コンパクトな向き付け不可能曲面の Torelli 群の正規生成系について

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結び目の数理
2018 年 12 月 24 日, 早稲田大学

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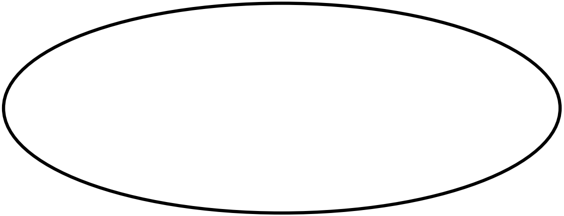

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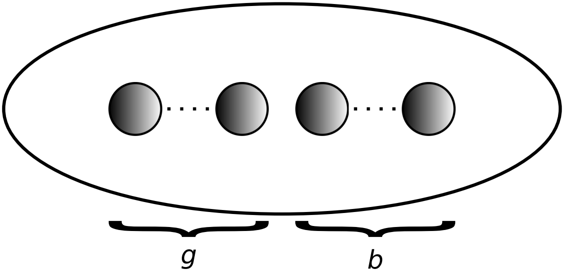

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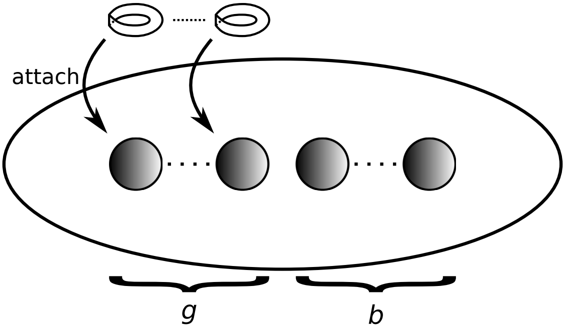

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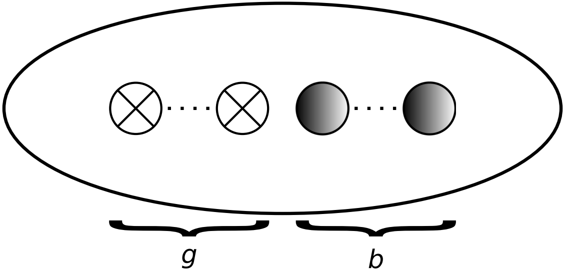

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N_g^b : a genus g compact non-orientable surface with b boundary components.
The mapping class group of N_g^b is defined as

$$\mathcal{M}(N_g^b) = \{f : N_g^b \stackrel{\text{diffeo.}}{\longrightarrow} N_g^b \mid f|_{\partial N_g^b} = \mathrm{id}\}/\mathrm{isotopy}.$$

The Torelli group of N_g^b is defined as

$$\mathcal{I}(N_g^b) = \ker(\mathcal{M}(N_g^b) \to \mathrm{Aut}(H_1(N_g^b; \mathbb{Z}))).$$

Background

Σ_g^b : a genus g compact orientable surface with b boundary components.

- A generating set for $\mathcal{I}(\Sigma_g^0)$ was found by Powell (1978).
- A finite generating set for $\mathcal{I}(\Sigma_g^0)$ was found by Johnson (1983).
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1. Find a generating set for $\mathcal{I}(N_g^b)$ for $b \geq 0$.
2. Can $\mathcal{I}(N_g^b)$ be finitely generated for $b \geq 0$?

Remark

$\mathcal{I}(N_g^0)$ is trivial for $g \leq 3$.

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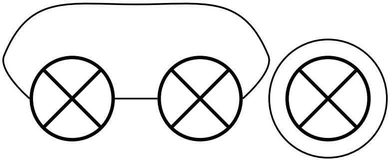


Figure: Two sided simple closed curves.

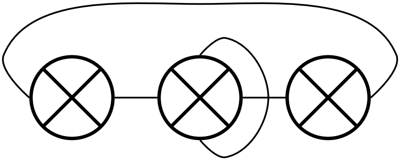
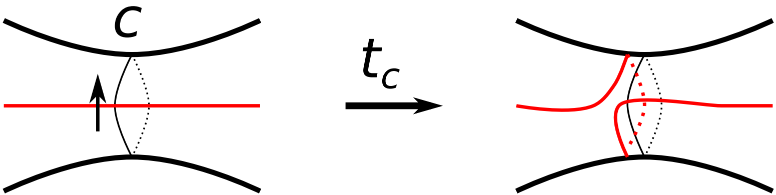


Figure: One sided simple closed curves.

