Crosscap number and knot projections

伊藤昇氏(東京大学)との共同研究

学習院中等科
瀧村祐介

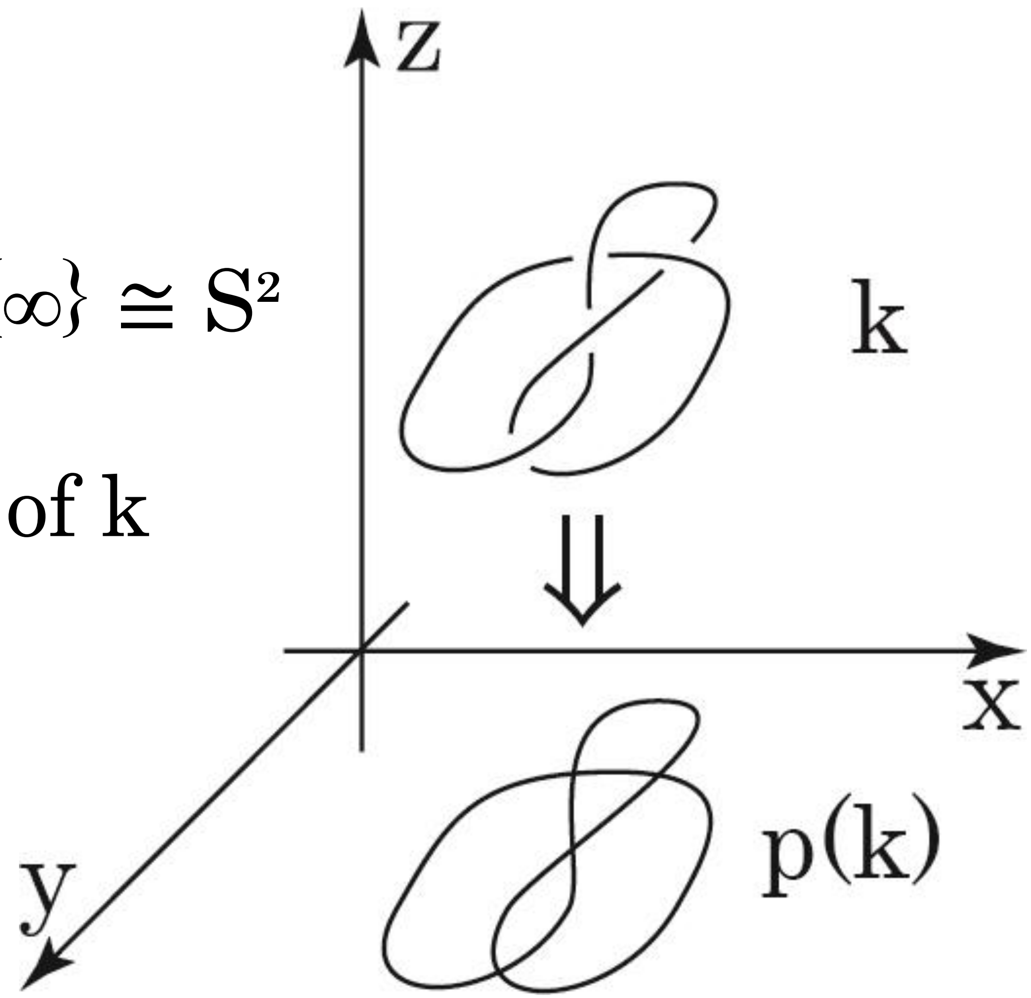
Definition

k: knot in $\mathbb{R}^3$

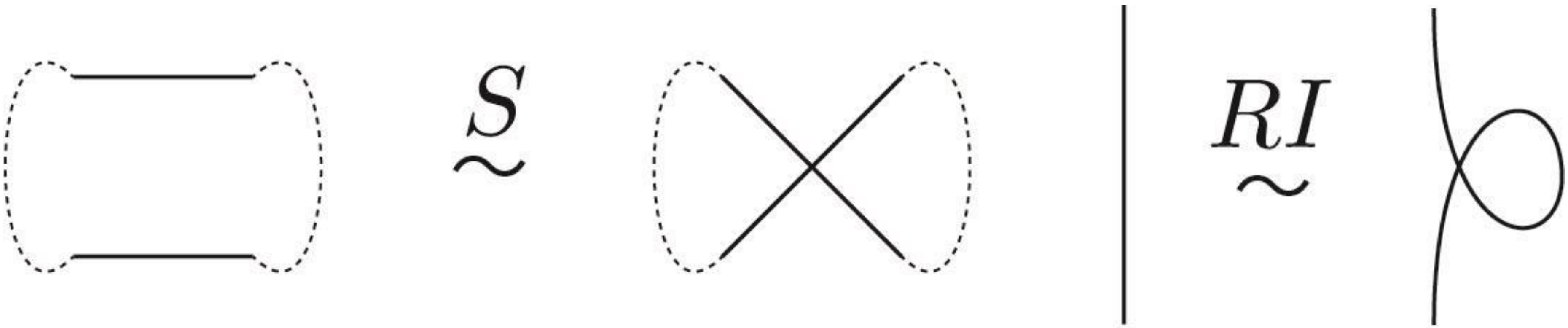
p : $\mathbb{R}^3 \rightarrow \mathbb{R}^2 \subset \mathbb{R}^2 \cup \{\infty\} \cong S^2$

p(k) : knot projection of k

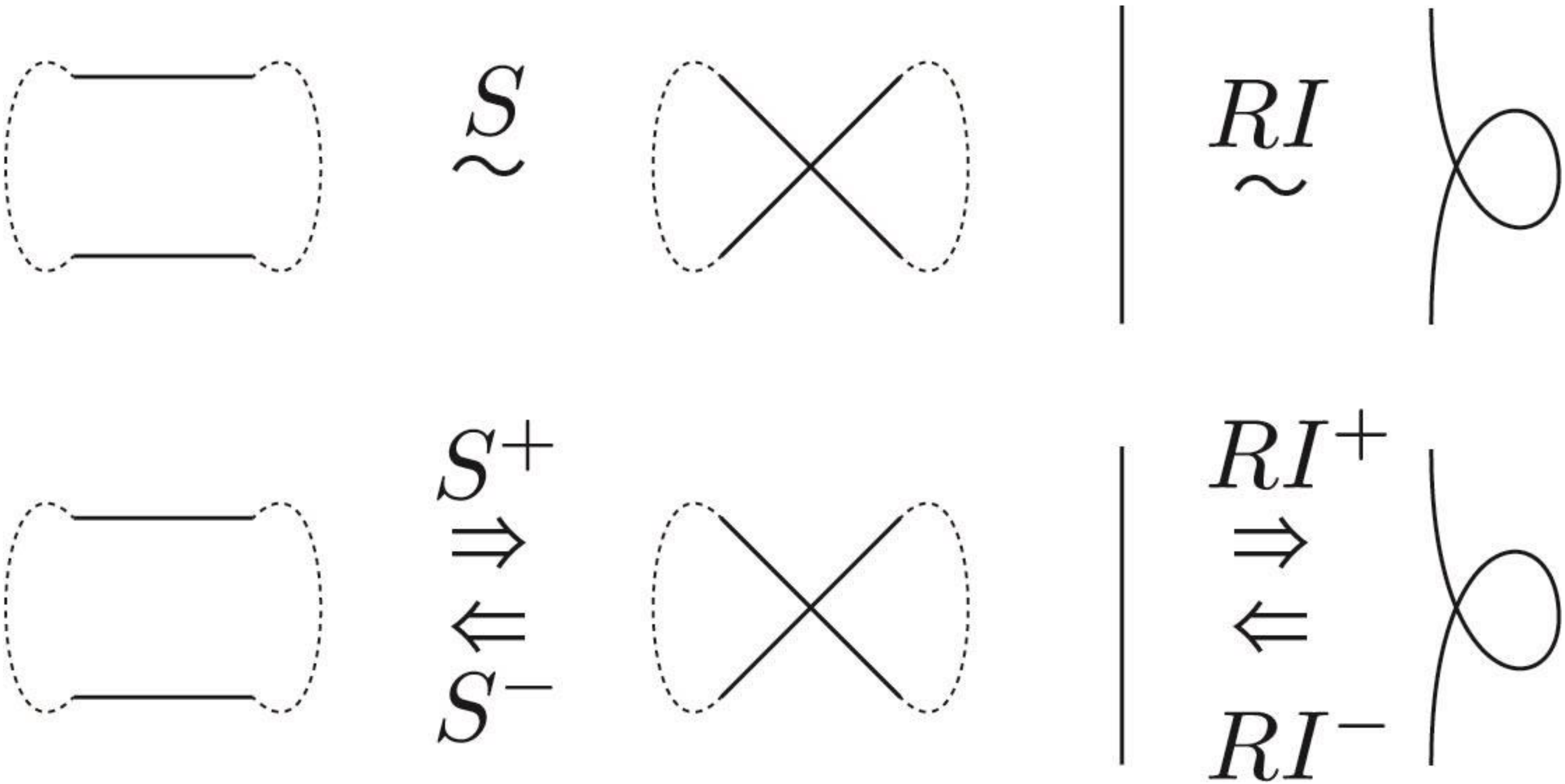
以後、P と表す



Definition



Definition

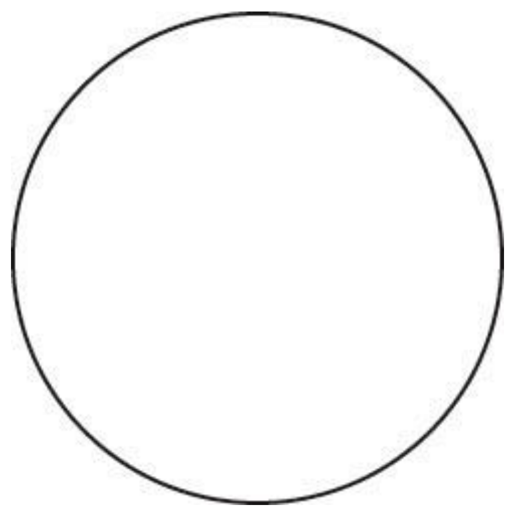


Proposition

任意の knot projection P は

有限回の S と RI で

simple closed curve に変形できる



※ O と表す

Definition

PをS, RIの有限列で
simple closed curveにするために
必要なSの最小回数をu(P)と表す

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P を S, RI の有限列で
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P を S^-, RI^- の有限列で
simple closed curve にするために
必要な S^- の最小回数を $u^-(P)$ と表す

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PをS, RIの有限列で
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$u(P) \leq u^-(P)$

Definition

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P を S^-, RI^- の有限列で
simple closed curve にするために
必要な S^- の最小回数を $u^-(P)$ と表す

