# Quandle 2-cocycle 不変量と shadow 3-cocycle 不変量の関係について

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# Introduction

X:quandle

{X-coloring numbers}
cf.[Fenn-Rourke (1992)]

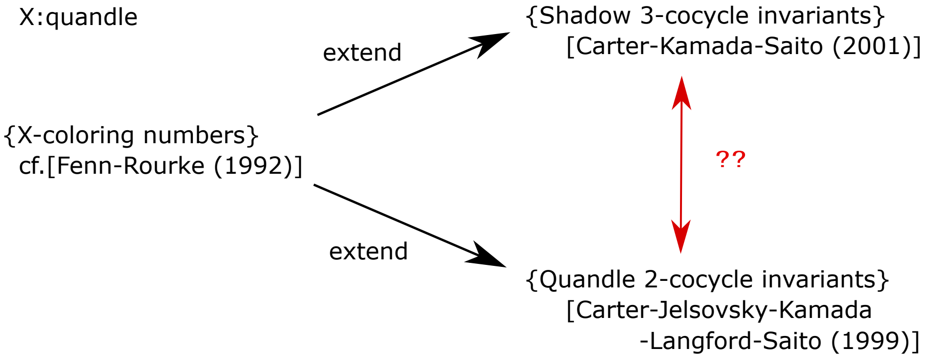
extend →

{Shadow 3-cocycle invariants}
[Carter-Kamada-Saito (2001)]

extend →

{Quandle 2-cocycle invariants}
[Carter-Jelsovsky-Kamada
-Langford-Saito (1999)]

# Introduction



# Quandle

## Definition (Quandle)

$X$: a non-empty set, $* : X \times X \to X$ : a binary operation

$X = (X, *)$: a **quandle**

$\overset{\text{def}}{\Longleftrightarrow}$

1. $x * x = x$ $(\forall x \in X)$.
2. A map $*x : X \to X$ $(\bullet \mapsto \bullet * x)$ is bijective $(\forall x \in X)$.
3. $(x * y) * z = (x * z) * (y * z)$ $(\forall x, y, z \in X)$.

$R_3 := (\{0, 1, 2\}, *)$

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 1 |
| 1 | 2 | 1 | 0 |
| 2 | 1 | 0 | 2 |

$S_4 := (\{0, 1, 2, 3\}, *)$

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 3 | 1 | 0 | 2 |
| 2 | 1 | 3 | 2 | 0 |
| 3 | 2 | 0 | 1 | 3 |

# Quandle coloring

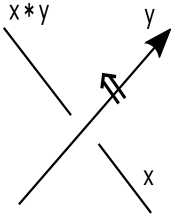
$X$: a finite quandle

$K$: an oriented knot, $D$: a diagram of $K$

## Definition

$$\mathrm{Col}_X(D) := \left\{ c : \{\text{arcs of } D\} \to X \;\middle|\; \text{condition } \boxed{\alpha} \right\}$$

condition $\boxed{\alpha}$: $x, y \in X$

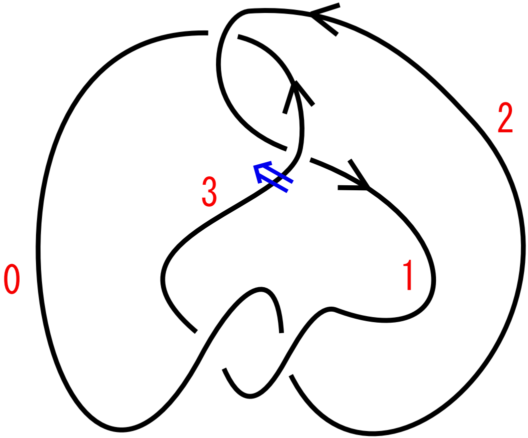



$|\mathrm{Col}_X(K)| := |\mathrm{Col}_X(D)|$ : ***X*-coloring number** of $K$

# Example

$X = S_4 = (\{0.1.2.3\}, *)$, $K = 4_1$

$|\mathrm{Col}_X(K)| = 16$



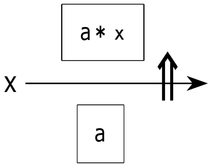

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 3 | 1 | 0 | 2 |
| 2 | 1 | 3 | 2 | 0 |
| 3 | 2 | 0 | 1 | 3 |

