On the minimal coloring number of the minimal diagram of torus links

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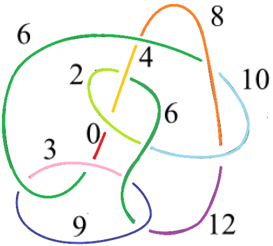
$\mathbb{Z}$-coloring

Let L be a link, and D a diagram of L.

$\mathbb{Z}$-coloring

A map $\gamma : \{\text{arcs of } D\} \to \mathbb{Z}$ is called a $\mathbb{Z}$-coloring on D if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of D with the over arc a and the under arcs b and c.
A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called a trivial $\mathbb{Z}$-coloring.
L is $\mathbb{Z}$-colorable if $\exists$ a diagram of L with a non-trivial $\mathbb{Z}$-coloring.



Let L be a $\mathbb{Z}$-colorable link.

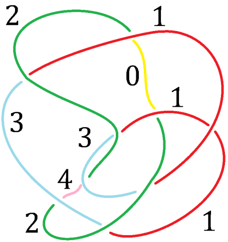
Minimal coloring number

[1] For a diagram D of L,
$\text{mincol}_{\mathbb{Z}}(D) := \min\{\#\text{Im}(\gamma) \mid \gamma : \text{non-tri. } \mathbb{Z}\text{-coloring on } D\}$

[2] $\text{mincol}_{\mathbb{Z}}(L) := \min\{\text{mincol}_{\mathbb{Z}}(D) \mid D : \text{a diagram of } L\}$

Simple $\mathbb{Z}$-coloring

γ : a $\mathbb{Z}$-coloring on a diagram D of a non-trivial $\mathbb{Z}$-colorable link L
If $\exists\ d \in \mathbb{N}$ s.t. at each crossings in D, the differences between the colors of the over arcs and the under arcs are d or 0, then we call γ a simple $\mathbb{Z}$-coloring.



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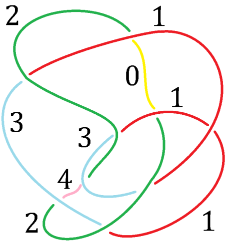

Theorem 1 [Ichihara-M., JKTR, 2017]

Let L be a non-splittable $\mathbb{Z}$-colorable link. If there exists a simple $\mathbb{Z}$-coloring on a diagram of L, then $\text{mincol}_{\mathbb{Z}}(L) = 4$.

Theorem 2 [M., to apper JKTR, Zhang-Jin-Deng]

Any $\mathbb{Z}$-colorable link has a diagram admitting a simple $\mathbb{Z}$-coloring.

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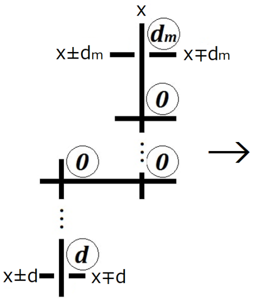
Colorally

L : a $\mathbb{Z}$-colorable link

$$\mathrm{mincol}_{\mathbb{Z}}(L) = \begin{cases} 2 \ (L : \text{splittable}) \\ 4 \ (L : \text{non-splittable}) \end{cases}$$

One of key moves of the proof



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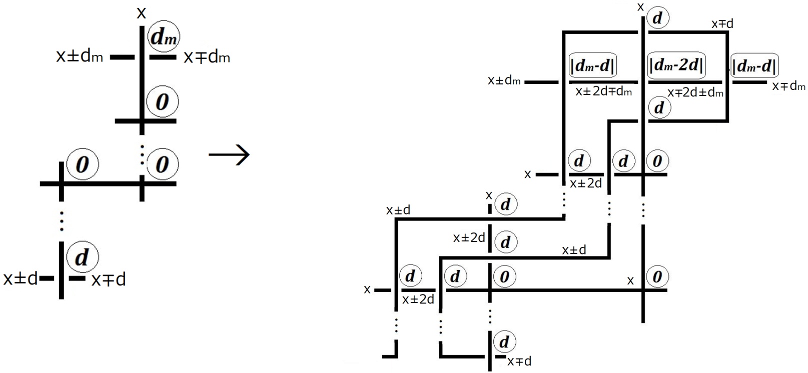

