Stable double point numbers of pairs of spherical curves

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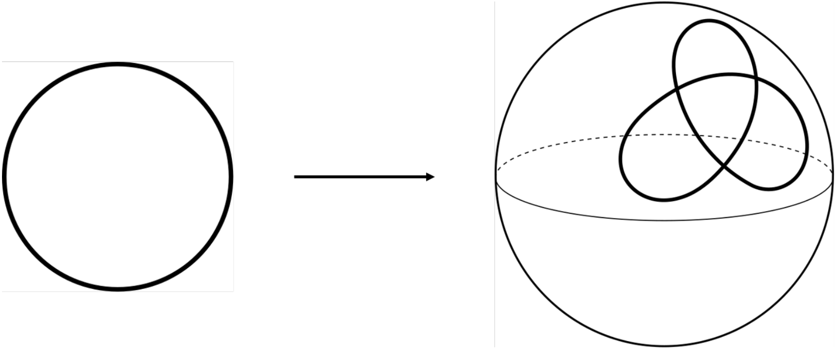
2018/12/23

Introduction

Spherical curve

A spherical curve is the image of a generic immersion of a circle into a 2-sphere.



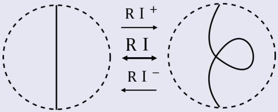
Deformations of type RI, RII, RIII

P, P' : spherical curves

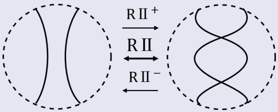
Definition 1

P' is obtained from P by

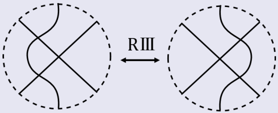
deformation of type RI if :



deformation of type RII if :



deformation of type RIII if :



Fact

$\forall$ pair of spherical curves P, P',
$\exists$ a sequence of spherical curves

$$P = P_0 \to P_1 \to \cdots \to P_n = P'$$

s.t. P_{i+1} is obtained from P_i $(i = 0, 1, \ldots, n-1)$ by a deformation of type RI, RII, RIII, or ambient isotopy.

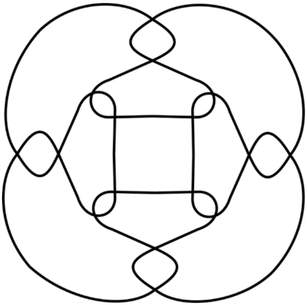
Conjecture (Östlund)

$\forall P$: plane curve,
Trivial plane curve is obtained from P by deformations of type RI, RIII.

Conjecture (Östlund)'

$\forall P$: spherical curve,
Trivial spherical curve is obtained from P by deformations of type RI, RIII.

Counterexample. (Hagge-Yazinski,2014 + Ito-Takimura)



P_{HY}

T. Hagge and J. Yazinski, On the necessity of Reidemeister move 2 for simplifying immersed planner curves, *Banach Center Publ.* 103 (2014), 101-110.

N. Ito and Y. Takimura, RII number of knot projections, preprint.

This result leads us :

Problem

Study the pairs of spherical curves that are (not) transformed from one to the other by deformations of type RI, RIII.

F.H.I.K.M propose a formulation for studying the problem.

Y. Funakoshi, M. Hashizume, N. Ito, T. Kobayashi, and H. Murai, A distance on the equivalence classes of spherical curves generated by deformations of type RI, *J. Knot Theory Ramifications*, Vol.27, No.12, 1850066, 2018.

Notation

$\mathcal{C}$: the set of the ambient isotopy classes of the spherical curves

Definition 2

$v, v' \in \mathcal{C}$
$v \sim_{RI} v'$ (v' is RI-equivalent to v)
$\overset{\text{def}}{\Longleftrightarrow} \exists P,\ P'$: representatives of v, v' s.t.

P' is obtained from P by a sequence of deformations of type RI and ambient isotopies.

Notation

$\tilde{\mathcal{C}} := \mathcal{C}/\sim_{RI}$
$[P] (\in \tilde{\mathcal{C}})$: the equivalence class containing P

The 1-complex $\tilde{\mathcal{K}}_3$

$\tilde{\mathcal{K}}_3$: the 1-complex s.t.

$\cdot \{v \mid v :$ vertex of $\tilde{\mathcal{K}}_3\} \longleftrightarrow \tilde{\mathcal{C}}$

$\cdot$ v, v' $(\in \tilde{\mathcal{C}})$ are joined by an edge

$\Leftrightarrow \exists P, P'$: representatives of v, v' s.t.