$u(P) \leq u^-(P)$

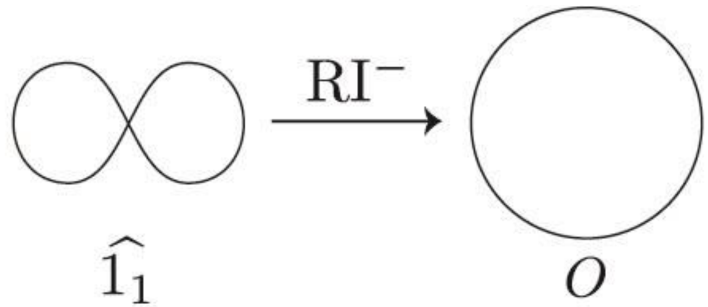
Proposition

・$u(P)$ は加法性が成り立たない

・$u^{-}(P_1 \sharp P_2) = u^{-}(P_1) + u^{-}(P_2)$

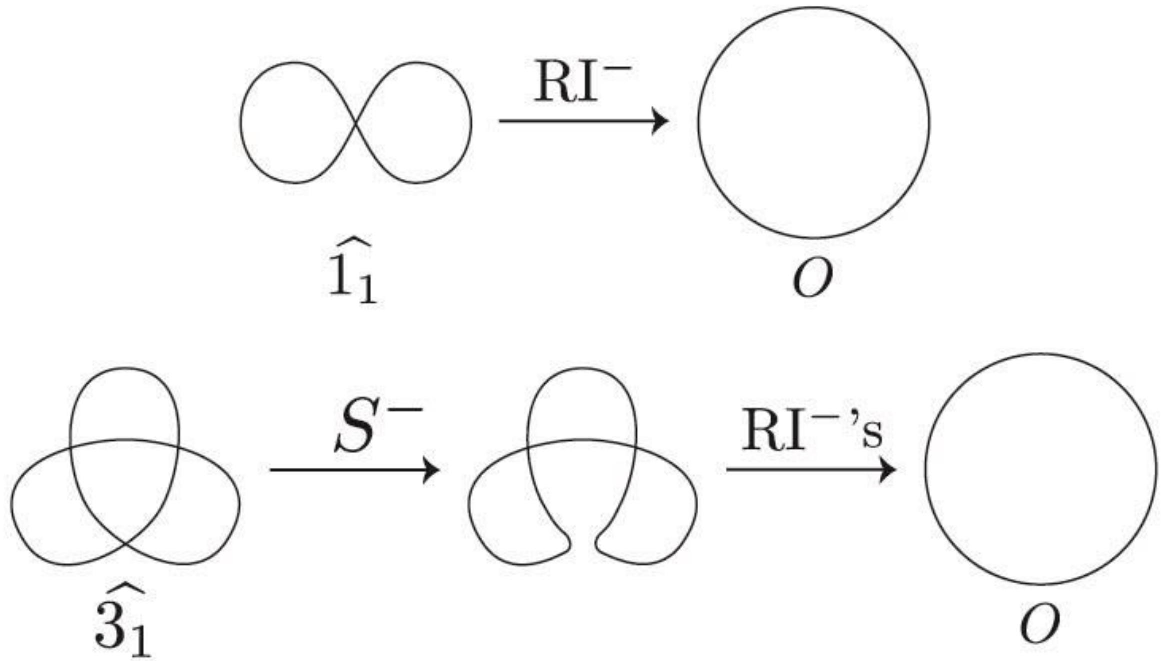
Example

Example



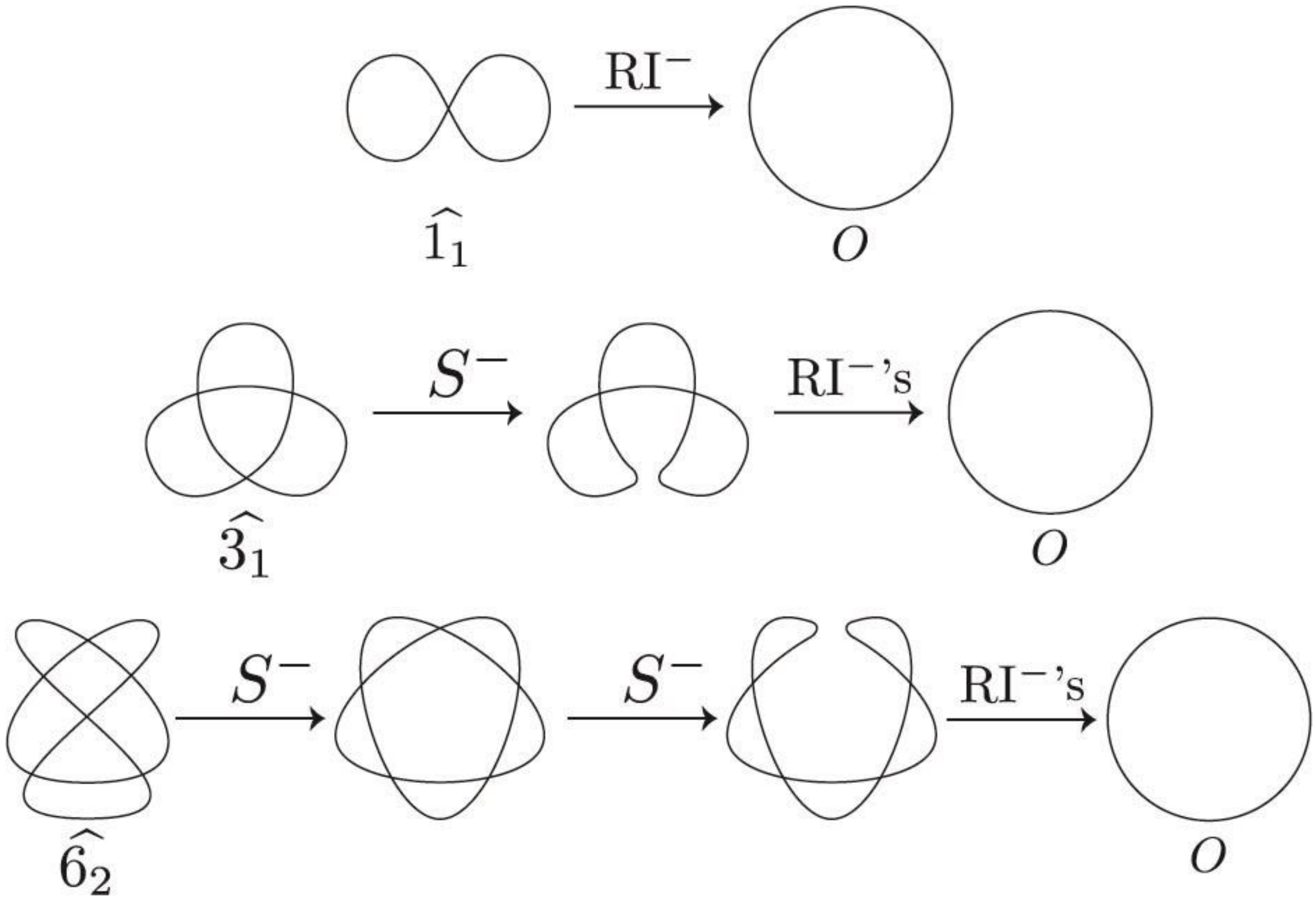
$$u^{-}(\widehat{1}_1) = 0$$

Example



$$u^-(\widehat{1}_1)=0 \qquad u^-(\widehat{3}_1)=1$$

Example



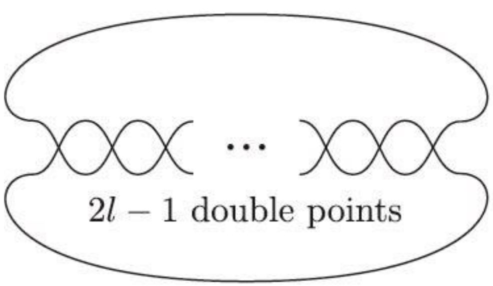
$u^-(\widehat{1}_1) = 0$ $u^-(\widehat{3}_1) = 1$ $u^-(\widehat{6}_2) = 2$

Definition

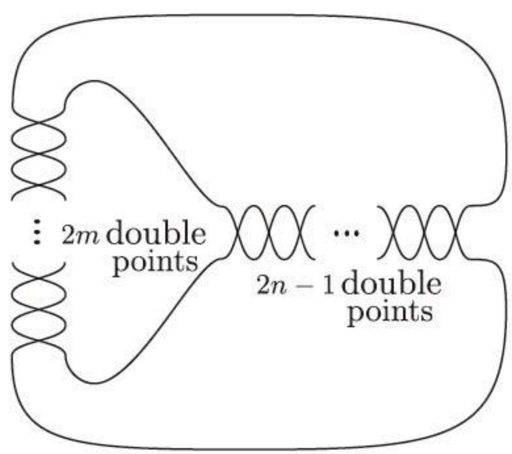
T : (2, 2l−1)-torus knot projections の集合 (l ≧ 2)

R : (2m, 2n−1)−rational knot projections の集合 (m ≧ 1, n ≧ 2)

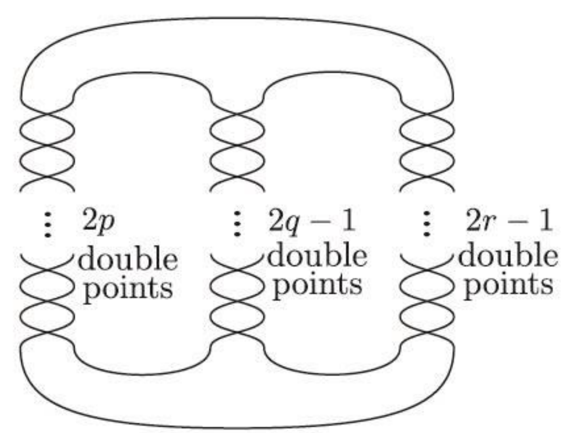
Z : (2p, 2q−1, 2r−1)−pretzel knot projections の集合 (p, q, r ≧ 1)



(2, 2l − 1)-torus knot projection



(2m, 2n − 1)-rational knot projection

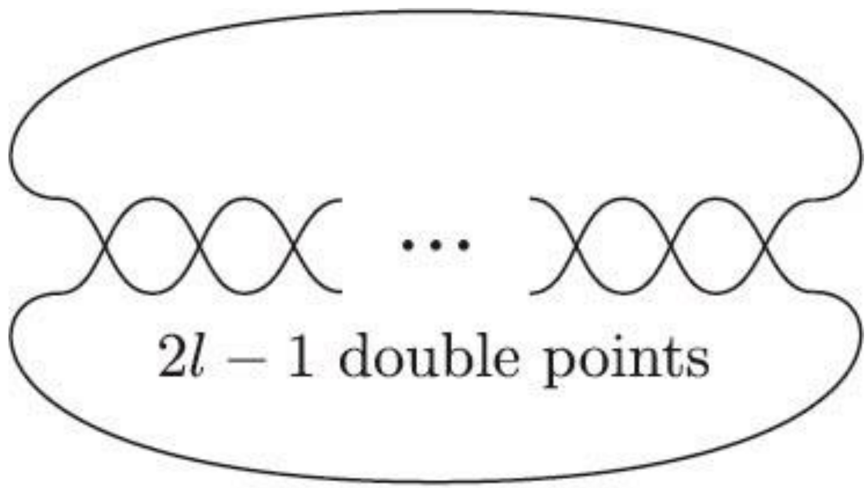


(2p, 2q − 1, 2r − 1)-pretzel knot projection

Theorem 1 (1) (Ito-T., 2018, IJM)

P : reduced knot projection

$u^{-}(P) = 1 \Leftrightarrow P \in T$

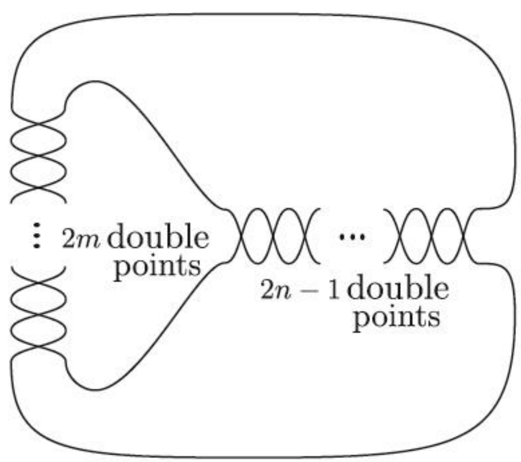


$(2, 2l-1)$-torus knot projection

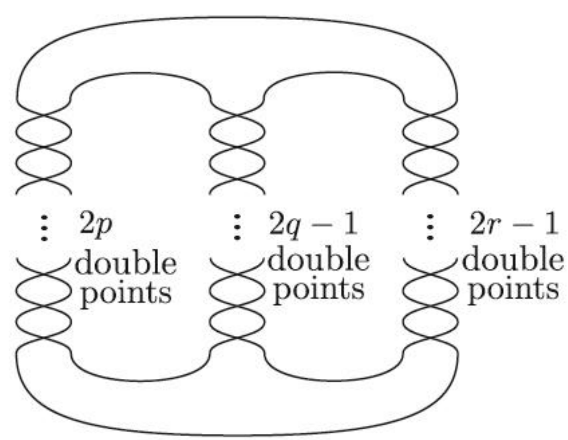
Theorem 1 (2) (Ito-T., 2018, IJM)

