Crosscap number two alternating knots

伊藤昇（東大数理）

瀧村祐介氏（学習院中等科）との
共同研究

結び目の数理, 2018/12/23, 早稲田大学

crosscap numberの歴史（敬称略）

定義　$C(K) = \min\{1-\chi(\Sigma) \mid \Sigma : \text{non-ori. surface}, \partial(\Sigma) = K.\}$

Clark (1978)　定義の導入とC(K)=1の決定

村上斉-安原 (1995) 加法性が成り立つ必要十分条件

寺垣内 (2004) torus knot のcrosscap number

寺垣内-平澤 (2006) 2-bridge knotのcrosscap number

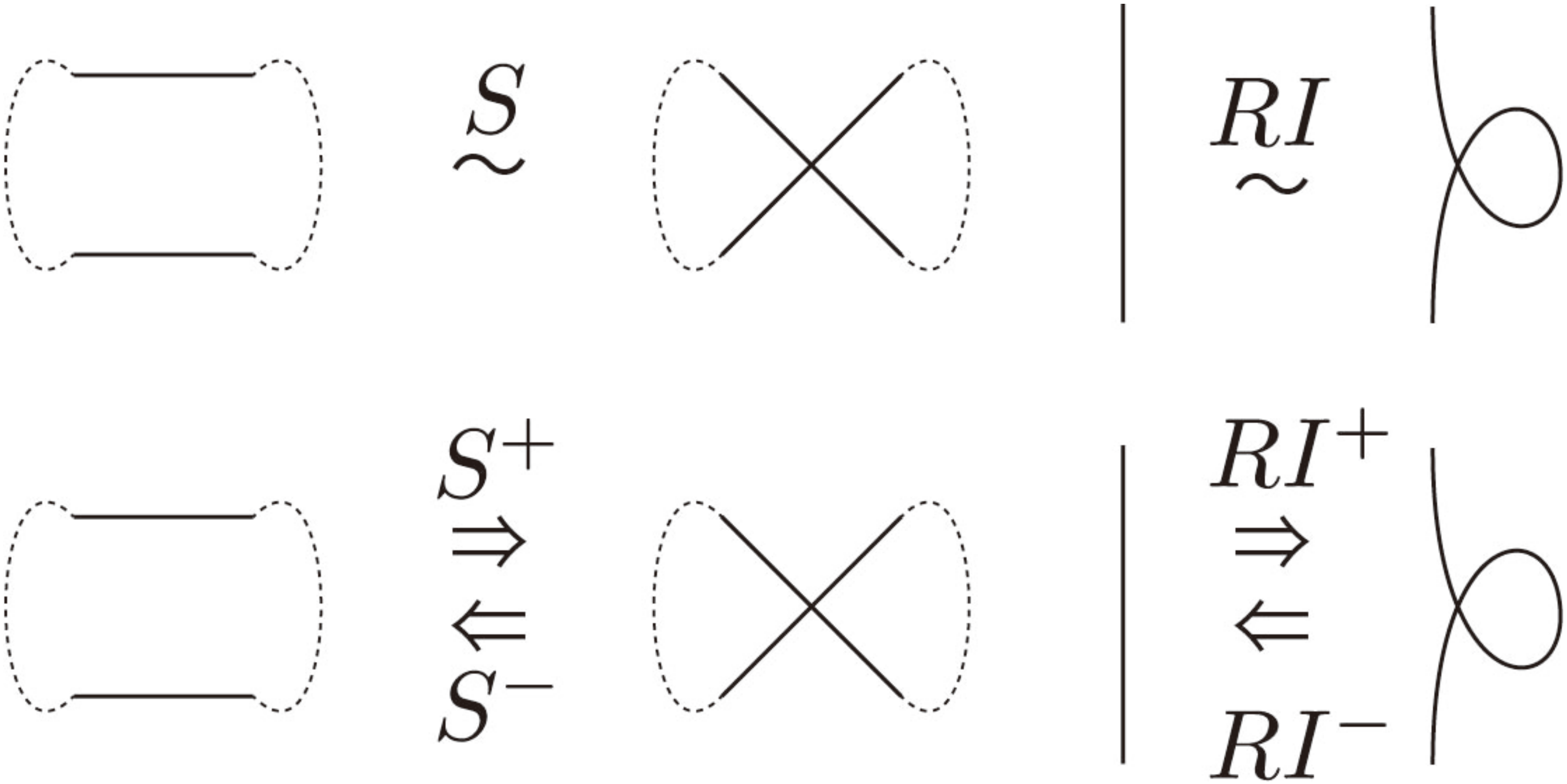
市原-水嶋 (2006) pretzel knot のcrosscap number