Dehn twist

For a two sided simple closed curve c, the Dehn twist t_c is defined as



Main result

Theorem (Hirose-K. ($b = 0$), K. ($b \geq 1$))

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_α, $t_\beta t_{\beta'}^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
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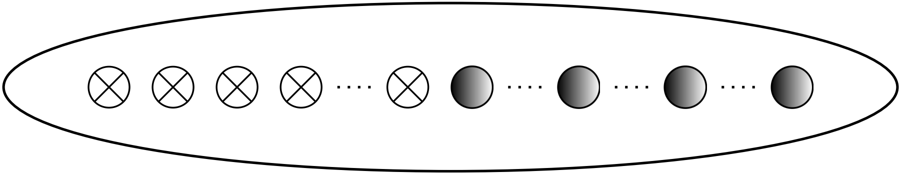
- t_γ *(only if $g = 4$).*



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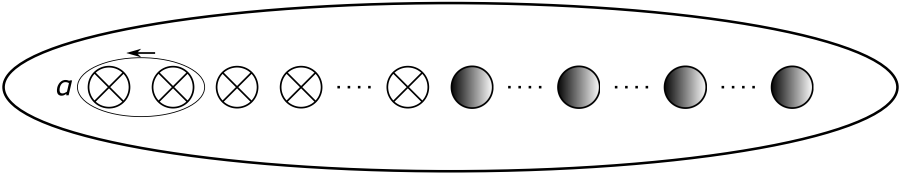

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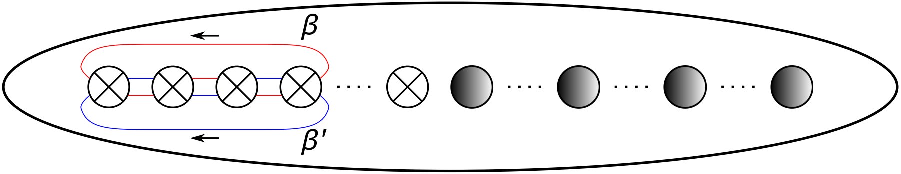

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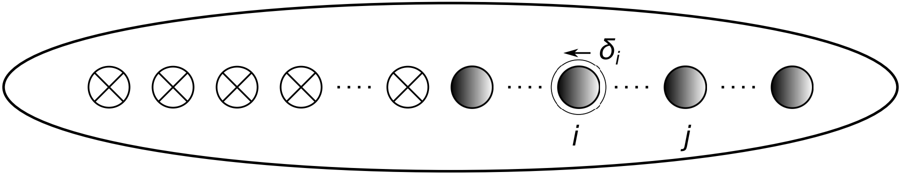

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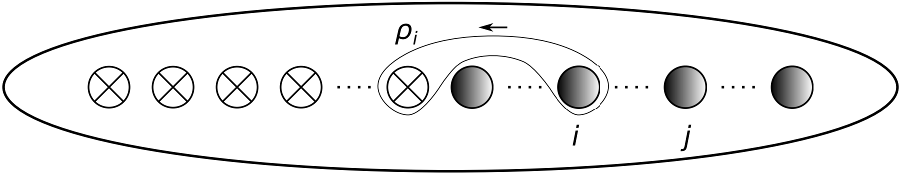

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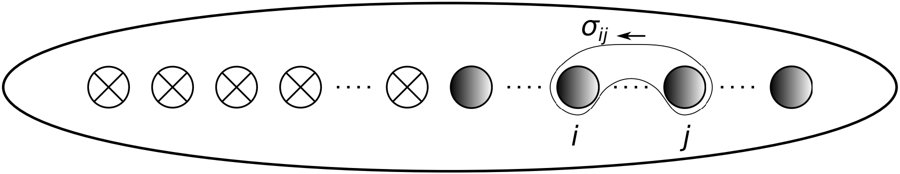

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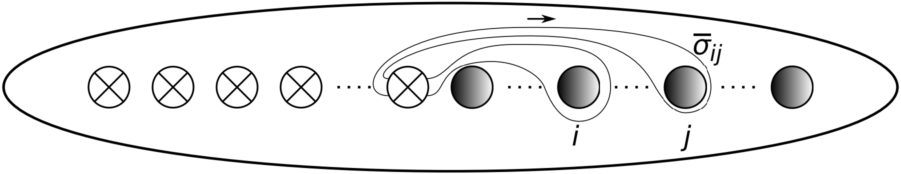