P : reduced knot projection

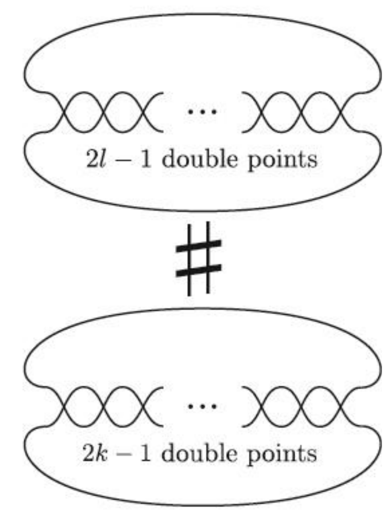
$$u^{-}(P) = 2 \Leftrightarrow P \in R \cup Z \text{ or } P = t \sharp t$$
$$(t \in T)$$



(2m, 2n − 1)-rational knot projection

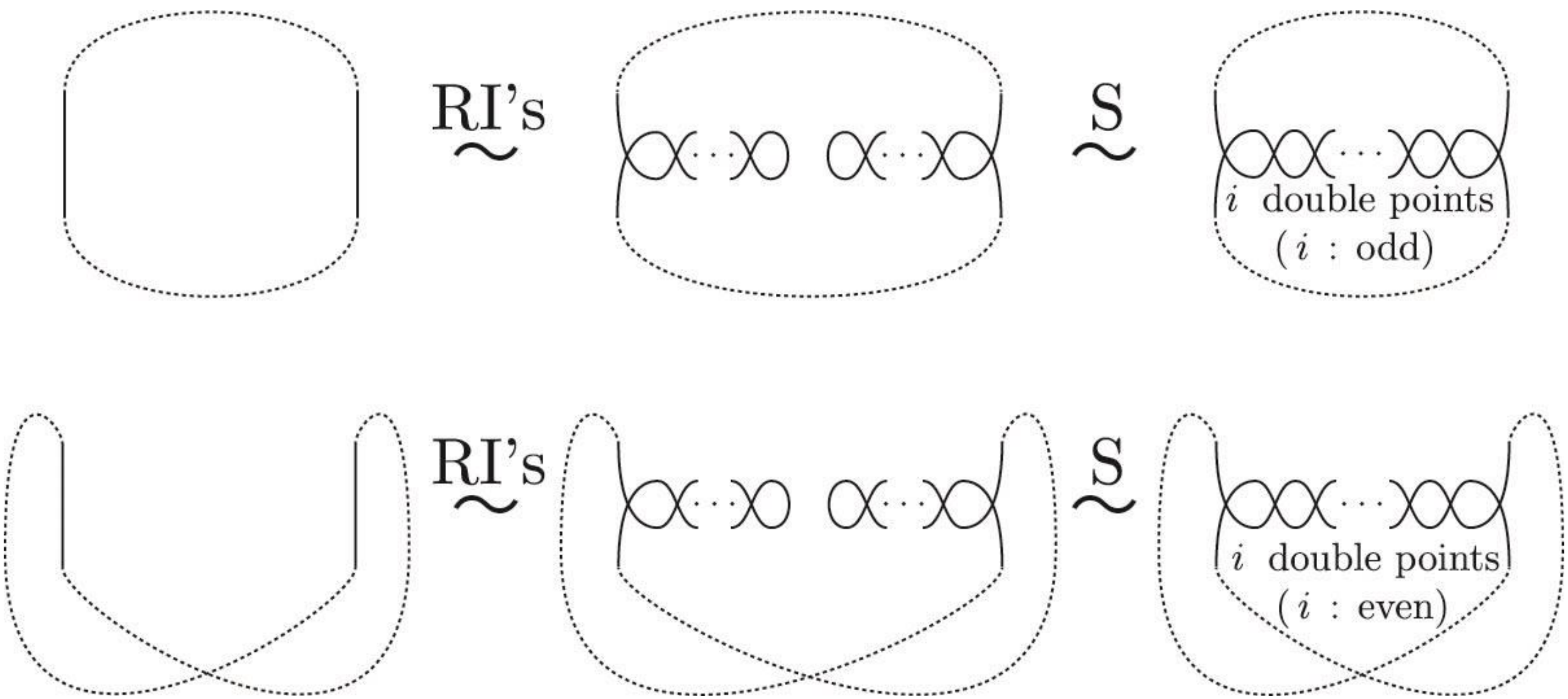


(2p, 2q − 1, 2r − 1)-pretzel knot projection

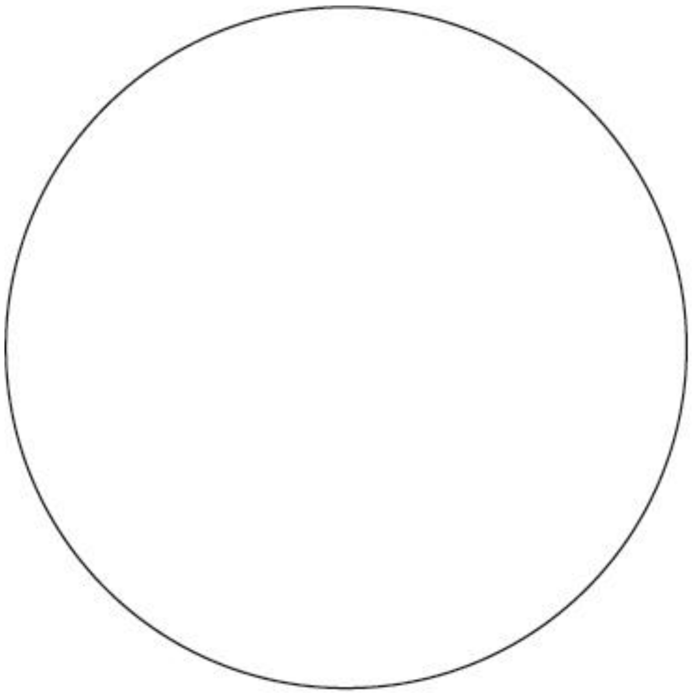


Lemma

1回の S と有限回の RI で
次の局所変形が可能



Proof of Theorem 1



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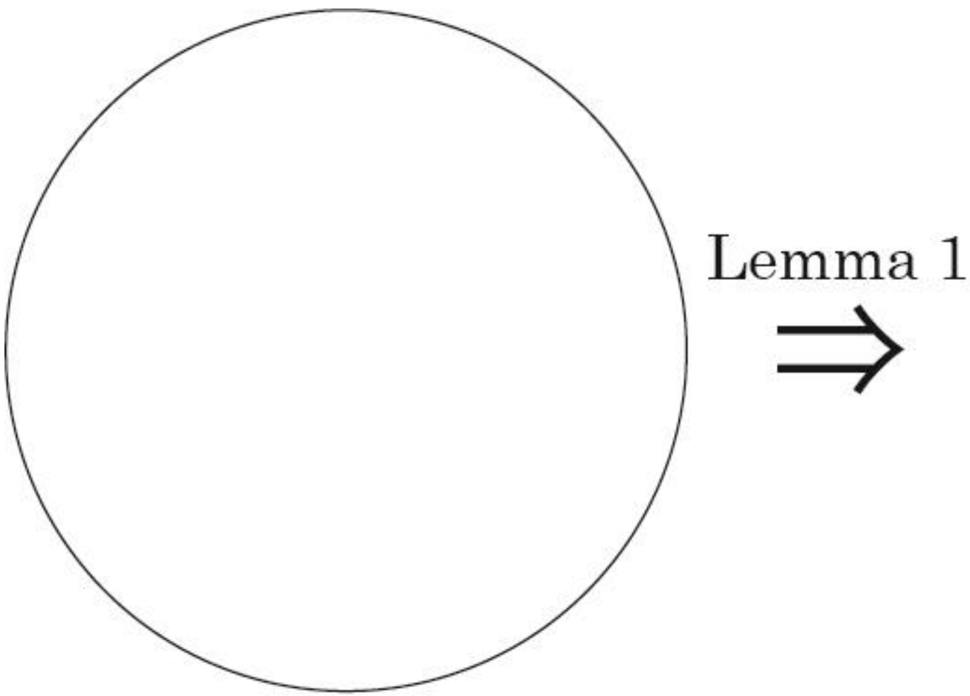
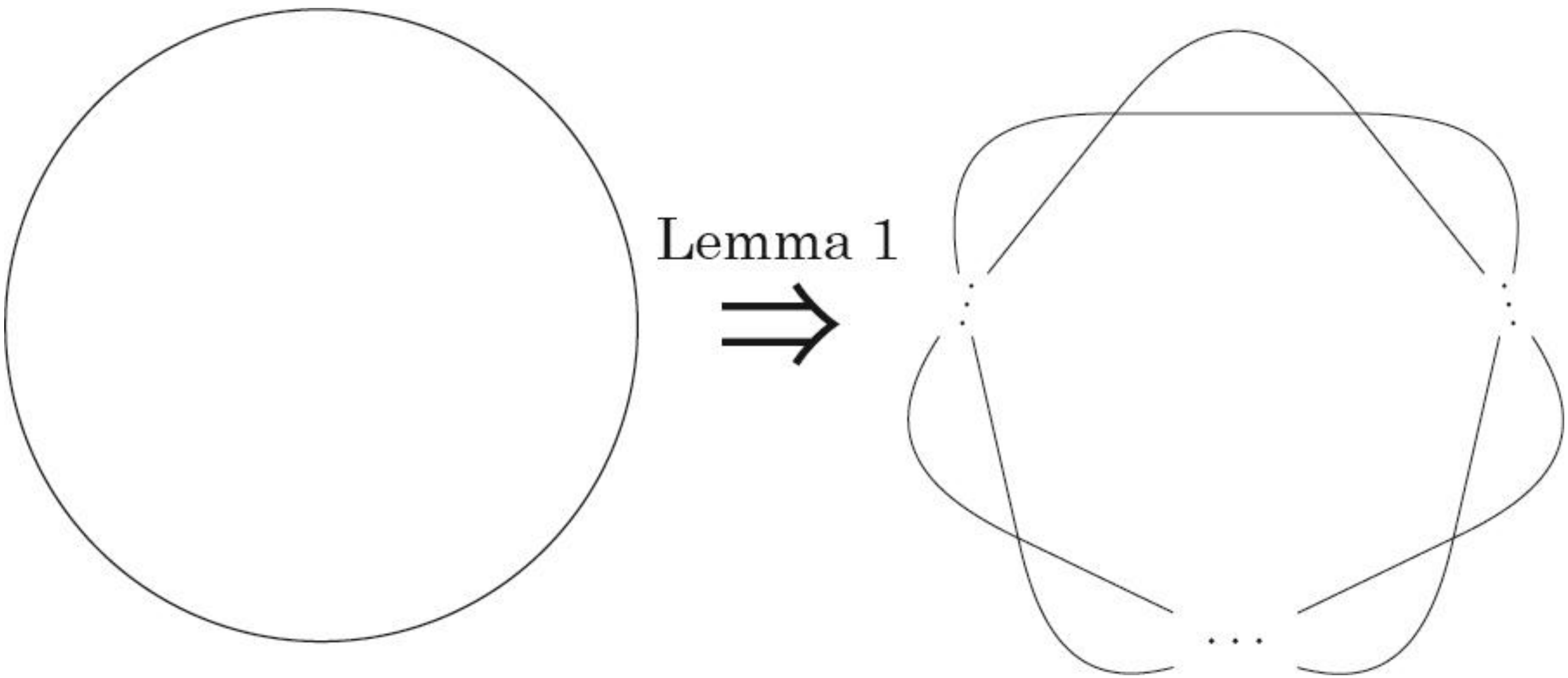
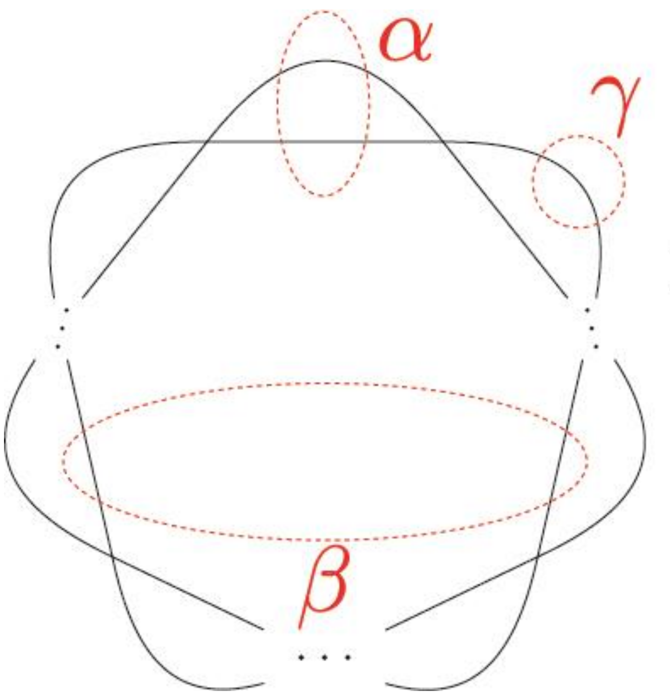
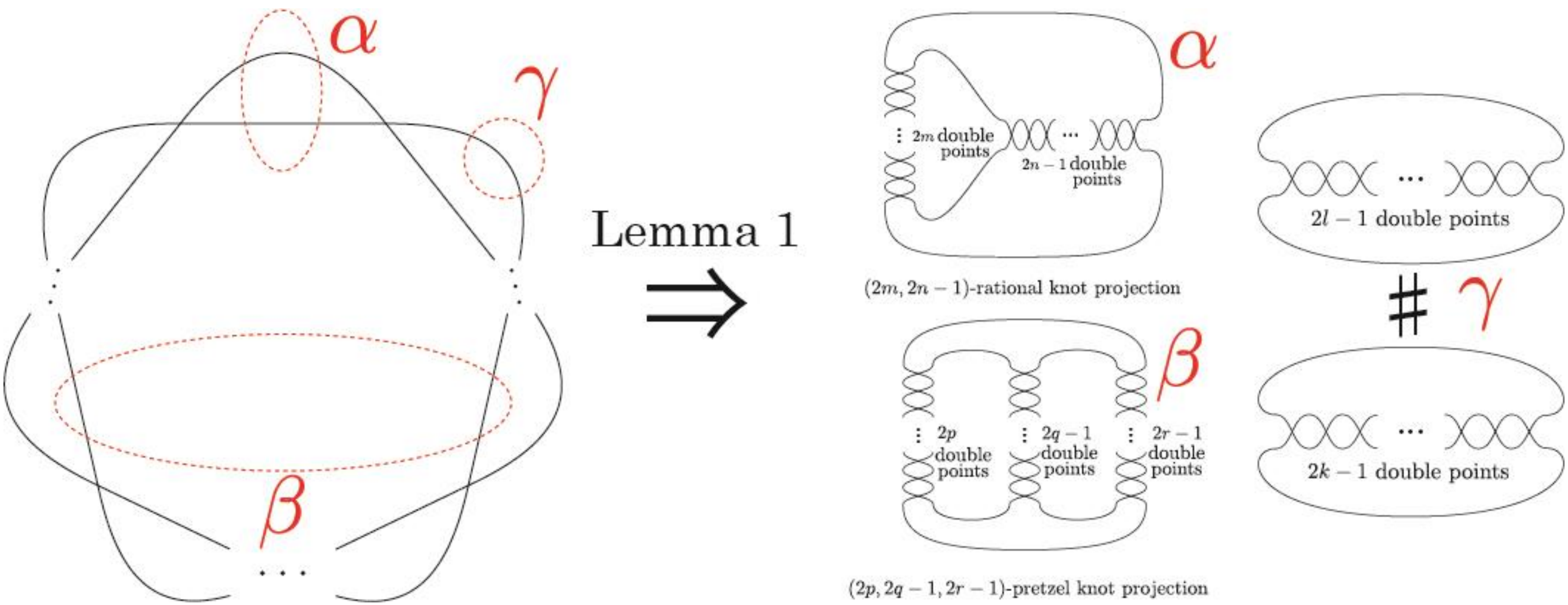