Adams-Kindred (2013) alternating knotのcrosscap number の理論的な決定

Kalfagianni-Lee (2016) (colored) Jones polynomialとの関係

今回の結果(2018) crosscap two alternating knotsの決定

Definition



$\overset{S}{\sim}$
$\overset{RI}{\sim}$
$S^+ \Rightarrow$
$\Leftarrow S^-$
$RI^+ \Rightarrow$
$\Leftarrow RI^-$

Definition

P を S, RI の有限列で
simple closed curve にするために
必要な S の最小回数を u(P) と表す

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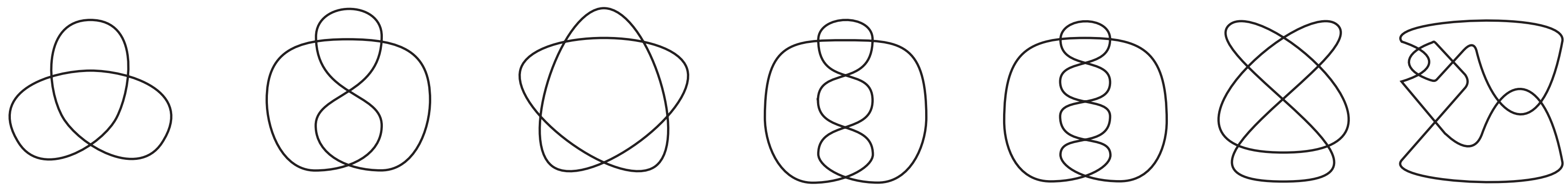
P を S^-, RI^- の有限列で
simple closed curve にするために
必要な S^- の最小回数を $u^-(P)$ と表す

u (P) (瀧村氏は8交点まで作成)

