Most graphs are knotted

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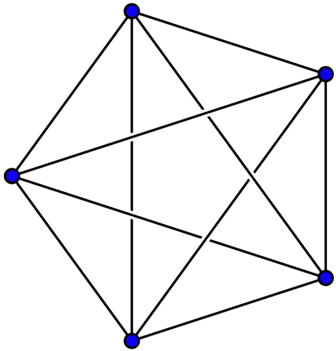
Graph and Embedding

Graph

an ordered pair $G = (V, E)$ comprising a set V of vertices together with a set E of edges.

We always assume that graphs are simple (no loops or multiple edges), and identify the combinatorial object with the associated 1-dim. cell complex.

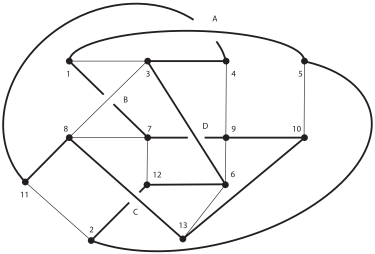
Embedding of G into $\mathbb{R}^3$



Knotted Graph

Intrinsically Knotted

A graph is called intrinsically knotted (IK), if every tame embedding in $\mathbb{R}^3$ contains a non-trivially knotted cycle.



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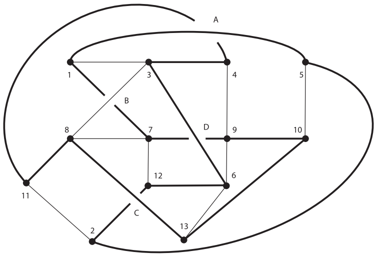

We want to ask;

Are “random” graphs intrinsically knotted?

What's a “random” graph?

Random Graph

Let $|V(G)| = n$ denote the order or number of vertices of a graph, $N = \binom{n}{2}$ the number of edges in the complete graph of order n.

Models 1 & 2

1 (Erdős-Rényi) Choose a graph $G(n, M)$ uniformly at random from the set of labelled graphs with n vertices and M edges. There are $\binom{N}{M}$ such graphs and the probability of choosing a particular graph is $\binom{N}{M}^{-1}$.

2 (Gilbert) For each of the possible N edges, we select it as an edge of the graph $G(n, p)$ independently with probability p.

Random Graph (cont'd)

Models 2.5 & 3

2.5 If $p = \frac{1}{2}$ in Gilbert's model, then every one of the 2^N labelled graphs on n vertices is equally likely. The probability of choosing a particular labelled graph with $|V(G)| = n$ is then 2^{-N}.

3 (Unlabelled version of Model 2.5) Let Γ_n denote the number of unlabelled graphs on n vertices. Choose a graph from this set uniformly at random. The probability of choosing a particular unlabelled graph with $|V(G)| = n$ is Γ_n^{-1}.

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Answer 2.

In all four models, the probability that a graph is intrinsically knotted goes to one as the number of vertices increases.

Result 1

Theorem 1.

In Model 2.5 or 3, there is a constant n_{IK} such that, when $n \geq n_{IK}$, MOST order n graphs are intrinsically knotted (i.e., at least half of such graphs are IK).

We can show that $13 \leq n_{IK} \leq 18$, but leave open the question of the exact value of n_{IK}.

Key Fact

Proposition.

A graph G with $|V(G)| = n \geq 7$ and $|E(G)| \geq 5n - 14$ is IK.

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[Mader, 1968]
If $|V(G)| = n \geq 7$ and $|E(G)| \geq 5n - 14$, then G has a K_7 minor.

Since K_7 is IK [Conway-Gordon], any graph with a K_7 minor is IK.
□

Proof of Thm 1. (Model 2.5)

We show that, if $n \geq 18$, then most graphs of order n are IK.

Pair off each order n graph G with its complement $\overline{G}$.

At least one of these two has at least $\frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$ edges.

If $n \geq 18$, we see that $n(n-1)/4 > 5n - 14$.

By Proposition, G or $\overline{G}$ is IK.

□

Result 2

Theorem 2.

In all four models, the probability that a graph is IK goes to 1 as the number of vertices increases.

Proof of Thm 2. (Model 2)

Assume $0 < p \leq 1$ in Model 2.
The probability that a graph is not IK is bounded by the probability that it has at most $5n - 15$ edges:

$$\begin{aligned} \text{Prob}(G \text{ not IK}) &\leq \text{Prob}(\|G\| \leq 5n-15) \\ &= \sum_{k=0}^{5n-15} \binom{N}{k} p^k (1-p)^{N-k} \leq e^{-2t^2 N}. \end{aligned}$$

The last inequality is due to Hoeffding, with $t = p - (5n-15)/N$, and shows that the probability approaches 0 as n goes to infinity.

□

Thank you for your attention!

I wish you
a Merry Christmas
and
a Happy New Year !!