Proof of Theorem 1




α
γ
β

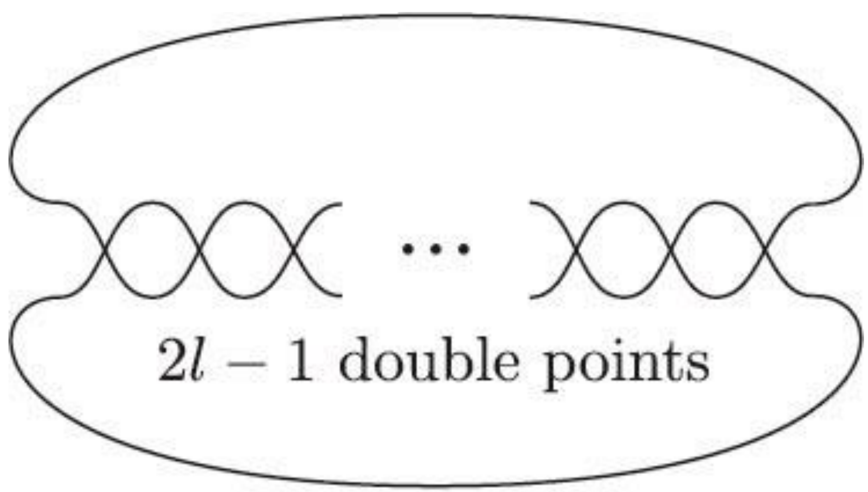


α
γ
β
Lemma 1
⇒
2m double points
2n − 1 double points
(2m, 2n − 1)-rational knot projection
α
2p double points
2q − 1 double points
2r − 1 double points
(2p, 2q − 1, 2r − 1)-pretzel knot projection
β
2l − 1 double points
♯ γ
2k − 1 double points

Theorem 1 (1) (Ito-T., 2018, IJM)

P : reduced knot projection

$u^{-}(P) = 1 \Leftrightarrow P \in T$

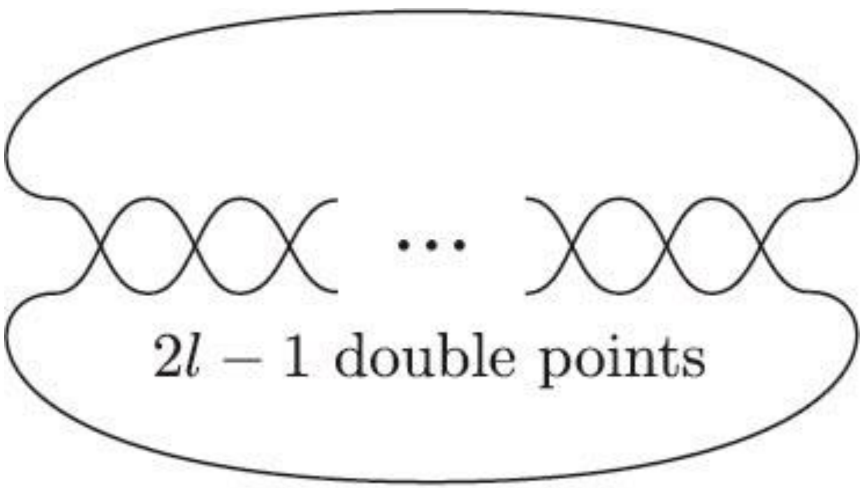


$(2, 2l-1)$-torus knot projection

Theorem 1 (1) (Ito-T., 2018, IJM)

P : reduced knot projection

$u^{-}(P) = 1 \Leftrightarrow P \in T \Leftrightarrow u(P) = 1$



$(2, 2l-1)$-torus knot projection

Theorem 1 (2) (Ito-T., 2018, IJM)

P : reduced knot projection

$u^{-}(P) = 2 \Leftrightarrow P \in R \cup Z$ or $P = t \# t$
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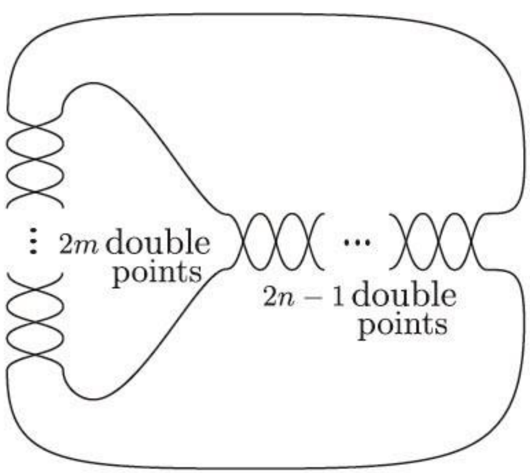
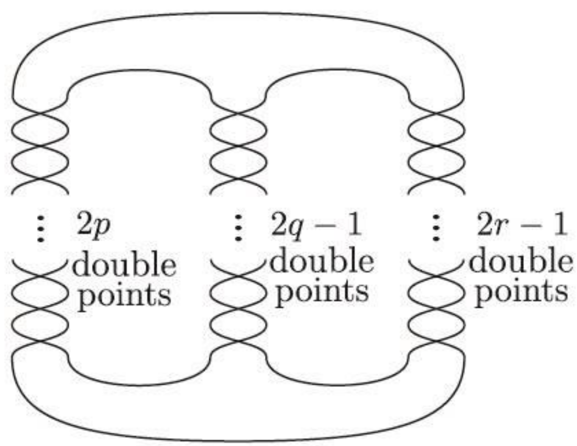
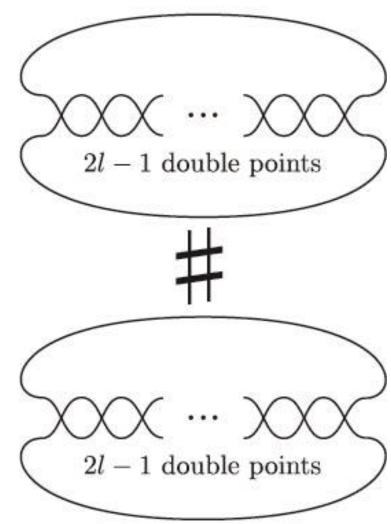


(2m, 2n − 1)-rational knot projection



(2p, 2q − 1, 2r − 1)-pretzel knot projection

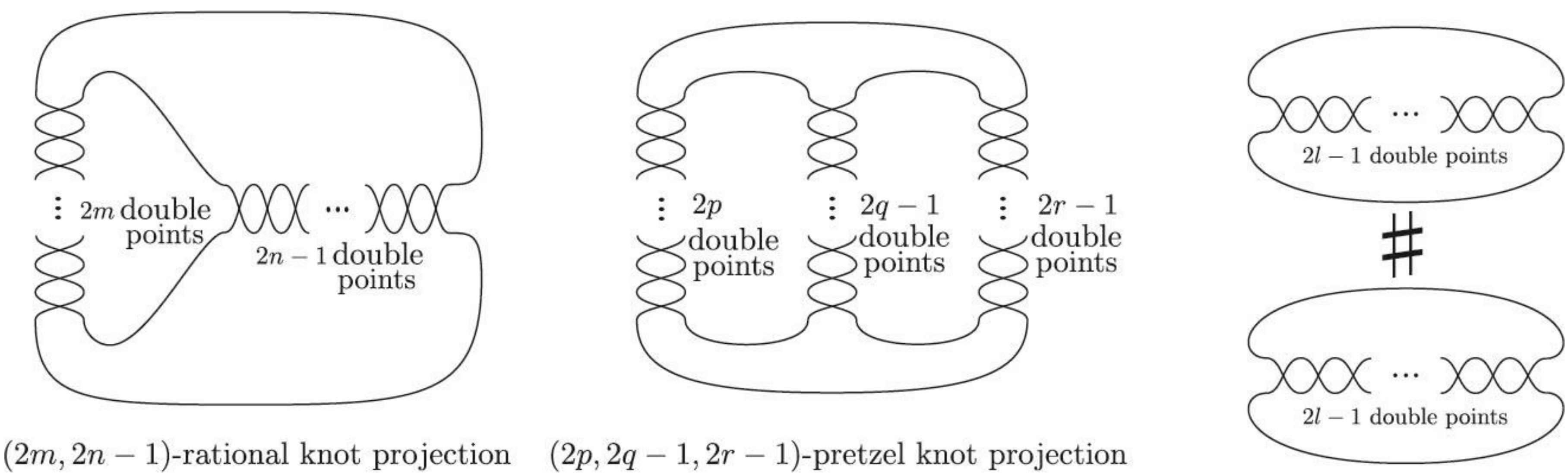


Theorem 1 (2) (Ito-T., 2018, IJM)