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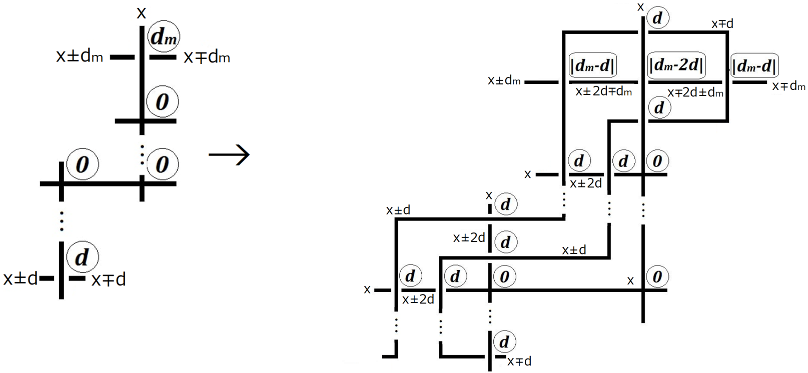

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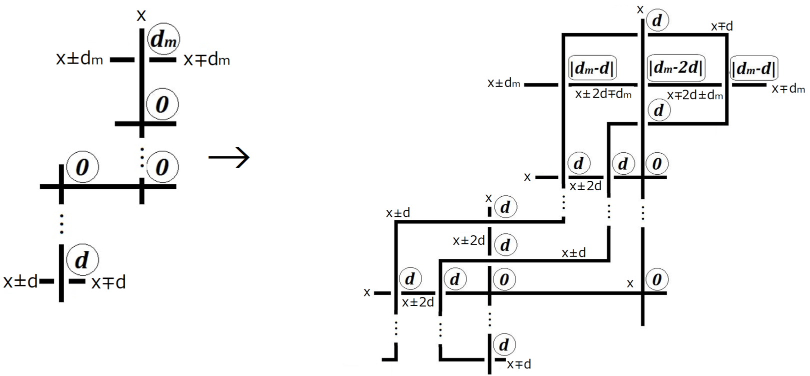


→ The obtained diagrams are often complicated.

Problem

$\mathrm{mincol}_{\mathbb{Z}}(D_m) = ?$ for a minimal diagram D_m of a $\mathbb{Z}$-colorable link.

Theorem 3 [Ichihara-M., Proc.Inst.Nat.Sci., Nihon Univ., 2018]

[1] For an even integer $n \geq 2$, the pretzel link $P(n, -n, \cdots, n, -n)$ with at least 4 strands has a minimal diagram D_m s.t. $\mathrm{mincol}_{\mathbb{Z}}(D_m) = n + 2$.

[2] For an integer $n \geq 2$, the pretzel link $P(-n, n+1, n(n+1))$ has a minimal diagram D_m s.t. $\mathrm{mincol}_{\mathbb{Z}}(D_m) = n^2 + n + 3$.

[3] For even integer $n > 2$ and non-zero integer p, the torus link $T(pn, n)$ has a minimal diagram D_m s.t. $\mathrm{mincol}_{\mathbb{Z}}(D_m) = 4$.

Theorem 4 [Ichihara-Ishikawa-M., In progress]

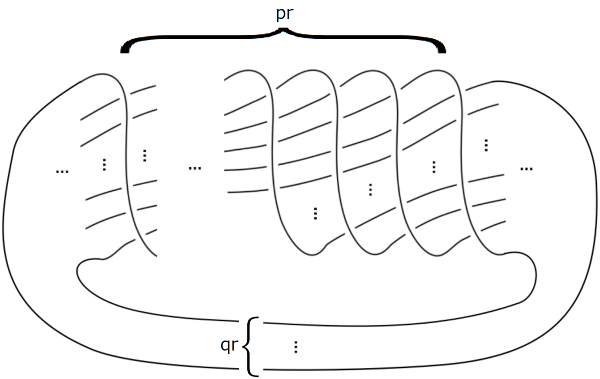
Let p, q and r be non-zero integers such that $|p| \geq q \geq 1$ and $r \geq 2$. If pr or qr are even, the torus link $T(pr, qr)$ has a minimal diagram D_m s.t.

$$\mathrm{mincol}_{\mathbb{Z}}(D_m) = \begin{cases} 4 \ (r : \text{even}) \\ ''5'' \ (r : \text{odd}) \end{cases}$$

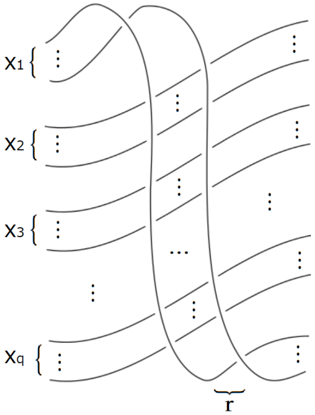
Remark

A torus link $T(pr, qr)$ is $\mathbb{Z}$-colorable if and only if pr or qr are even.