$\exists$ a sequence

$$P = P_0 \to P_1 \to \cdots \to P_n = P'$$

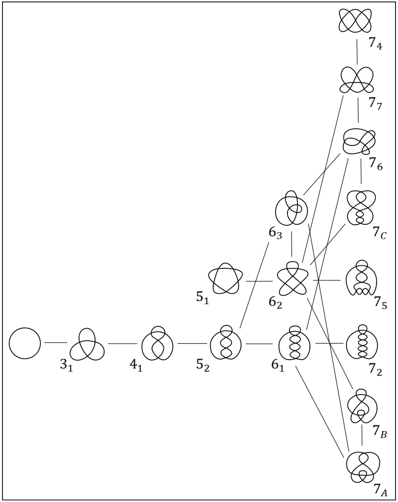
consisting of

- $\cdot$ exactly one deformation of type RIII,
- $\cdot$ deformations of type RI, and
- $\cdot$ ambient isotopies.

$\tilde{\mathcal{K}}_3$

$[P_{HY}]$	$[\bigcirc]$
$\vert$	$\vert$
$\vdots$	$\vdots$

$\tilde{\mathcal{K}}_3$ is not connected.



N. Ito and Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections, *Osaka J.Math*, **52**(2015), 617-646 .

Preliminaries

Double point number $d(v)$

$v \in \tilde{\mathcal{C}}$ (: the vertex of $\tilde{\mathcal{K}}_3$),

$$d(v) := \min\{\sharp \text{ of double points of } P \mid P \in v\}$$

We call $d(v)$ the *double point number* of v.

Double point number $d([P])$

P : spherical curve

P is RI-*minimal* $\overset{\text{def}}{\Longleftrightarrow}$ Each region of P is not a 1-gon.

In general, P is not RI-minimal.

Fact $\forall P \xrightarrow{RI^-} \cdots \xrightarrow{RI^-}$ RI-minimal spherical curve

reduced(P) denotes such spherical curve.

Then we have

Lemma 3

$$d([P]) = \sharp \textit{ of the double points of reduced}(P)$$

Stable double point number $sd(P, P')$

$(P,\ P')$: a pair of spherical curves

Notation

- $\mathcal{L}(P, P')$: the set of the paths in $\tilde{\mathcal{K}}_3$ connecting $[P]$ and $[P']$
- $V(L)$: the set of the vertices of $L \in \mathcal{L}(P, P')$

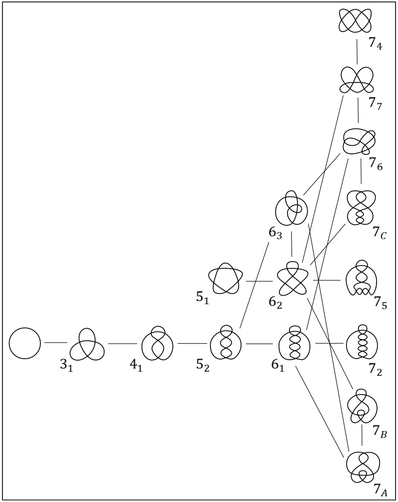
Definition 4

If $[P]$ and $[P']$ are contained in the same component of $\tilde{\mathcal{K}}_3$,

$$sd(P, P') = \min_{L \in \mathcal{L}(P,P')} \left\{ \max_{v \in V(L)} \{d(v)\} \right\}$$

If $[P]$ and $[P']$ are not contained in the same component of $\tilde{\mathcal{K}}_3$,

$$sd(P, P') := \infty$$



N. Ito and Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections, *Osaka J.Math*, **52**(2015), 617-646 .

Stable double point number $sd(P, P')$

Proposition 1

Let (P, P') be a pair of spherical curves such that $[P] \neq [P'], d([P']) \leq d([P])$.
Suppose that each region of P is not a 1-gon or triangle.
Then we have :