P : reduced knot projection

$u^{-}(P) = 2 \Leftrightarrow P \in R \cup Z$ or $P = t \# t$

$\Leftrightarrow u(P) = 2$ $(t \in T)$

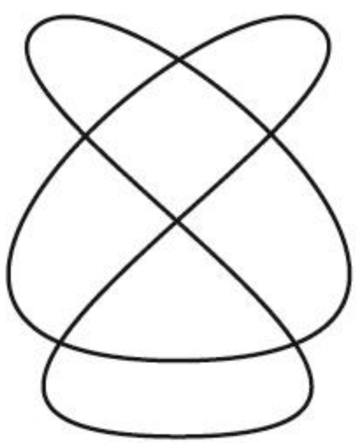


Definition

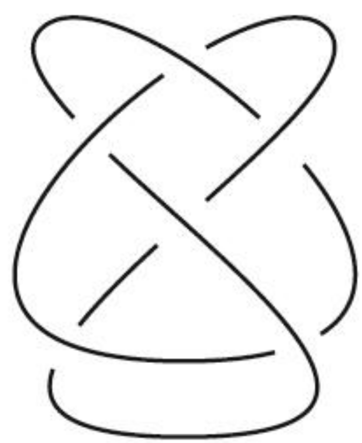
P : knot projection

D_P : P に交点の上下の情報を与えて得られる knot diagram

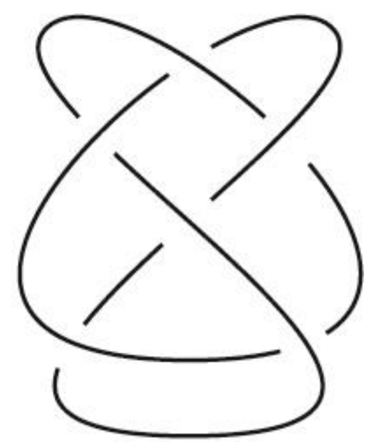
$K(D_P)$: D_Pによって決まる knot



P



D_P



$K(D_P)$

Definition

K : knot

C(K) : K の crosscap number

Definition

K : knot

C(K) : K の crosscap number

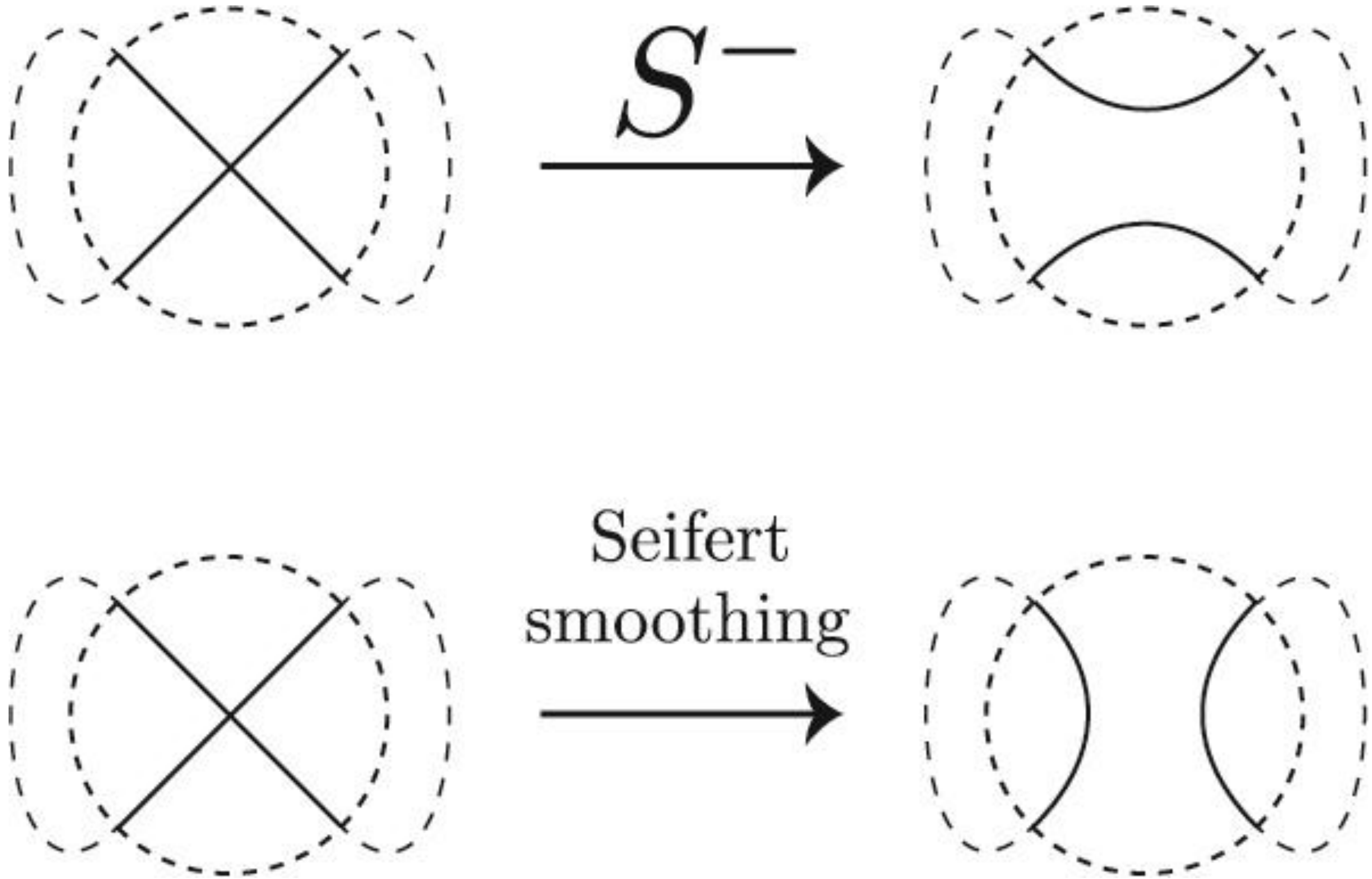
$C(K) := \min\{1 - \chi(\Sigma) \mid \Sigma : \text{a non-orientable surface with } \partial\Sigma = K\}$

Theorem 2 (Ito-T., 2018, IJM)

P : knot projection

$$C(K(D_P)) \leq u^-(P)$$

Definition



Fact

交点数が n である knot diagram に沿って

half-twisted bands をつけて得られる曲面は

2^n 通り存在する

(全ての交点の splice, smoothing の
やり方は2^n 通り存在する)

Fact

交点数が n である knot diagram において

2^n 通りの曲面 $\begin{cases} \text{・全ての交点で Seifert smoothing} \\ \text{・1回でも } S^- \text{ を含む} \end{cases}$

Fact

交点数が n である knot diagram において

2^n 通りの曲面

- ・全ての交点で Seifert smoothing
  orientable surface
- ・1回でも S^- を含む
  non orientable surface

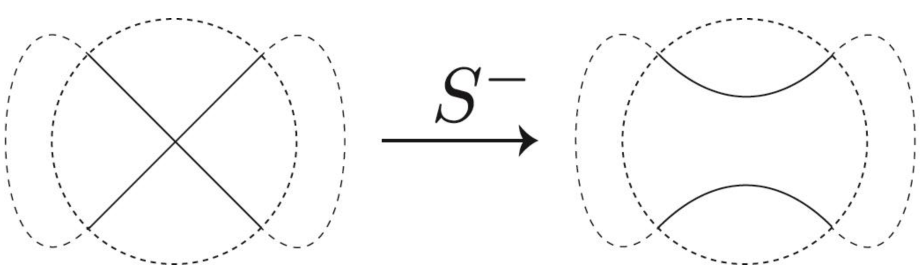
Definition

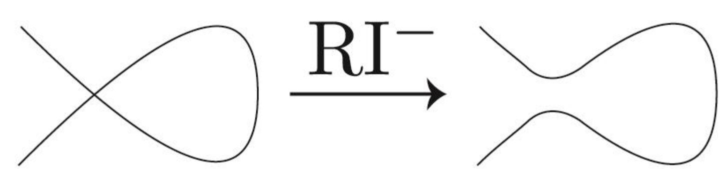
P : knot projection (交点数を n(P) とする)

P から O までの列 (S^- と RI^-)

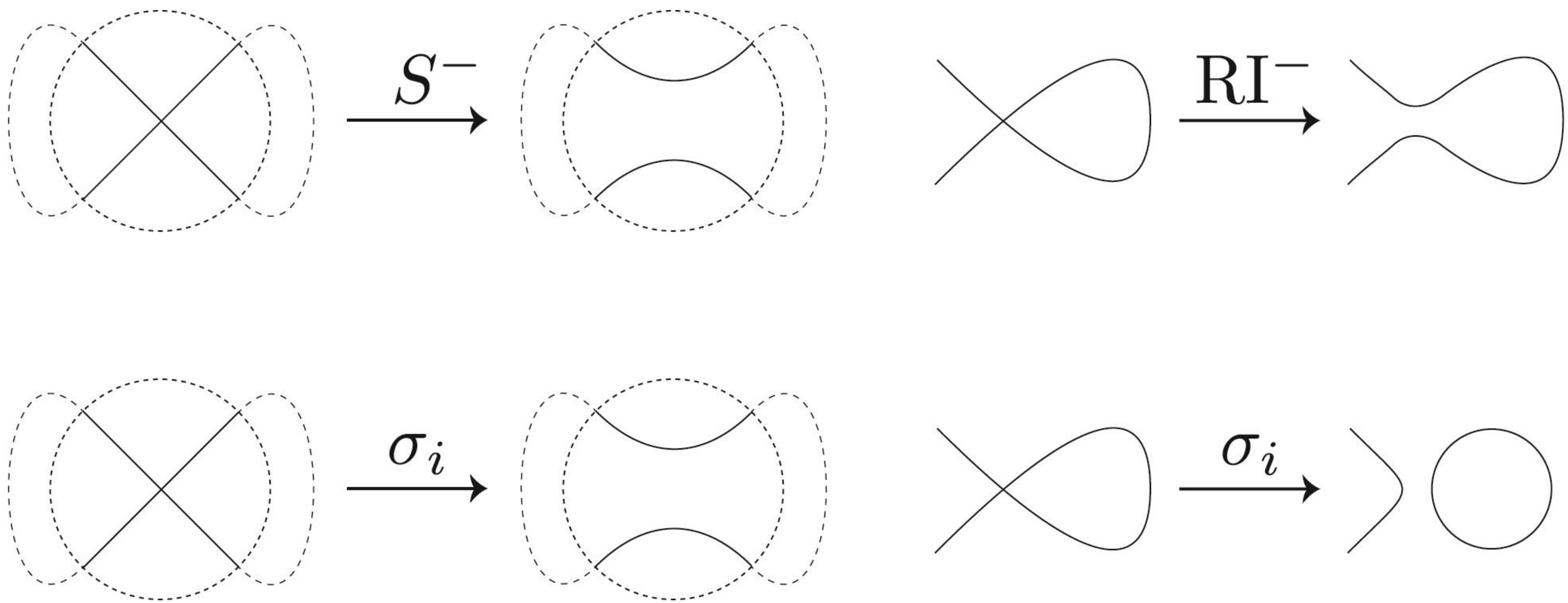
$$
\begin{array}{ccccccccc}
 & \mathrm{Op}_1 & & \mathrm{Op}_2 & & \mathrm{Op}_3 & \cdots & \mathrm{Op}_{n(P)} & \\
P = P_0 & \rightarrow & P_1 & \rightarrow & P_2 & \rightarrow & \cdots & \rightarrow & P_{n(P)} = O
\end{array}
$$