# Shadow coloring

## Definition

$$\mathrm{SCol}_X(D) := \Big\{ \, c^s : \{\text{arcs of } D\} \cup \{\text{regions of } D\} \to X \;\Big|\; \text{condition } \boxed{\alpha} + \boxed{\beta} \, \Big\}$$

<u>condition $\boxed{\beta}$</u>:

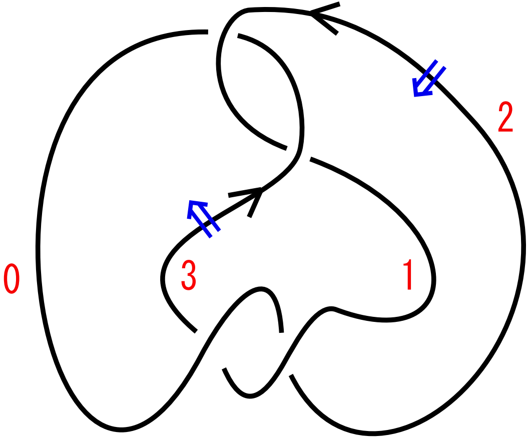



## Important remark

$$\begin{array}{ccc} \mathrm{SCol}_X(D) & \xleftrightarrow{1:1} & \mathrm{Col}_X(D) \times X \\ c^s & \longmapsto & \left(c^s|_{\{\text{arcs of D}\}},\, c^s(\infty\text{-}region)\right) \end{array}$$

# Example

$X = S_4 = (\{0.1.2.3\}, *)$, $K = 4_1$




| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 3 | 1 | 0 | 2 |
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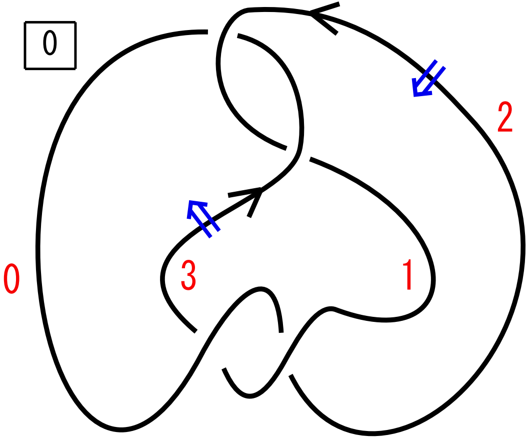
# Example

$X = S_4 = (\{0.1.2.3\}, *)$, $K = 4_1$




| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 3 | 1 | 0 | 2 |
| 2 | 1 | 3 | 2 | 0 |
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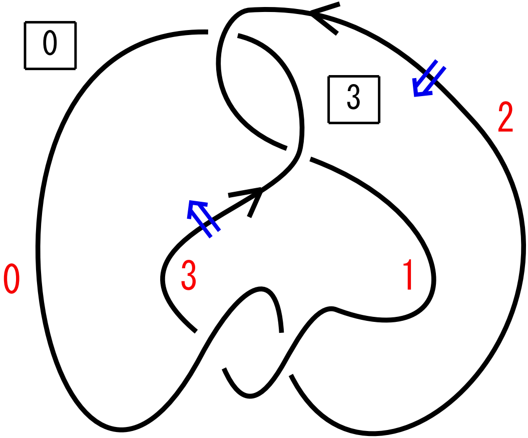
# Example

$X = S_4 = (\{0.1.2.3\}, *)$, $K = 4_1$




| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
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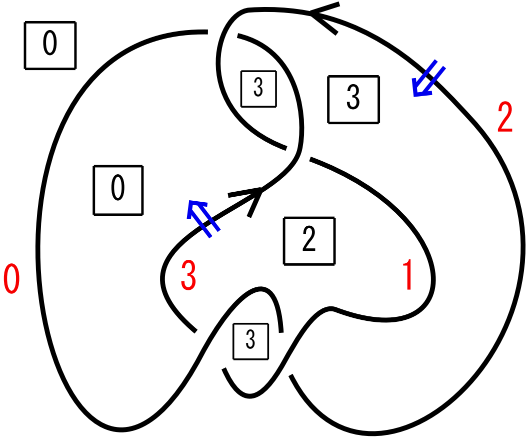


