Gluck twists on branched twist spins

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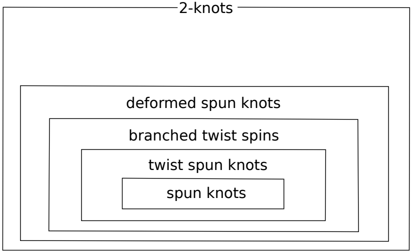
Introduction

A 2-knot (S^2-knot) is an *smoothly* embedded S^2 in S^4.

$\mathcal{K} = \mathcal{K}' \overset{\text{def}}{\Longleftrightarrow} \mathcal{K}'$ is transformed into $\mathcal{K}$ via an ambient isotopy of S^4.



$(S^4, \mathcal{K}) \rightsquigarrow (\Sigma(\mathcal{K}), \mathcal{K}')$: Gluck twist

Main theorem

We determine $\mathcal{K}'$ when $\mathcal{K}$ is a branched twist spin.

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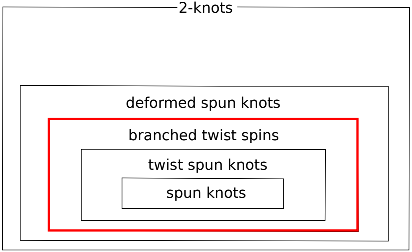

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Contents

S^1-actions on S^4

$\tau, \tau' : S^1 \times S^4 \to S^4$: S^1-actions

τ is called locally smooth
$\Leftrightarrow \forall$ orbit $\exists$ nbd. s.t. each isotropy group acts ortho. on each slice disk.

τ and τ' are called weak equivalent
$\Leftrightarrow \exists H \in \mathrm{Diff}(S^4)$, $\exists \alpha \in \mathrm{Aut}(S^1)$ s.t. $H(\tau(\theta, x)) = \tau'(\alpha(\theta), x)$.

Orbits of an S^1-action

- free orbit
- exceptional orbit of $\mathbb{Z}_m$-type $\overset{\text{def}}{\Longleftrightarrow}$ isotoropy group $\cong \mathbb{Z}_m$
- fixed point

Classification of S^1-actions

Theorem (Fintushel, Pao '78)

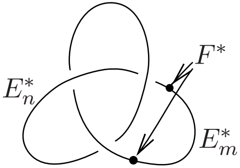
$$\{\ S^1 \curvearrowright S^4 : \text{locally smooth effective}\} \ / \ \text{weak equivalence}$$
$$\updownarrow \exists \text{ bijection}$$
$$\{\{D^3\}, \{S^3\}, \{S^3, m\}, \{(S^3, K), m, n\}\},$$
where m, n are coprime positive integers.

- D^3, S^3 : orbit space
- m, n : type of exceptional orbit
- K : union of exceptional orbits and fixed points

Definition of branched twist spins

- $p : S^4 \to S^3$: orbit map
- E_m, E_n : the sets of exceptional orbits
- F : the fixed point set

  $p(E_m) := E_m^*$, $p(E_n) := E_n^*$, $p(F) := F^*$



Definition (Branched twist spin)

Let K be the 1-knot $E_m^* \cup E_n^* \cup F^*$. The (m, n)-branched twist spin of K is defined as $K^{m,n} := E_n \cup F \subset S^4$.

Remark

We can take K as any 1-knot.

- $\{D^3\}$: spun knot $\cdots$ fibered (depend on K)
- $\{S^3\}$: 1-twist spun knot $\cdots$ fibered
- $\{S^3, m\}$: m-twist spun knot $\cdots$ fibered
- $\{(S^3, K), m, n\}$: branched twist spin $\cdots$ fibered

Gluck twist

- $\mathcal{K} \subset S^4$: 2-knot
- $\mathrm{Nbd}(\mathcal{K}) \cong S^2 \times D^2$: a nbd. of $\mathcal{K}$

  The isotopy class of orientation preserving homeomorphisms on $S^2 \times S^1$ is isom. to $\mathbb{Z}_2$.
- γ : non-trivial generator of $\mathbb{Z}_2$

Definition

The Gluck twist is an operation of removing $\mathrm{int}\mathrm{Nbd}(\mathcal{K})$ from S^4 and regluing $S^2 \times D^2$ by γ.

Remark

$\gamma^2 =$ identity.

Known results

$\Sigma(\mathcal{K})$: manifold obtained from S^4 by the Gluck twist along $\mathcal{K}$.

By a Freedman's work, it is known that $\Sigma(\mathcal{K})$ is homeomorphic to S^4. However we do not know whether $\Sigma(\mathcal{K})$ is diffeomorphic to S^4 or not in general.

Theorem (Gordon '76)

$\Sigma(K^{m,1})$ is diffeomorphic to S^4.