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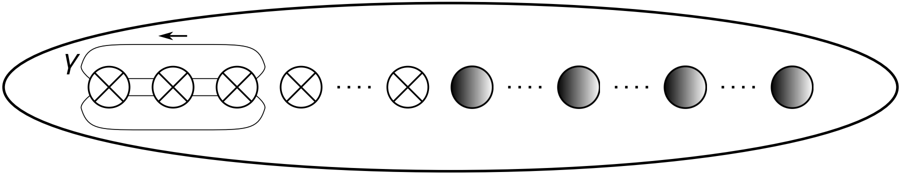

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The case of a closed surface

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The level 2 mapping class group of N_g^b is defined as

$$\Gamma_2(N_g^b) = \ker(\mathcal{M}(N_g^b) \to \mathrm{Aut}(H_1(N_g^b; \mathbb{Z}/2\mathbb{Z}))).$$

The level-2 principal congruence subgroup of $GL(n; \mathbb{Z})$ is defined as

$$\Gamma_2(n) = \ker(GL(n; \mathbb{Z}) \to GL(n; \mathbb{Z}/2\mathbb{Z})).$$

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Lemma

We have the short exact sequence

$$1 \to \mathcal{I}(N_g^0) \to \Gamma_2(N_g^0) \to \Gamma_2(g-1) \to 1.$$

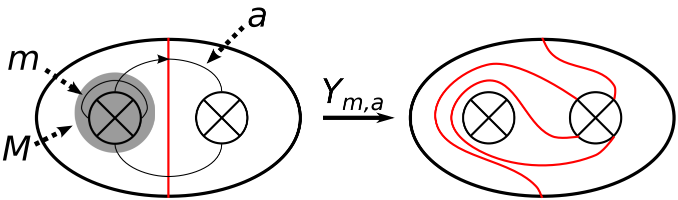
In general, if there is a short exact sequence

$$1 \to G \to \langle X \mid Y \rangle \xrightarrow{\phi} \langle \phi(X) \mid Z \rangle \to 1,$$

then G is normally generated by $\{\tilde{z} \mid z \in Z, \phi(\tilde{z}) = z\}$.

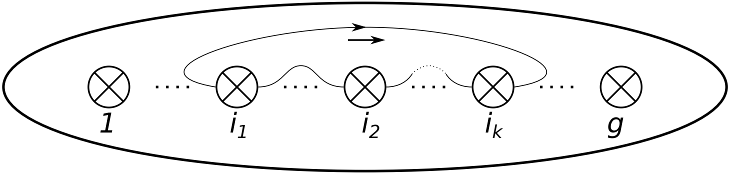
Crosscap slide

m : a one sided simple closed curve,
a : a two sided oriented simple closed curve,
(m and a intersect transversely at only one point)
M : a regular neighborhood of m.
The crosscap slide $Y_{m,a}$ is defined as



Generating sets for $\Gamma_2(N_g^0)$

For $1 \leq i_1 < i_2 < \cdots < i_k \leq g$, $\alpha_{i_1,\dots,i_k}$ is defined as



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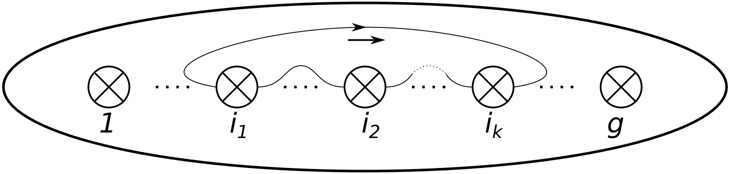

Theorem (Szepietowski (2013))

For $g \geq 4$, $\Gamma_2(N_g^0)$ is finitely generated by

1. $Y_{\alpha_i,\alpha_{i,j}}$ *for* $1 \leq i \leq g-1$, $1 \leq j \leq g$ *and* $i \neq j$,
2. $t^2_{\alpha_{i,j,k,l}}$ *for* $1 \leq i < j < k < l \leq g$.

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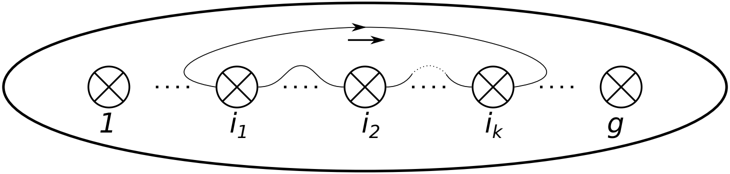

Theorem (Hirose-Sato (2014))

For $g \geq 4$, $\Gamma_2(N_g^0)$ is minimally generated by

1. $Y_{\alpha_i,\alpha_{i,j}}$ *for* $1 \leq i \leq g-1$, $1 \leq j \leq g$ *and* $i \neq j$,
2. $t^2_{\alpha_{1,j,k,l}}$ *for* $1 < j < k < l \leq g$.