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$|\mathrm{Col}_X(D)| = 16$
$|\mathrm{SCol}_X(D)| = 64$
$(= 16 \times 4)$

# Quandle cocycles

$X$: a quandle, $A$: an abelian group

## Definition (Quandle 2-cocycle)

$\varphi : X^2 \to A$: a **quandle 2-cocycle**

$\overset{\text{def}}{\Longleftrightarrow}$

1. $\varphi(x, x) = 0 \ (\forall x \in X)$
2. $\varphi(x, y) + \varphi(x * y, z) = \varphi(x, z) + \varphi(x * z, y * z)$ $(\forall x, y, z \in X)$

## Definition (Quandle 3-cocycle)

$\psi : X^3 \to A$: a **quandle 3-cocycle**

$\overset{\text{def}}{\Longleftrightarrow}$

1. $\psi(x, x, y) = \psi(x, y, y) = 0 \ (\forall x, y \in X)$
2. $\psi(x, y, z) + \psi(x, z, w) + \psi(x * z, y * z, y * z, w)$
   $= \psi(x * y, z, w) + \psi(x, y, w) + \psi(x * w, y * w, z * w)$ $(\forall x, y, z, w \in X)$

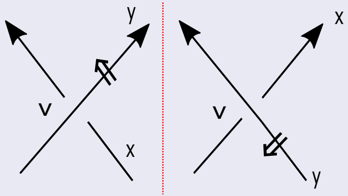
# Quandle 2-cocycle invariant

$K$: an oriented knot, $D$: a diagram of $K$

## Definition [CJKLS (1999)]

$X$: a quandle, $A$: an abelian group, $c \in Col_X(D)$
$\varphi : X^2 \to A$: a quandle 2-cocycle

$$W_\varphi(v, c) := \begin{cases} +\varphi(x, y) & (v : positive) \\ -\varphi(x, y) & (v : negative) \end{cases}$$




$W_\varphi(D, c) := \sum_v W_\varphi(v, c) \in A$
$\Phi_\varphi(D) := \{W_\varphi(D, c) \mid c \in Col_X(D)\}$

$\Phi_\varphi(K) := \Phi_\varphi(D)$: a **quandle 2-cocycle invariant**

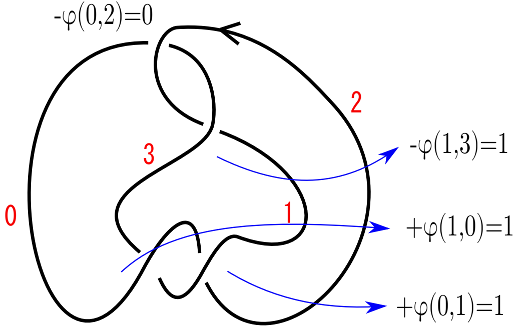
# Example

$K = 4_1$

$\varphi : S_4 \times S_4 \to \mathbb{Z}_2, \quad \varphi(x,y) = \begin{cases} 1 & (x \neq y \text{ and } x, y \in \{0,1,3\}) \\ 0 & (otherwise) \end{cases}$

$W_\varphi(D,c)$
$= -\varphi(0,2) - \varphi(1,3)$
$\quad +\varphi(1,0) + \varphi(0,1)$
$= 1$

$\Phi_\varphi(K) = \{0_4, 1_{12}\}$
$(|\mathrm{Col}_{S_4}(K)| = 16)$



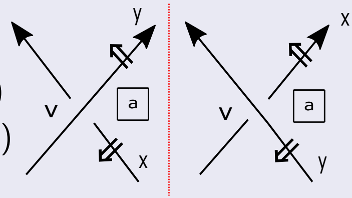
# Shadow 3-cycle invariant

$K$: an oriented knot, $D$: a diagram of $K$

## Definition [CKS (2001)]

$X$: a quandle, $A$: an abelian group, $c^s \in SCol_X(D)$
$\psi : X^3 \to A$: a quandle 3-cocycle

$$W^s_\psi(v, c^s) := \begin{cases} +\psi(a, x, y) & (v : positive) \\ -\psi(a, x, y) & (v : negative) \end{cases}$$