$$sd(P, P') \geq d([P]) + 1$$

M. Hashizume and N. Ito, New deformations on spherical curves and Östlund conjecture, preprint.

Stable double point number $sd(P, P')$

Question

$\forall (P, P')$,

$$sd(P, P') \leq \max\{d([P]), d([P'])\} + 1 \ ?$$

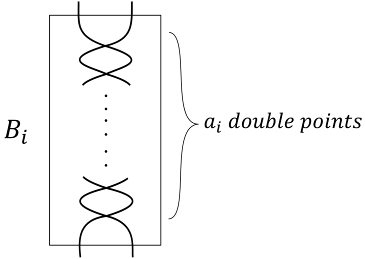
In particular,

$$sd(P, \bigcirc) \leq d([P]) + 1 \ ?$$

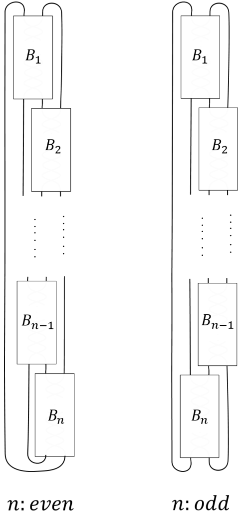
Results

2-bridge spherical curve

$a_1, a_2, \ldots, a_n$ $(n \geq 1)$: an n-tuple of positive integers



$C(a_1, a_2, \ldots, a_n)$ is called the 2-bridge spherical curve (of type $(a_1, a_2, \ldots, a_n)$)



2-bridge spherical curve

By Ito-Takimura it is shown that

$$sd(C(a_1, \ldots, a_n), \bigcirc) < \infty$$

The first result of this talk is

Proposition 2

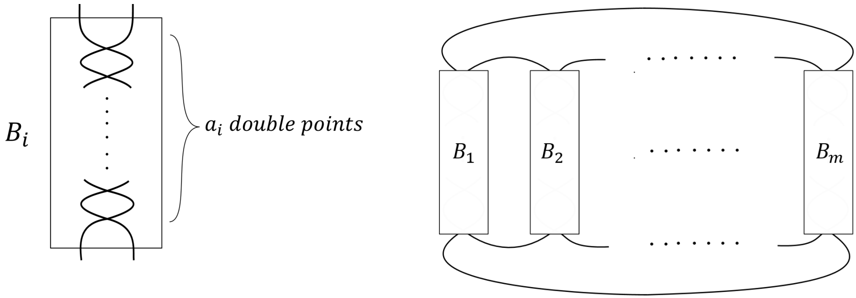
For each 2-bridge spherical curve $C(a_1, \ldots, a_n)$, we have

$$sd(C(a_1, \ldots, a_n), \bigcirc) = d([C(a_1, \ldots, a_n)]), \text{ or } d([C(a_1, \ldots, a_n)]) + 1$$

N. Ito and Y. Takimura, RII number of knot projections, preprint .

Pretzel spherical curve

$a_1, a_2, \ldots, a_m$ $(m \geq 3)$: an m-tuple of positive integers



$P(a_1, a_2, \ldots, a_m)$ is called the pretzel spherical curve (of type $(a_1, a_2, \ldots, a_m)$).

Pretzel spherical curve

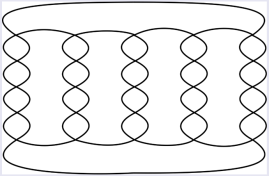
By Ito-Takimura it is shown that

$$sd(P(a_1, \ldots, a_n), \bigcirc) < \infty$$

The second result of this talk is

Theorem 5

$$sd(P(5,5,5,5,5), \bigcirc) = 27$$
$$(= d([P(5,5,5,5,5)]) + 2)$$



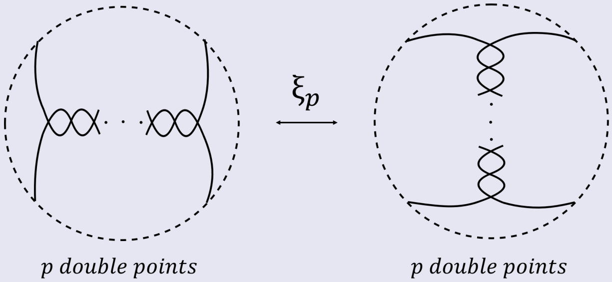
N. Ito and Y. Takimura, RII number of knot projections, preprint .