$$\mathrm{Op_i} = \ S^- \text{ or } RI^- \ (1 \le \mathrm{Op_i} \le n(P))$$



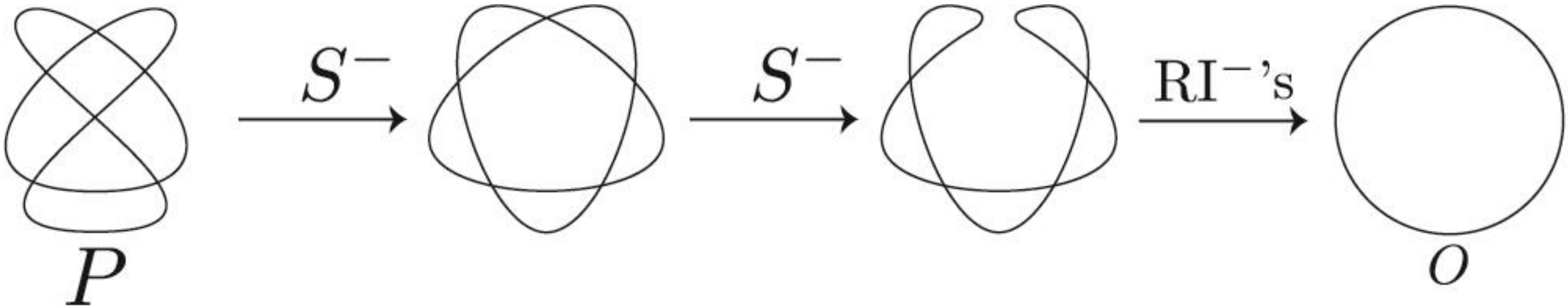


S^- と RI^- に対応して σ_i を定義する
$(1 \leq i \leq n(P))$

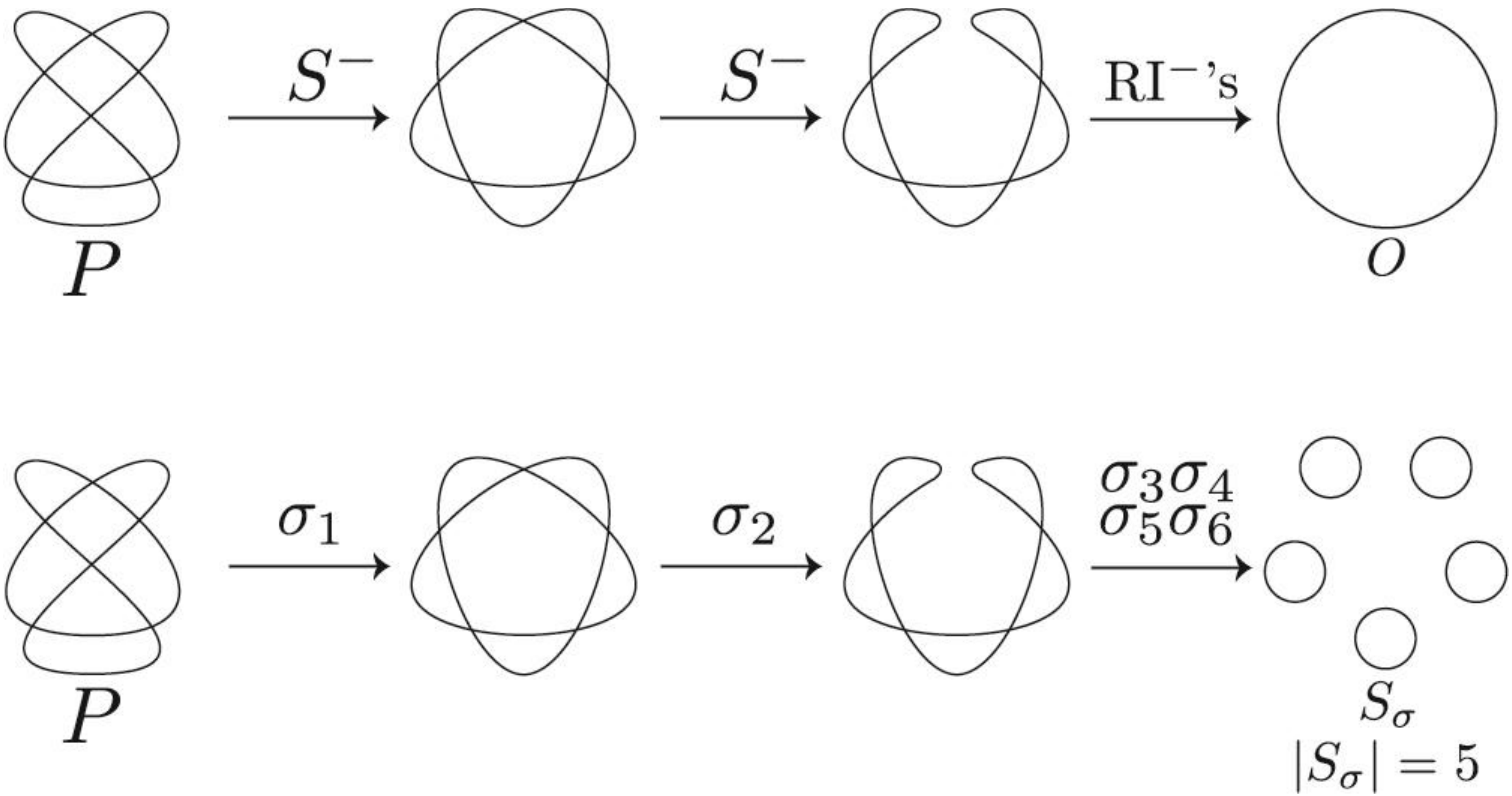


S_σ： P に $\sigma = (\sigma_1, \sigma_2 \cdots \sigma_{n(P)})$ を施して得られる circle の集合

$|S_\sigma|$： S_σ の circle の数



S^-
S^-
RI^-'s
P
O

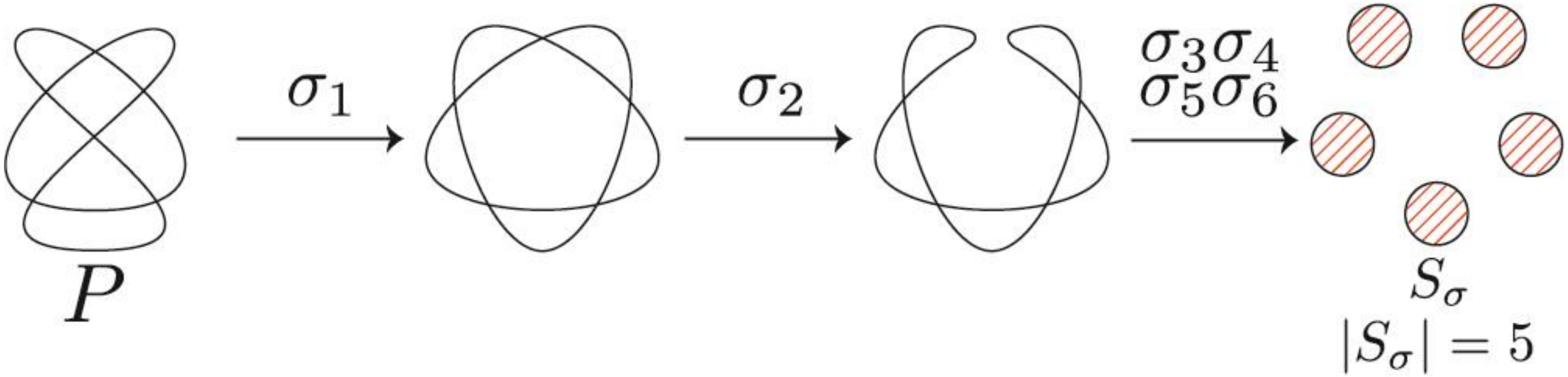


S^-
S^-
RI$^-$'s
P
O
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
P
S_σ
$|S_\sigma| = 5$

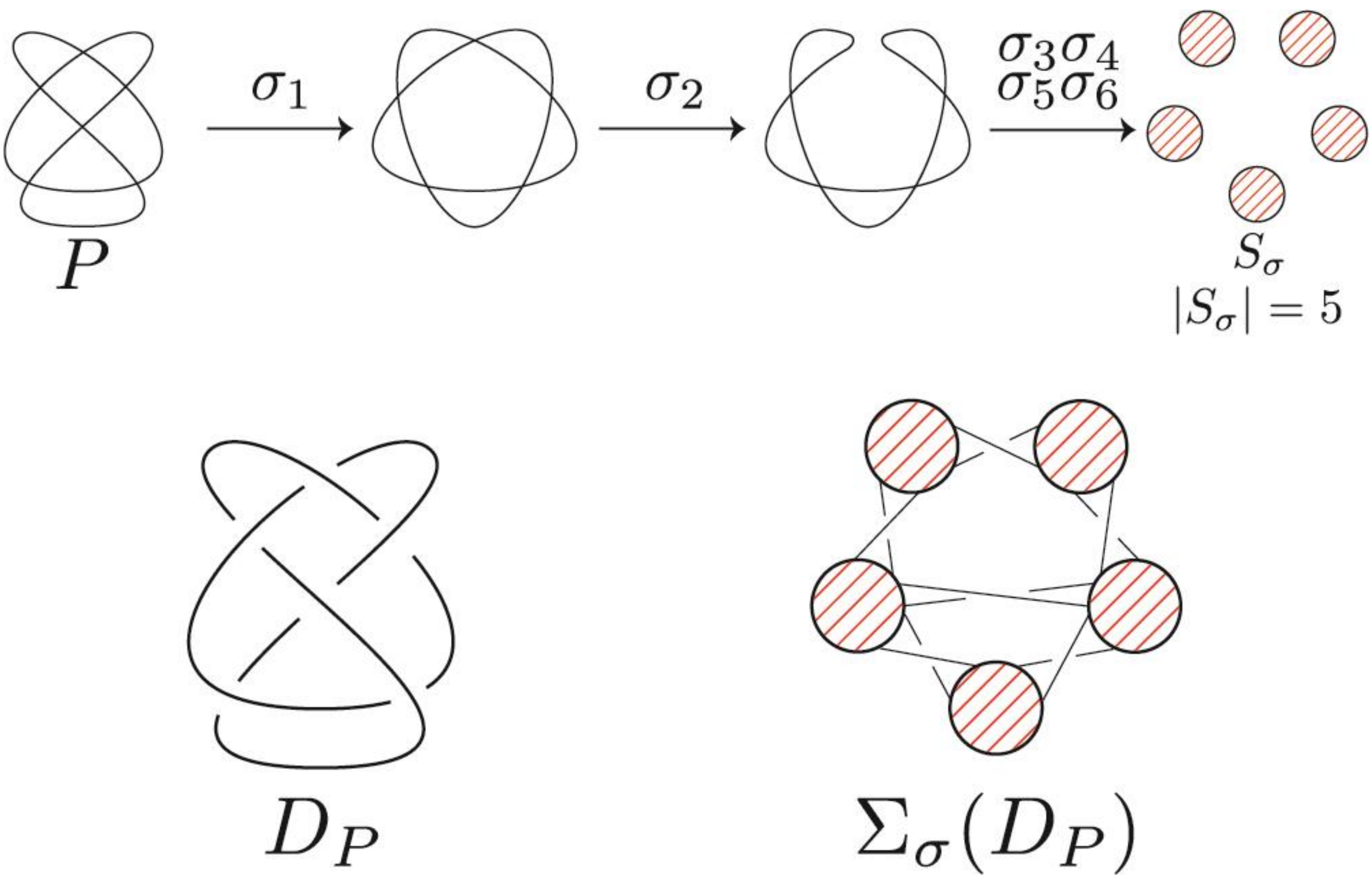
Definition

$\Sigma_\sigma(D_P)$：D_P において S_σ に half - twisted bands をつけて得られる曲面 (non-orientable)

Σ_0： $C(K(D_P)) = 1 - \chi(\Sigma_0)$ を満たす non-orientable surface



P
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
S_σ
$|S_\sigma| = 5$



P
σ_1
σ_2
$\sigma_3\sigma_4$
$\sigma_5\sigma_6$
S_σ
$|S_\sigma| = 5$
D_P
$\Sigma_\sigma(D_P)$

Proof of Theorem 2

$$\chi(\Sigma_0) \geq \chi(\Sigma_\sigma(D_P))$$

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$$\chi(\Sigma_0) \geq \chi(\Sigma_\sigma(D_P))$$

$$1 - C\big(K(D_P)\big) = \chi(\Sigma_0) \geq \chi(\Sigma_\sigma(D_P)) = |S_\sigma| - n(P)$$

