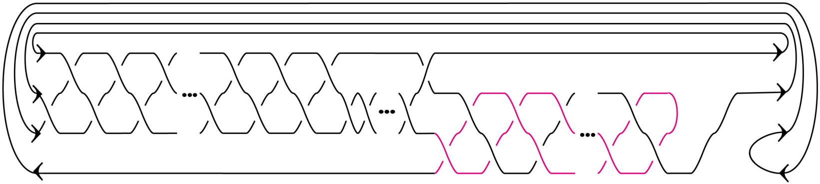
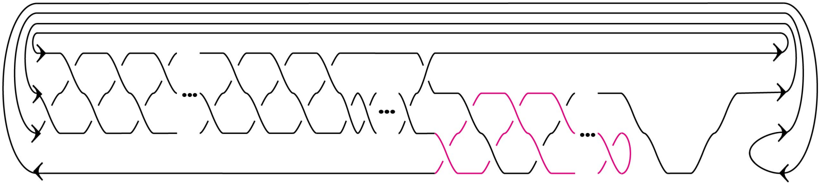


One crossing change.



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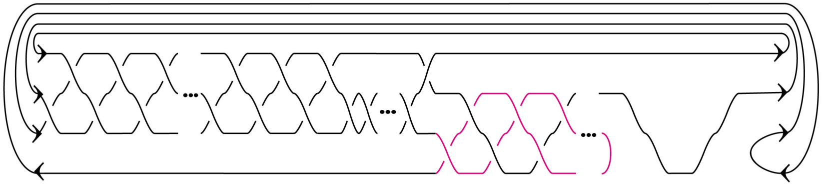
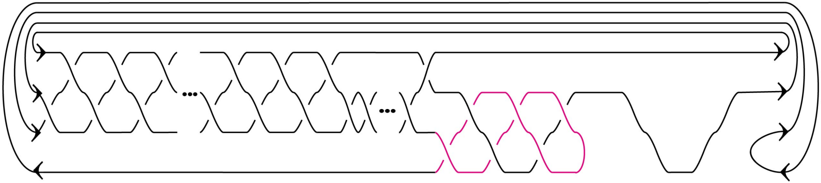


Alternation number

One crossing change.



Alternation number

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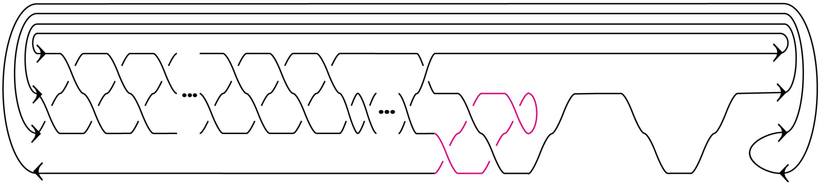
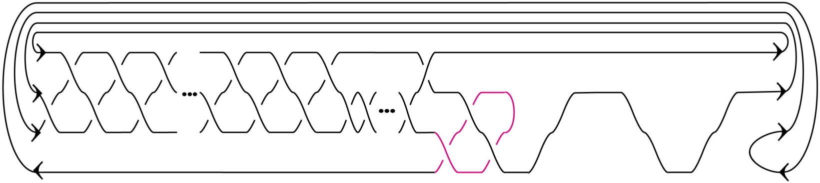


One crossing change.



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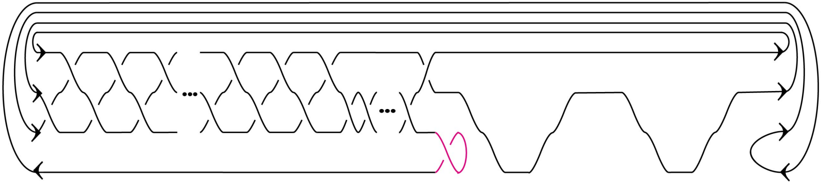

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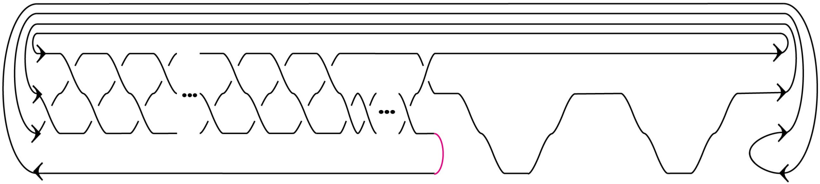