$W^s_\psi(D, c^s) := \sum_v W^s_\psi(v, c^s) \in A$
$\Phi^s_\psi(D) := \{W_\psi(D, c^s) \mid c^s \in SCol_X(D)\}$

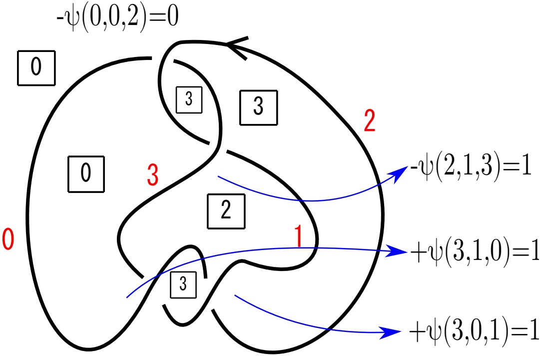
$\Phi^s_\psi(K) := \Phi^s_\psi(D)$: a **shadow 3-cocycle invariant**

# Example 1

$K = 4_1$
$\psi : S_4 \times S_4 \times S_4 \to \mathbb{Z}_2$: a specific quandle 3-cocycle

$$\begin{aligned} & W^s_\psi(D, c^s) \\ &= -\psi(0,0,2) - \psi(2,1,3) \\ &\quad +\psi(3,1,0) + \psi(3,0,1) \\ &= 1 \end{aligned}$$

$\Phi^s_\psi(K) = \{0_{16}, 1_{48}\}$
$(|\mathrm{SCol}_{S_4}(K)| = 64)$



# Example 2

$K = 3_1$
$\psi : R_3 \times R_3 \times R_3 \to \mathbb{Z}_3$
$\psi(x, y, z) = (x - y)(y - z)^2 z$ : Mochizuki's 3-cocycle

$\Phi^s_\psi(K) = \{0_9, 1_{18}\}$
$\Phi^s_\psi(K^*) = \{0_9, 2_{18}\}$

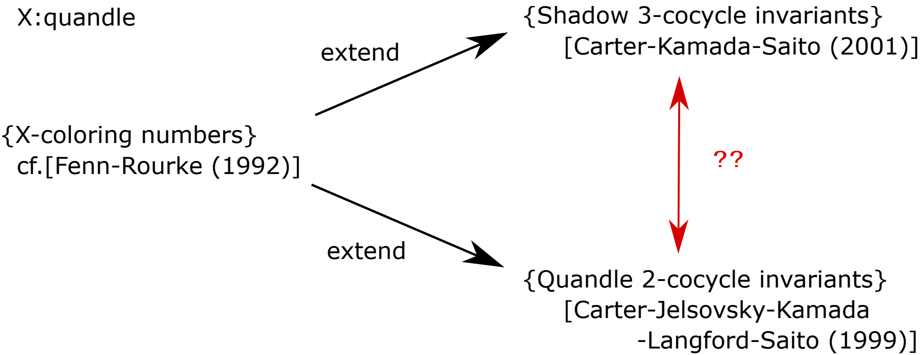
$\therefore 3_1 \nsim 3_1^*$

**Remark**

$\forall \varphi : R_3 \times R_3 \to \mathbb{Z}_3$ : a quandle 2-cocycle, $\Phi_\varphi(K) = \{0_9\}$

$(|\mathrm{Col}_{R_3}(K)| = 9,\ |\mathrm{SCol}_{R_3}(K)| = 27)$

X:quandle


{Shadow 3-cocycle invariants}
[Carter-Kamada-Saito (2001)]
extend
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??
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# Theorem 1

## Theorem 1

$X$: a finite quandle, $A$: an abelian group

$\varphi : X^2 \to A$: a quandle 2-cocycle

(1) $\psi_\varphi : X^3 \to A$

$\psi_\varphi(x, y, z) = \varphi(y, z) - \varphi(x, z) + \varphi(x, y)$ is a quandle 3-cocycle.