Presentations for $\Gamma_2(g-1)$

$$\Gamma_2(N_g^0) \ni Y_{\alpha_i,\alpha_{i,j}}, t^2_{\alpha_{i,j,k,l}} \mapsto Y_{ij}, T_{ijkl} \in \Gamma_2(g-1).$$

Proposition (cf. Fullarton (2014), K. (2015))

$\Gamma_2(g-1)$ *is generated by* Y_{ij} *and* T_{1jkl}*, and has the relators*

1. Y_{ij}^2 *for* $1 \leq i \leq g-1$ *and* $1 \leq j \leq g$*,*
2. $[Y_{ik}, Y_{jk}]$ *for* $1 \leq i,j \leq g-1$ *and* $1 \leq k \leq g$*,*
3. $[Y_{ij}, Y_{ik}Y_{jk}]$ *for* $1 \leq i,j \leq g-1$ *and* $1 \leq k \leq g$*,*
4. $[Y_{ij}, Y_{kl}]$ *for* $1 \leq i,k \leq g-1$ *and* $1 \leq j,l \leq g$*,*
5. $(Y_{ij}Y_{ik}Y_{il})^2$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j,k,l \leq g$*,*
6. $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$ *for* $1 \leq i,j,k \leq g-1$*,*
7. $T_{1jkl} \cdot$ *(a product of* Y_{ij}*'s),*

where $[X,Y] = X^{-1}Y^{-1}XY$ *and* i,j,k,l *are all different.*

Remark

For $g \geq 4$, $\Gamma_2(N_g^0)$ is generated by

1. $Y_{\alpha_i,\alpha_{i,j}}$ for $1 \leq i \leq g-1$, $1 \leq j \leq g$ and $i \neq j$,
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$\Gamma_2(g-1)$ is generated by Y_{ij} and T_{1jkl}, and has the relators

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$$1 \to \mathcal{I}(N_g^0) \to \Gamma_2(N_g^0) \to \Gamma_2(g-1) \to 1$$

Let $Y_{\alpha_i,\alpha_{i,j}} = Y_{i;j}$ and $t^2_{\alpha_{i,j,k,l}} = T_{i,j,k,l}$.

Corollary

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by followings in $\Gamma_2(N_g^0)$,

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$(\mathcal{I}(N_g^0) \lhd \mathcal{M}(N_g^0)$, $\mathcal{I}(N_g^0) \lhd \Gamma_2(N_g^0)$, $\Gamma_2(N_g^0) < \mathcal{M}(N_g^0)$.)

Let $Y_{\alpha_i,\alpha_{i,j}} = Y_{i;j}$ and $t^2_{\alpha_{i,j,k,l}} = T_{i,j,k,l}$.

Corollary

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by followings in $\mathcal{M}(N_g^0)$,

1. $Y_{i;j}^2$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j \leq g$,
2. $[Y_{i;k}, Y_{j;k}]$ *for* $1 \leq i, j \leq g-1$ *and* $1 \leq k \leq g$,
3. $[Y_{i;j}, Y_{i;k}Y_{j;k}]$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j, k \leq g$,
4. $[Y_{i;j}, Y_{k;l}]$ *for* $1 \leq i, k \leq g-1$ *and* $1 \leq j, l \leq g$,
5. $(Y_{i;j}Y_{i;k}Y_{i;l})^2$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j, k, l \leq g$,
6. $(Y_{j;i}Y_{i;j}Y_{k;j}Y_{j;k}Y_{i;k}Y_{k;i})^2$ *for* $1 \leq i, j, k \leq g-1$,
7. $T_{1,j,k,l} \cdot$ *(a product of* $Y_{i;j}$*'s) for* $1 < j < k < l \leq g$,

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We checked that these are products of conjugations of t_α, $t_\beta t_{\beta'}^{-1}$ and t_γ.

We have

Theorem (Hirose-K. (2016))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by

- t_α, $t_\beta t_{\beta'}^{-1}$ *and*
- t_γ *(only if $g = 4$).*

The case of a surface with boundary

Theorem (K.)