Theorem (Pao '76)

$\Sigma(K^{m,n})$ is diffeomorphic to S^4.

$\mathcal{K}^*$: dual of $\mathcal{K}$ (i.e. 2-knot obtained from $\mathcal{K}$ by the Gluck twist.)

Question. Determine $\mathcal{K}$ and $\mathcal{K}^*$ are equivalent or not.

Theorem (Gordon '76)

If m is odd, then $K^{m,1} \neq K^{m,1*}$.

Theorem (Plotnick '86)

For a fibered 2-knot $\mathcal{K}$ with odd monodromy, $\mathcal{K} \neq \mathcal{K}^*$.

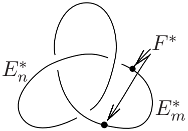
Theorem (Hillman,Plotnick '88)

Let K is a torus or hyperbolic 1-knot.
If $m > n$ and $m \geq 3$, then $K^{m,n} \neq K^{m,n*}$

Main result

Main Theorem (F.)

$K^{m,n*} = K^{m,m+n}$.



Sketch of Proof.

First step. Decompose S^4 according to the S^1-action into four pieces:

$$S^4 = \text{Nbd(fixed pts.)} \cup \text{Nbd}(E_n) \cup \text{Nbd}(E_m) \cup \text{Nbd(free orbits)}$$
$$= \underbrace{(B_1^4 \sqcup B_2^4) \cup (V_n \times E_n^*)}_{\text{nbd. of } K^{m,n*}} \cup \underbrace{(V_m \times E_m^*) \cup ((S^3 \setminus \text{intNbd}(K)) \times S^1)}_{\text{knot complement of } K^{m,n*}}$$

Here V_n is the solid torus whose core is an exceptional orbit of $\mathbb{Z}_n$-type.

Second step. Replace the S^1-action on $(B_1^4 \sqcup B_2^4) \cup (V_n \times E_n^*)$ so that the action is compatible with that on $(V_m \times E_m^*) \cup ((S^3 \setminus \text{intNbd}(K)) \times S^1)$ by the Gluck twist.

Sketch of proof

Third step. Check that $K^{m,n*}$ consists of exceptional orbits of $\mathbb{Z}_m$-type and $\mathbb{Z}_{m+n}$-type.
Due to Fintushel and Pao, it is sufficient to show that $K^{m,n*}$ contains an exceptional orbit of $\mathbb{Z}_{m+n}$-type. □

Theorem (Fintushel, Pao) [Recall]

$$\{\ S^1 \curvearrowright S^4 : \text{locally smooth effective}\} \ / \text{ weak equivalence}$$
$$\updownarrow \exists \text{ bijection}$$
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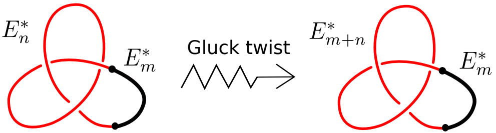
where m, n are coprime and positive integers.



Application(On equiv. classes of b.t.s.)

Applying Main theorem to Plotnick's theorem, the following corollary holds.

Corollary (F.)

If m is odd, $K^{m,n}$ and $K^{m,m+n}$ are not equivalent.

If m is even, we have the following sufficient condition to distinguish branched twist spins.

Theorem (F. '16)

Let $K_1^{m_1,n_1}$ and $K_2^{m_2,n_2}$ be branched twist spins constructed from 1-knots K_1 and K_2. Suppose that m_1 is even.
(1) If m_2 is even and $|\Delta_{K_1}(-1)| \neq |\Delta_{K_2}(-1)|$, then $K_1^{m_1,n_1}$ and $K_2^{m_2,n_2}$ are not equivalent.
(2) If m_2 is odd and $|\Delta_{K_2}(-1)| \neq 1$, then $K_1^{m_1,n_1}$ and $K_2^{m_2,n_2}$ are not equivalent.

Application(Montesinos twin)

Definition (Montesinos twin)

A Montesinos twin is a pair of 2-knots which meet transversely twice.

By definition, a pair of branched twist spins $K^{m,n}$ and $K^{n,m}$ is a Montesinos twin. From Main theorem, we see that there exist exceptional orbits of $\mathbb{Z}_n$-type and $\mathbb{Z}_{m+n}$-type in $\Sigma(K^{n,m})$. Then we have the following:

Theorem (F.)

$K^{m+n,n}$ is obtained from $K^{m,n}$ by Gluck twist along $K^{n,m}$.

Summary

Main Theorem (F.)

$K^{m,n*} = K^{m,m+n}$.

Corollary (F.)

If m is odd, $K^{m,n}$ and $K^{m,m+n}$ are not equivalent.

Theorem (F.)

$K^{m+n,n}$ is obtained from $K^{m,n}$ by Gluck twist along $K^{n,m}$.