(2) $\Phi^s_{\psi_\varphi}(K) = |X|\,\Phi_\varphi(K)$

Notation

$A = \mathbb{Z}_3$

$3\{0_1, 1_3, 2_5\}$

$:= \{0_3, 1_9, 2_{15}\}$

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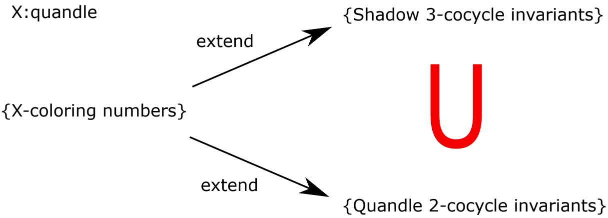



<u>Notation</u>

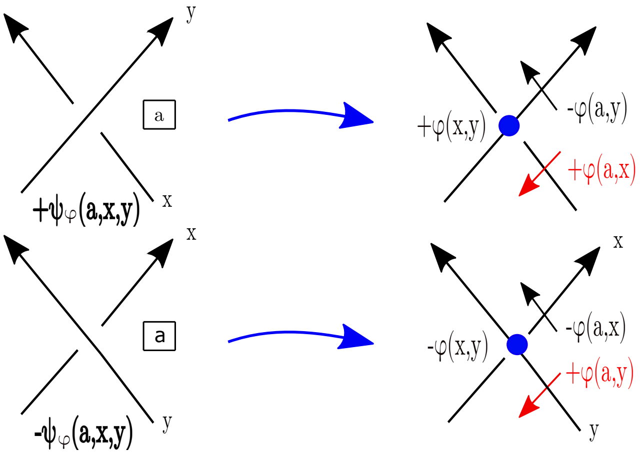
$A = \mathbb{Z}_3$

$3\{0_1, 1_3, 2_5\}$

$:= \{0_3, 1_9, 2_{15}\}$

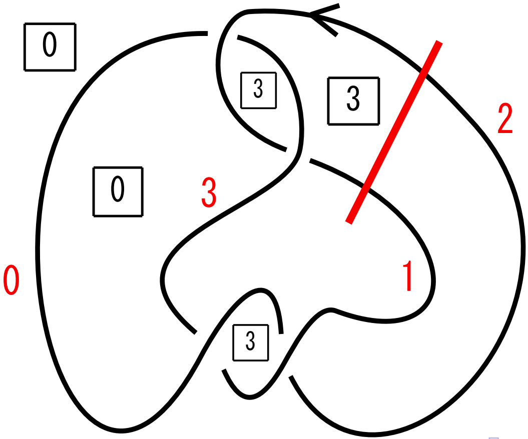
# Proof of Thm1 (2)

Interpret the weights on crossings as follow.
$\psi_\varphi(a,x,y) = \varphi(x,y) - \varphi(a,y) + \varphi(a,x)$

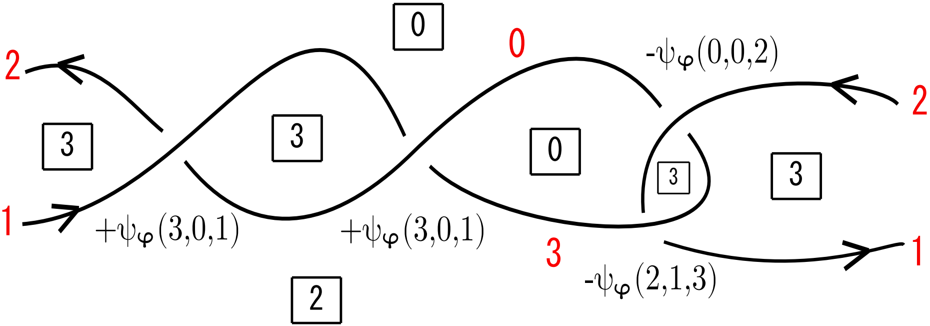


# Proof of Thm1 (2)

$K = 4_1, X = S_4,$
$\varphi : X \times X \to A$ : a quandle 2-cocycle
$c^s \in \mathrm{SCol}_X(D)$