Key Deformation

Deformation of type ξ_p :

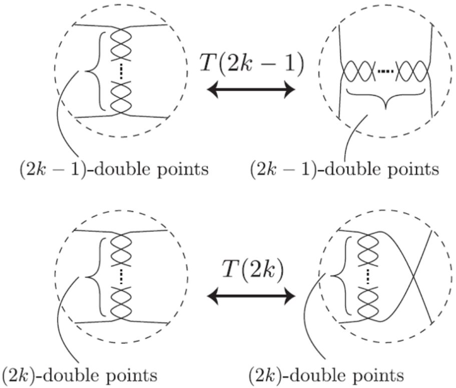
Definition 6

p : a positive odd integer



Remark

In [I-T], Ito-Takimura introduced $T(2k-1)$, $T(2k)$. We note that $T(2k-1)$ is exactly ξ_{2k-1}.



[I-T] N. Ito and Y. Takimura, RII number of knot projections, preprint.

P, P' : spherical curves

Proposition 3 (Lemma 2 of I-T)

$\forall p = 2k + 1\ (k \geq 1)$,

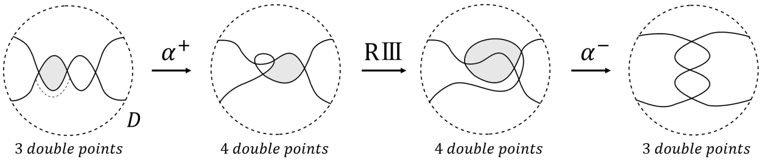
$$P \xrightarrow[in\ \ D]{\xi_p} P' \Rightarrow P \xrightarrow[in\ \ D]{RI's, RIII's} P'$$

[I-T] N. Ito and Y. Takimura, RII number of knot projections, preprint.

Remark of Proposition 3

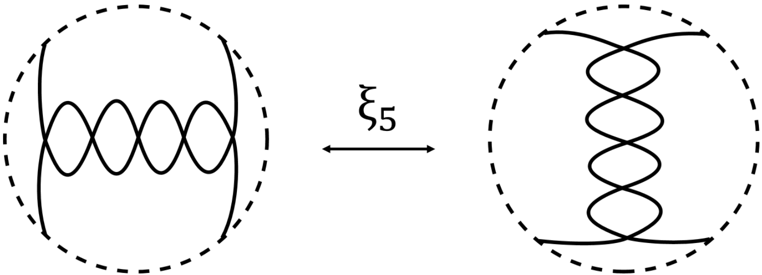
The maximal number of double points in D of the spherical curves that appear in the sequence is $p + (p-1)/2$ $(= p + k)$.

Example $(p = 3)$:



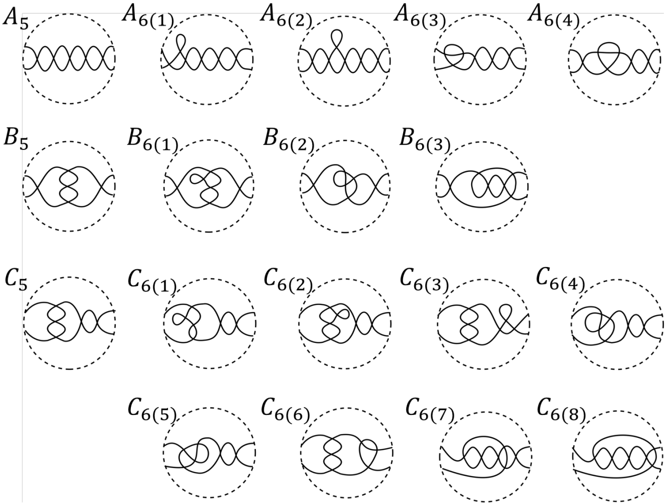
I have the impression that $p + (p-1)/2$ is best possible.

I could show that the statement holds for the case $p = 5$.
Fact : For the deformation



by RI, RIII the $\sharp$ of double points must be raised at least 7.

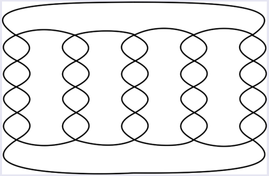
The proof of Fact is carried out by using exhaustion argument depictied as in the following.



Pretzel spherical curve

Theorem 7

$sd(P(5,5,5,5,5), \bigcirc) = 27$
$(= d([P(5,5,5,5,5)]) + 2)$



Conjecture 1

$p(\geq 3)$: *positive odd integer*

$$sd(P(p,p,p,p,p), \bigcirc) = 5p + (p-1)/2$$