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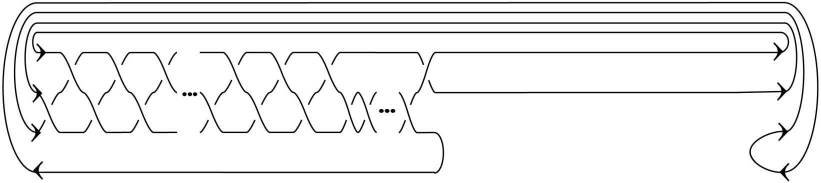
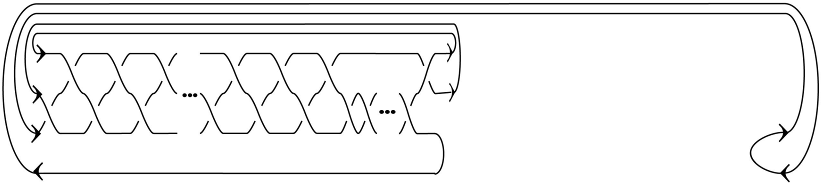


Alternation number

One crossing change.



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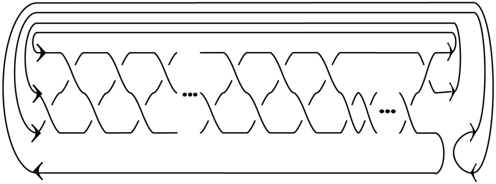
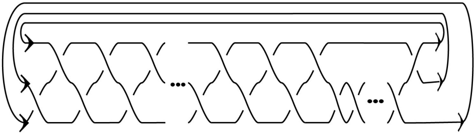


Alternation number

One crossing change.

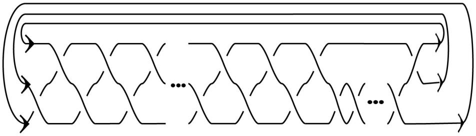


We obtain

$$(\sigma_1\sigma_2)^{3(m+1)}\sigma_1^{l}\sigma_2^{-1}$$

Alternation number

One crossing change.



We obtain

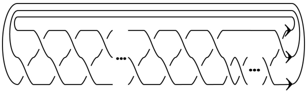
$$(\sigma_1\sigma_2)^{3(m+1)}\sigma_1^{l}\sigma_2^{-1}$$

which is conjugate to

$$(\sigma_1\sigma_2)^{3m}\sigma_1^{l+4}\sigma_2$$

Theorem (Kanenobu, 2010)

For positive integers m, r with r odd and $r \geq 5$, we have that the closure of the 3-braid $(\sigma_1\sigma_2)^{3m}\sigma_1^r\sigma_2$, denoted by $K_{m,r}$, has alternation number equal to m.



It was used the following inequality.

$$|\sigma(K_{m,r}) - s(K_{m,r})|/2 \leq alt(K_{m,r}). \quad (9)$$

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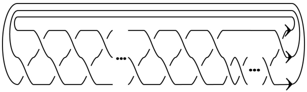

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$$\left|\sigma(K_{m,r}) - s(K_{m,r})\right|/2 = alt(K_{m,r}). \tag{10}$$

Then, $alt(K) \leq m + 1$.

Skein relation

$$0 \leq \sigma(D_+) - \sigma(D_-) \leq 2. \quad (11)$$

$$0 \leq s(D_+) - s(D_-) \leq 2. \quad (12)$$

D_+ is a diagram of K, $D_- = K_{m,r}$

$$m - 1 \leq \left| \sigma(K) - s(K) \right| /2 \leq m + 1. \quad (13)$$

Then, $alt(K) \geq m - 1$.

Therefore,

$$m - 1 \leq alt(K) \leq m + 1.$$

Theorem (G.)

For all pair m, n of positive integers there exists a family of knots

$$\mathcal{F}^{m,n} = \{N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^{l}\sigma_3^{-1}(\sigma_1\sigma_2)^{3n} \cdot c) \,|\, l \in \mathbb{N}, l \text{ is odd.}\}$$

such that, if $K \in \mathcal{F}^{m,n}$ then

$$dalt(K) = m + n \quad \text{and} \quad m - 1 \leq alt(K) \leq m + 1.$$

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Thank you for your attention