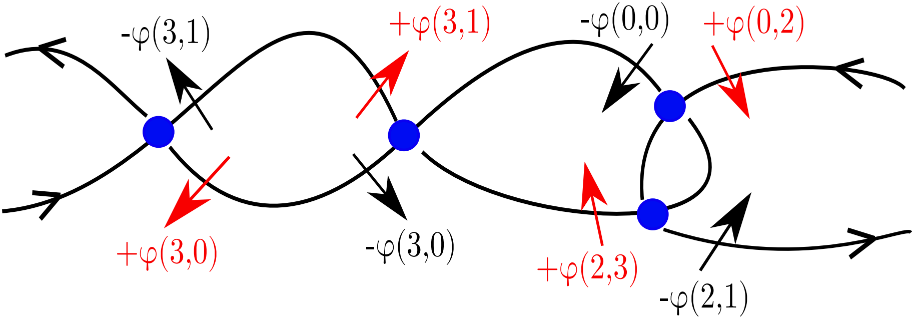
# Proof of Thm1 (2)




We want to prove $W^{s}_{\psi_\varphi}(D, c^s) = W_\varphi(D, c)$.

$c := c^s\big|_{\{\text{arcs of D}\}}$

# Proof of Thm1 (2)




## Remark

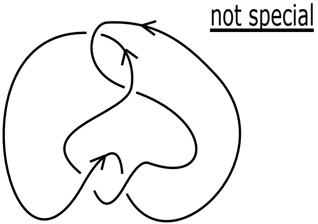
The sum of • is $W_\varphi(D, c)$

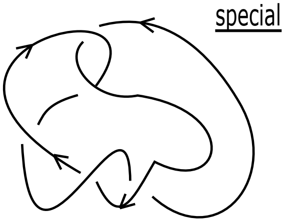
$$-\varphi(0,0) + \varphi(0,2) + \varphi(2,3) - \varphi(2,1) = 0 \text{ ??}$$

## Proposition [Murasugi (1965)]

Every knot posseses a special diagram.

Definition(Special diagram)



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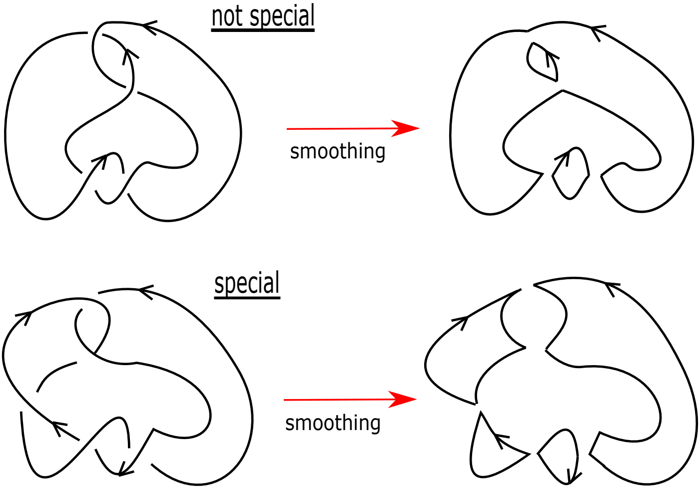

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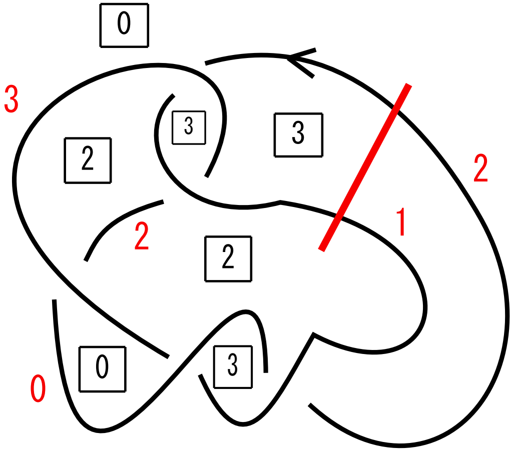

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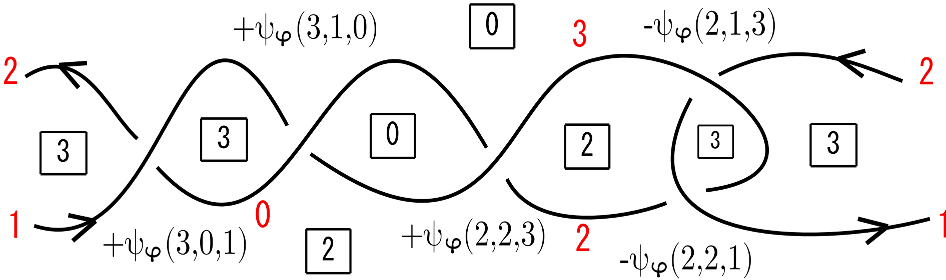