Proof of Theorem 2

$$\chi(\Sigma_0) \geq \chi(\Sigma_\sigma(\mathrm{D_P}))$$

$$1 - \mathrm{C}\big(\mathrm{K}(\mathrm{D_P})\big) = \chi(\Sigma_0) \geq \chi(\Sigma_\sigma(\mathrm{D_P})) = |\mathrm{S}_\sigma| - n(\mathrm{P})$$

$$|\mathrm{S}_\sigma| = \#\{Op_i \mid Op_i = \mathrm{RI}^-\} + 1$$

$$n(P) = \#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = \mathrm{RI}^-\}$$

Proof of Theorem 2

$$1 - C\big(K(D_P)\big) \geq |S_\sigma| - n(P)$$

Proof of Theorem 2

$$1 - C\big(K(D_P)\big) \geq |S_\sigma| - n(P)$$

$$= \big(\#\{Op_i \mid Op_i = RI^-\} + 1\big) - \big(\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = RI^-\}\big)$$

Proof of Theorem 2

$$1 - C\left(K(D_P)\right) \geq |S_\sigma| - n(P)$$

$$= (\#\{Op_i \mid Op_i = RI^-\} + 1) - (\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = RI^-\})$$

$$= 1 - \#\{Op_i \mid Op_i = S^-\}$$

Proof of Theorem 2

$$1 - C\big(K(D_P)\big) \geq |S_\sigma| - n(P)$$

$$= (\#\{Op_i \mid Op_i = RI^-\} + 1) - (\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = RI^-\})$$

$$= 1 - \#\{Op_i \mid Op_i = S^-\}$$

$$= 1 - u^-(P)$$

Proof of Theorem 2

$$1 - C(K(D_P)) \geq |S_\sigma| - n(P)$$

$$= (\#\{Op_i \mid Op_i = RI^-\} + 1) - (\#\{Op_i \mid Op_i = S^-\} + \#\{Op_i \mid Op_i = RI^-\})$$

$$= 1 - \#\{Op_i \mid Op_i = S^-\}$$

$$= 1 - u^-(P)$$

よって

$$C(K(D_P)) \leq u^-(P)$$

Theorem 2 (Ito-T., 2018, IJM)