For $g \geq 4$ and $b \geq 1$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_α, $t_\beta t_{\beta'}^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$, $t_{\bar{\sigma}_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_γ *(only if* $g = 4$*).*

Capping map and Forgetful maps

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \{f : N_g^{b-1} \stackrel{\text{diffeo.}}{\longrightarrow} N_g^{b-1} \mid f|_{\partial N_g^{b-1} \cup \{*\}} = \mathrm{id}\}/\mathrm{isotopy}$$

We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.

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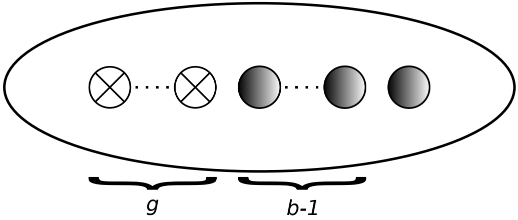

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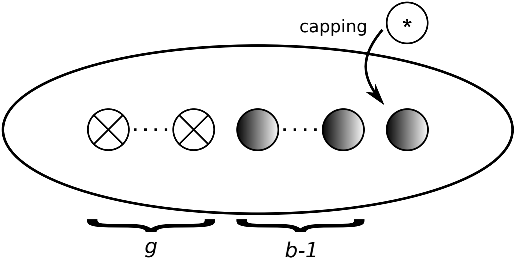

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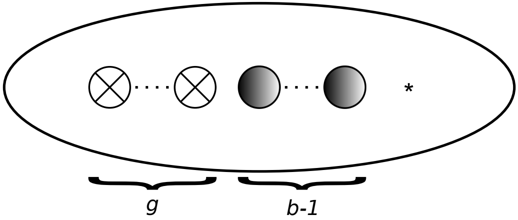

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The capping map $\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$ is the homomorphism induced by $N_g^b \hookrightarrow N_g^{b-1}$.

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The forgetful map $\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$ is the homomorphism induced by

$$\{f : N_g^{b-1} \xrightarrow{\text{diffeo.}} N_g^{b-1} \mid f|_{\partial N_g^{b-1} \cup \{*\}} = \mathrm{id}\}$$
$$\to \{f : N_g^{b-1} \xrightarrow{\text{diffeo.}} N_g^{b-1} \mid f|_{\partial N_g^{b-1}} = \mathrm{id}\}$$

$$1 \to \ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)} \to \mathcal{I}(N_g^b) \overset{\mathcal{C}_g^b}{\to} \mathcal{C}_g^b(\mathcal{I}(N_g^b)) \to 1$$

$$1 \to \ker \mathcal{F}_g^{b-1}|_{\mathcal{C}_g^b(\mathcal{I}(N_g^b))} \to \mathcal{C}_g^b(\mathcal{I}(N_g^b)) \overset{\mathcal{F}_g^{b-1}}{\to} \mathcal{I}(N_g^{b-1}) \to 1$$

$\mathcal{I}(N_g^b)$ is normally generated by

- $\ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)}$ and
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  - lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

$$1 \to \ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)} \to \mathcal{I}(N_g^b) \overset{\mathcal{C}_g^b}{\to} \mathcal{C}_g^b(\mathcal{I}(N_g^b)) \to 1$$

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$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b}, t_{ρ_b}, $t_{\sigma_{jb}}$, $t_{\bar{\sigma}_{jb}}$ and
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  - ~~$\ker \mathcal{F}_g^{b-1}|_{\mathcal{C}_g^b(\mathcal{I}(N_g^b))}$ and~~
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Remark

A normal generating set of $\mathcal{I}(N_g^b)$ is obtained by adding t_{δ_b}, t_{ρ_b}, $t_{\sigma_{jb}}$ and $t_{\bar{\sigma}_{jb}}$ to that of $\mathcal{I}(N_g^{b-1})$.

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There is the $\mathbb{Z}$-module homomorphism

$$J : \mathcal{I}(N_g^1) \to \wedge^3 H_1(\Sigma_g^1, \mathbb{Z}).$$

Theorem (Tsuji)

$$\dim(\mathbb{Q} \otimes \mathrm{Im} J) = \frac{(g-1)(g-2)(g-3)}{6} + \frac{g(g-1)^2}{2}.$$

Corollary

$$\dim(\mathbb{Q} \otimes \mathcal{I}(N_g^1)^{\mathrm{ab}}) \geq \frac{(g-1)(g-2)(g-3)}{6} + \frac{g(g-1)^2}{2}.$$

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Thus the number of generators of $\mathcal{I}(N_g^1)$ is at least $\frac{(g-1)(g-2)(g-3)}{6} + \frac{g(g-1)^2}{2}$.

Thank you for your attention!