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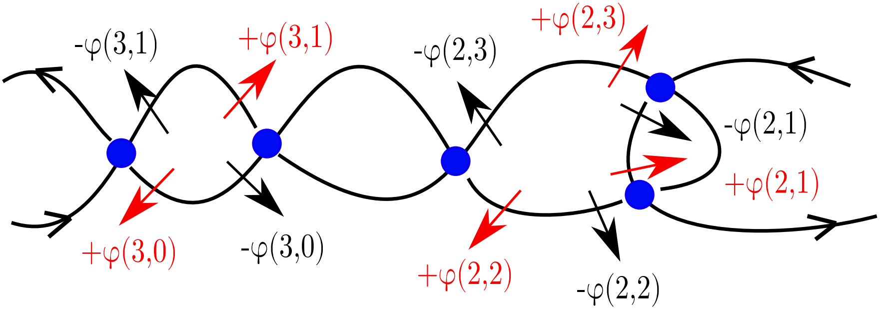

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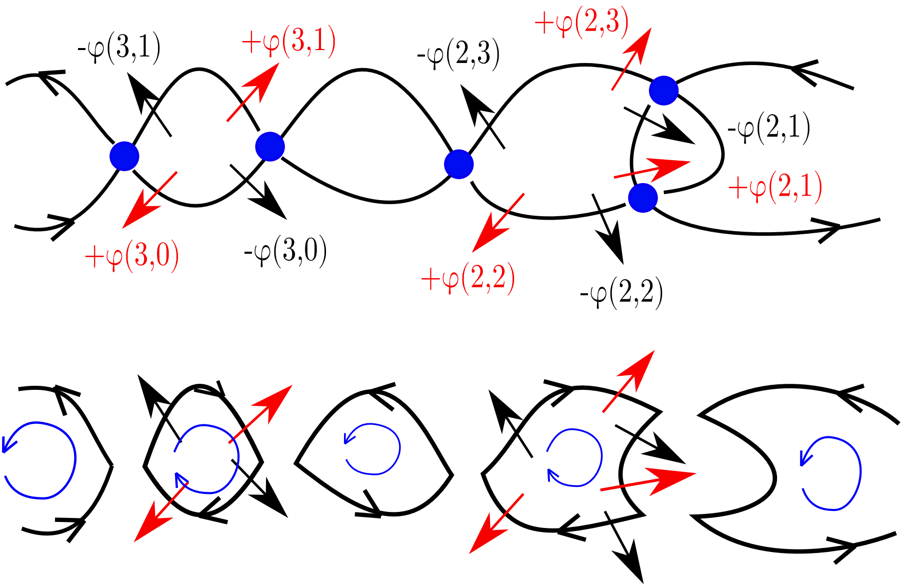

# Theorem 2

## Theorem 2

$X$: a quandle, $A$: an abelian group
$\sigma_\sharp : C^Q_n(X) \to C^Q_{n-1}(X)$, $(x_1, ..., x_n) \mapsto \sum_{i=1}^n (-1)^{n+i}(x_1, .., \hat{x}_i, .., x_n)$
Then
$\sigma_\sharp : C^Q_*(X) \to C^Q_{*-1}(X)$ is a chain map.

$\sigma_\sharp$ induces the cochain map $\sigma^\sharp : C_Q^{*-1}(X;A) \to C_Q^*(X;A)$.
$\hookrightarrow \varphi$ :(n-1)-cocycle $\Rightarrow \sigma^\sharp(\varphi)$ :n-cocycle

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Then

$\sigma_\sharp : C_*^Q(X) \to C_{*-1}^Q(X)$ is a chain map.

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$\hookrightarrow \varphi$ :(n-1)-cocycle $\Rightarrow \sigma^\sharp(\varphi)$ :n-cocycle

## Theorem 1 (1)

$\varphi : X^2 \to A$: a quandle 2-cocycle

$\psi_\varphi(x, y, z) = \varphi(y, z) - \varphi(x, z) + \varphi(x, y)$ is a quandle 3-cocycle.

$$\psi_\varphi = \sigma^\sharp(\varphi)$$