P : knot projection

$$C(K(D_P)) \leq u^-(P)$$

Question 1

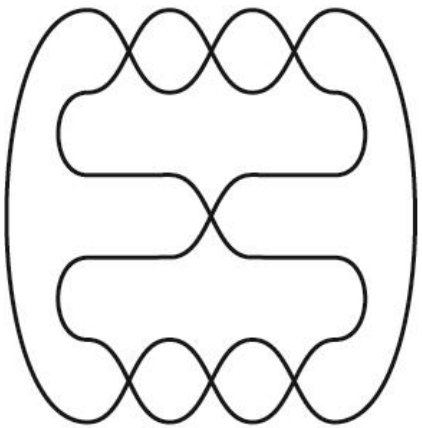
P : knot projection

$$C(K(D_P)) \leq u(P)$$

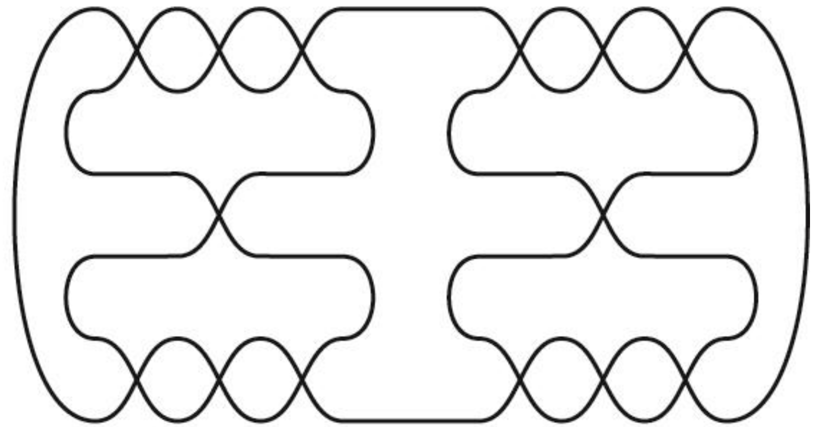
は成り立つか？

$u^{-}(P)$ と $u(P)$ のギャップ

$u^-(P)$ と $u(P)$ のギャップ

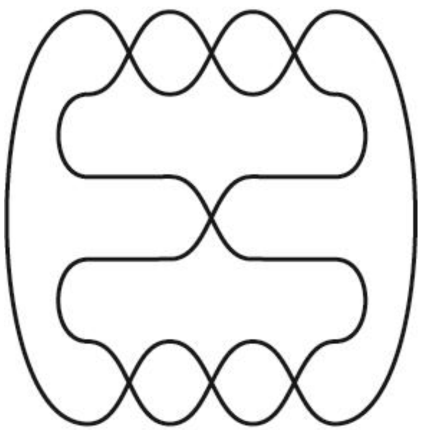


$\widehat{7_4}$

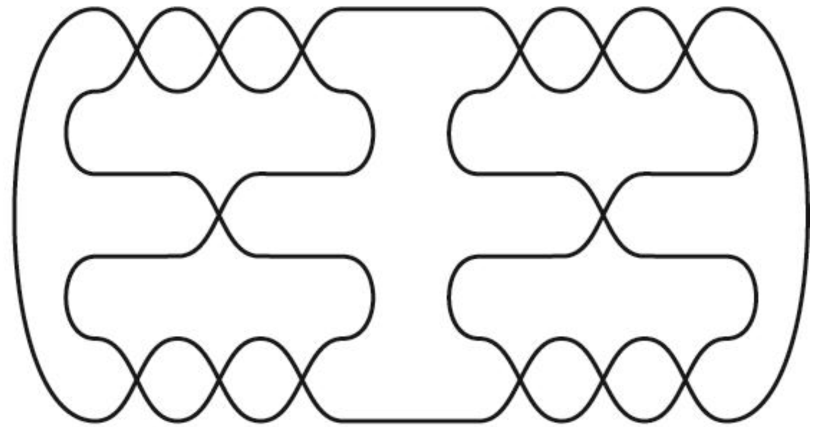


$\widehat{7_4}\sharp\widehat{7_4}$

$u^-(P)$ と $u(P)$ のギャップ



$\widehat{7_4}$

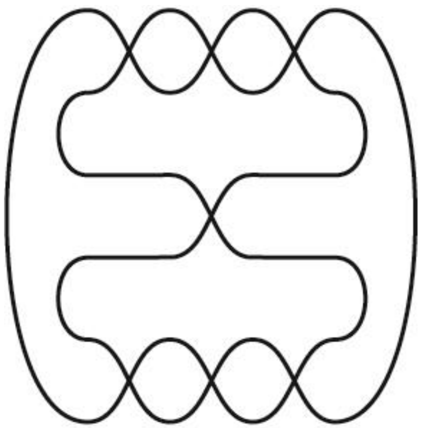


$\widehat{7_4} \sharp \widehat{7_4}$

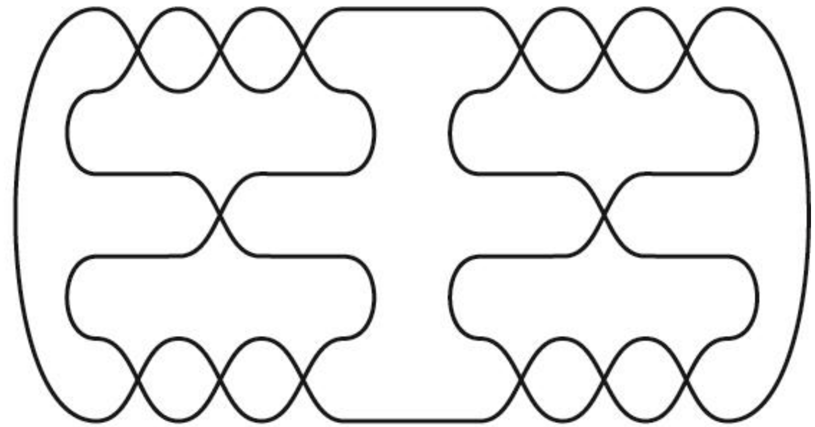
$u^-(7_4) = 3$

$u\,(7_4) = 3$

$u^-(P)$ と $u(P)$ のギャップ



$\widehat{7_4}$



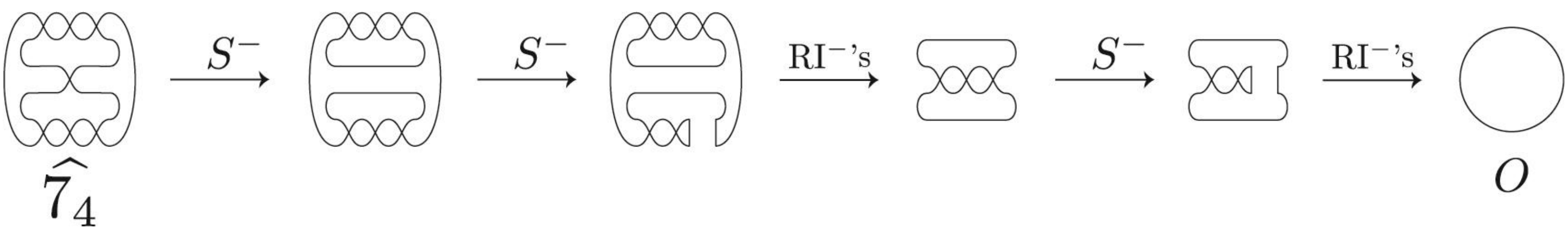
$\widehat{7_4} \sharp \widehat{7_4}$

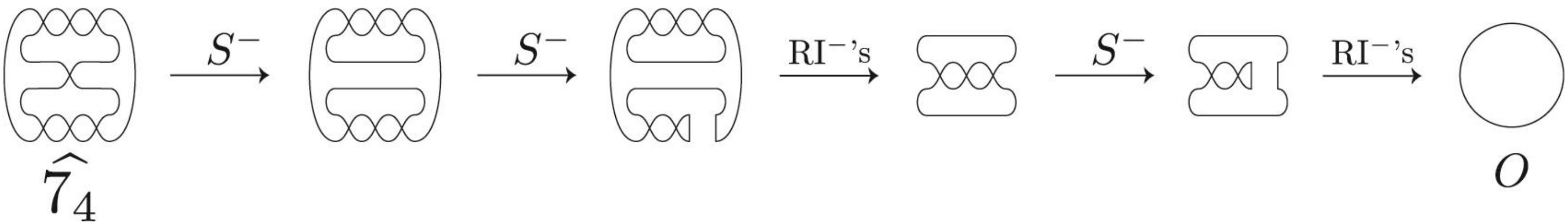
$u^-(7_4) = 3$

$u\,(7_4) = 3$

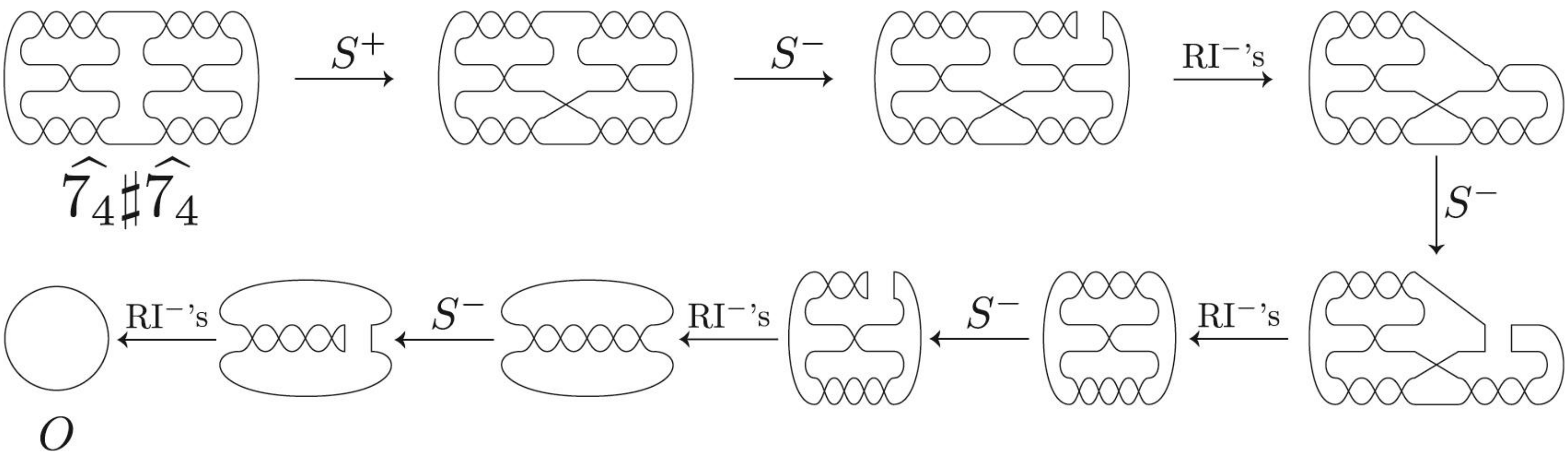
$u^-(7_4 \# 7_4) = 6$

$u\,(7_4 \# 7_4) \leq 5$


S^-
S^-
RI$^-$'s
S^-
RI$^-$'s
$\widehat{7}_4$
O

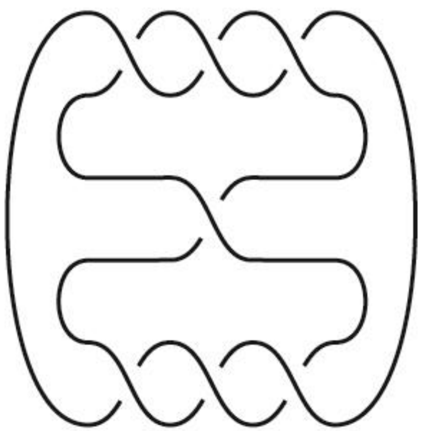


$\widehat{7}_4$
S^-
S^-
RI$^-$'s
S^-
RI$^-$'s
O



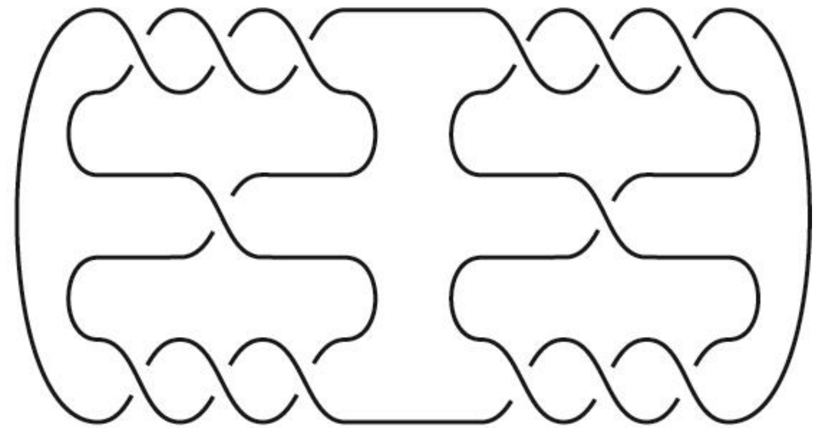
$\widehat{7}_4 \sharp \widehat{7}_4$
S^+
S^-
RI$^-$'s
S^-
RI$^-$'s
S^-
RI$^-$'s
S^-
RI$^-$'s
O

Fact (H. Murakami-Yasuhara, 1995)



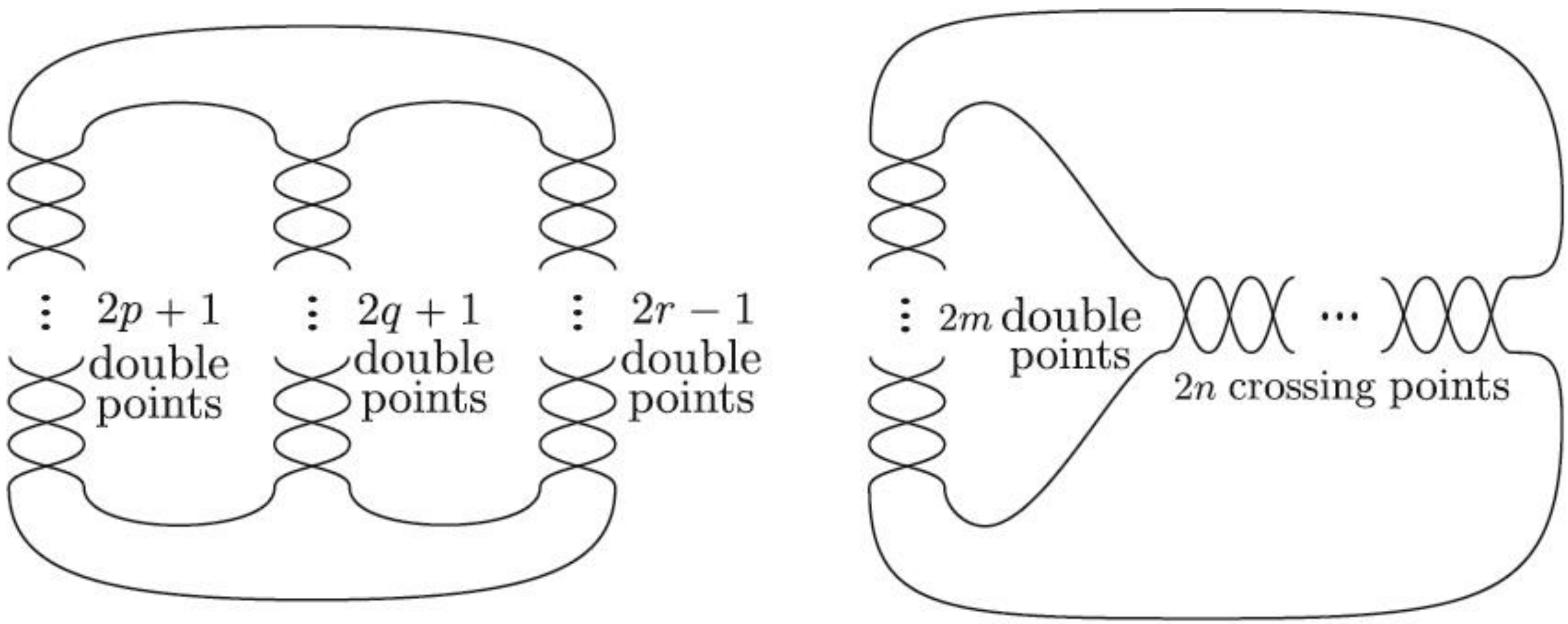
7_4

$C(7_4) = 3$



$7_4 \sharp 7_4$

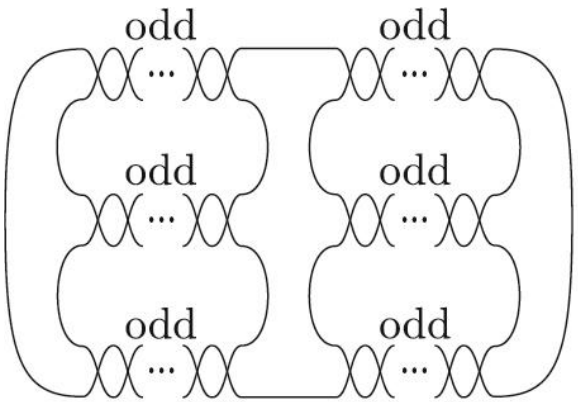
$C(7_4 \# 7_4) = 5$



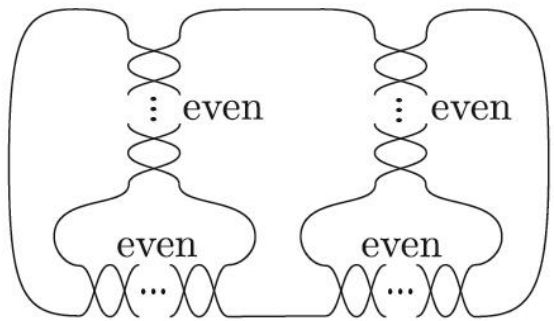
P   *P′*

$$u^{-}(P) = u^{-}(P') = 3$$

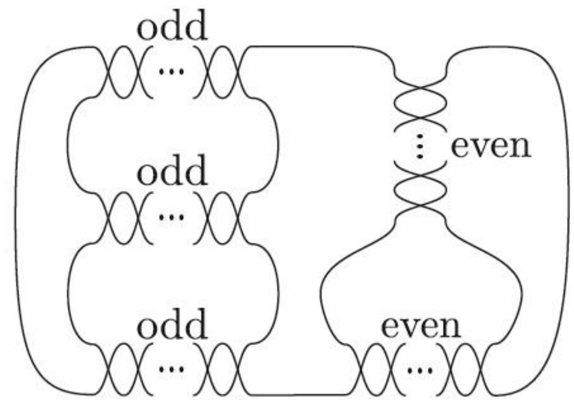
$$u\,(P) = u\,(P') = 3$$



$P \sharp P$



$P' \sharp P'$



$P \sharp P'$

$u^-(P \# P) = 6 \quad u^-(P' \# P') = 6 \quad u^-(P \# P') = 6$

$u\,(P \# P) \leq 5 \quad u\,(P' \# P') \leq 5 \quad u\,(P \# P') \leq 5$

Question 2

P : prime knot projection

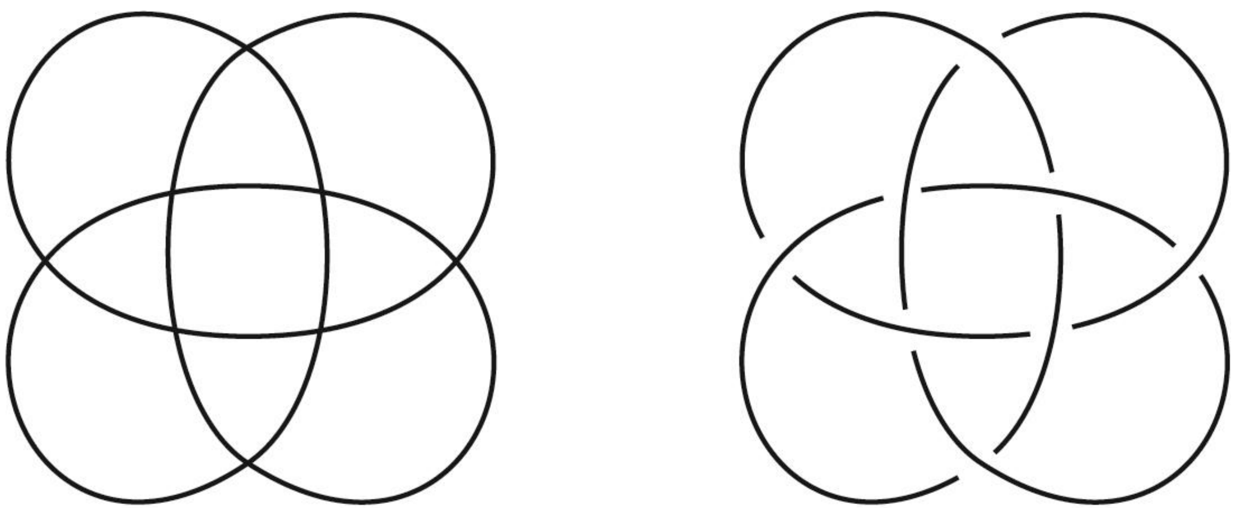
$$u(P) < u^{-}(P)$$

となる P は存在するか？

Definition

P：knot projection

$K^{alt}(P)$：P に交点の上下の情報を alternate に与えて得られる knot



P　　　$K^{alt}(P)$

Question 3

P : prime knot projection

$$C(K^{alt}(P)) < u^{-}(P)$$

となる P は存在するか？

Question 3

P：prime knot projection

$$C(K^{alt}(P)) < u^{-}(P)$$

となる P は存在するか？

(P が 8交点以下の場合
$C(K^{alt}(P)) = u^{-}(P)$)

Thank you for listening