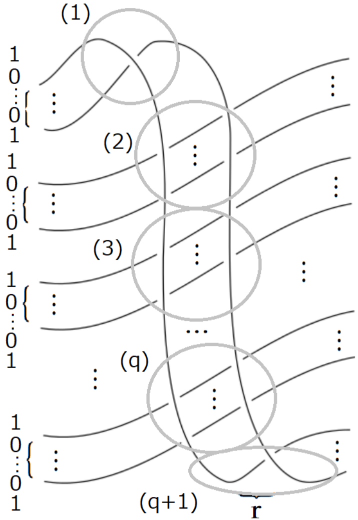
[Proof of Theorem 4 (In the case r:even)]
Let D be the following minimal diagram of $T(pr, qr)$.



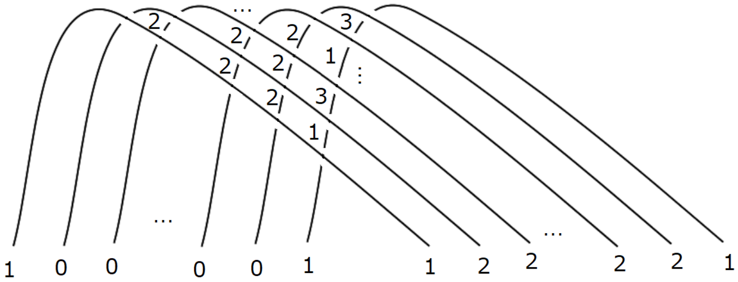
In the following, we will find a $\mathbb{Z}$-coloring γ on D by assigning colors on the arcs of D.
We devide such arcs into q subfamilies $\mathbf{x}_1, \cdots, \mathbf{x}_q$.



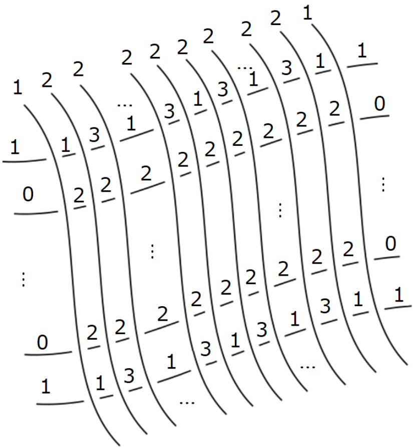
We first find a local $\mathbb{Z}$-coloring γ. In the case r is even, we start with setting $\gamma(\mathbf{x}_i) = (\gamma(x_{i,1}), \gamma(x_{i,2}), \cdots, \gamma(x_{i,r}))$
$= (1, 0, \cdots, 0, 1)$ for any i.



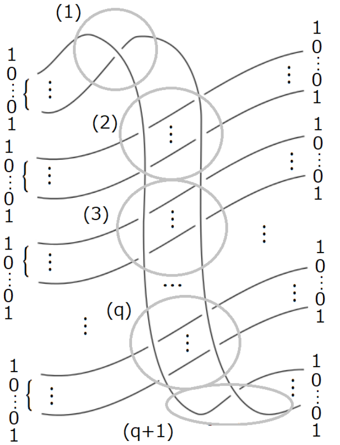
We can extend γ on the arcs in the regions (1) and $(q+1)$.



We can extend γ on the arcs in the regions $(2), (3), \cdots, (q)$.



Now, γ can be extended on all the arcs in the region depicted as follows.



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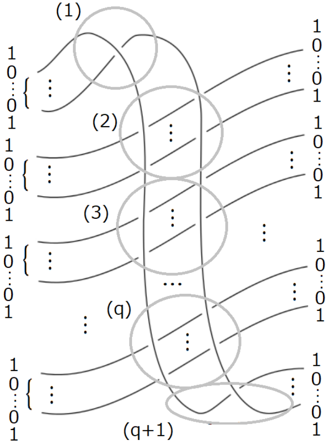

Since D is composed of p copies of the local diagram, it concludes that D admits a $\mathbb{Z}$-coloring with only four colors 0, 1, 2 and 3.

Thank you
for your attention.