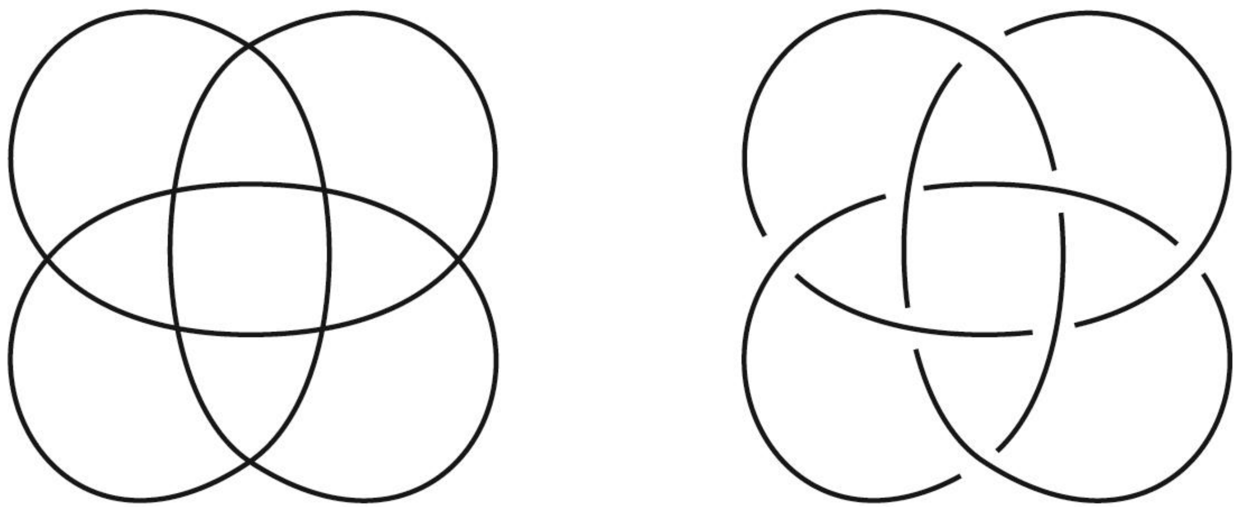


1　2　1　2　2　2　3

Definition

P : knot projection

$\mathrm{K}^{\mathrm{alt}}(\mathrm{P})$: P に交点の上下の情報を alternate に与えて得られる knot



P　　　$K^{alt}(P)$

Observation 1

P : prime knot projection

どうも u(P) と $C(K^{alt}(P))$が異なる例がみつからないようだ。

u(P)の微調整

u(P)には adding band S^+ と removing band S^-

の2typeがある。→シャープそうだが曲面が掴みにくい

→当面は removing band u-(P)を用いよう！

（実際, primeだと差が今の所見つからない。）

Theorem 2 (Takimura-I., IJM, 2018)

P : knot projection

$$C(K(D_P)) \leq u^{-}(P)$$

結び目不変量 u(K) , u-(K)の導入

（by IJMのレフリーのアドバイスに依るnotation）

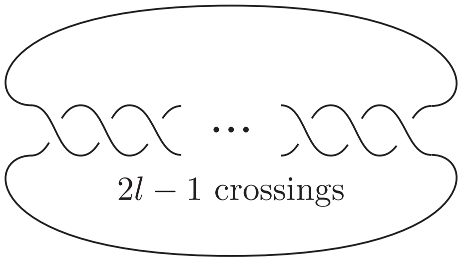
K : knot, P: a projection of K

Z(K) : the set of knot projections of K

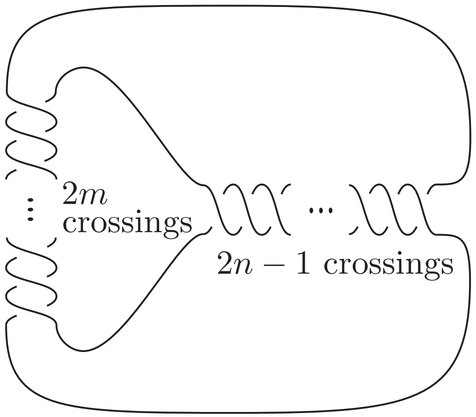
$$u(K) = \min_{P \in Z(K)} u(P),$$

$$u\text{-}(K) = \min_{P \in Z(K)} u\text{-}(P)$$

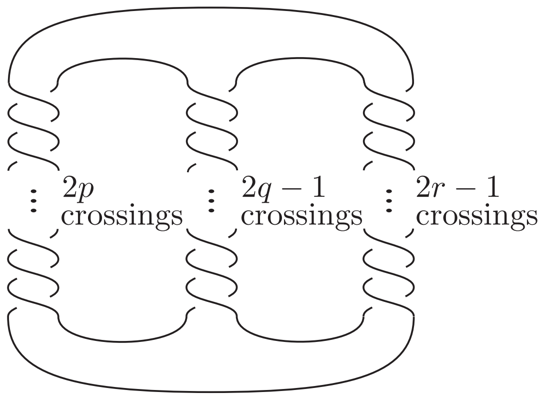
Sets $\mathcal{T}$, $\mathcal{R}$, $\mathcal{P}$



$(2, 2l-1)$-torus knot



$(2m, 2n-1)$-rational knot



$(2p, 2q-1, 2r-1)$-pretzel knot

$\mathcal{T}$ $\mathcal{R}$ $\mathcal{P}$

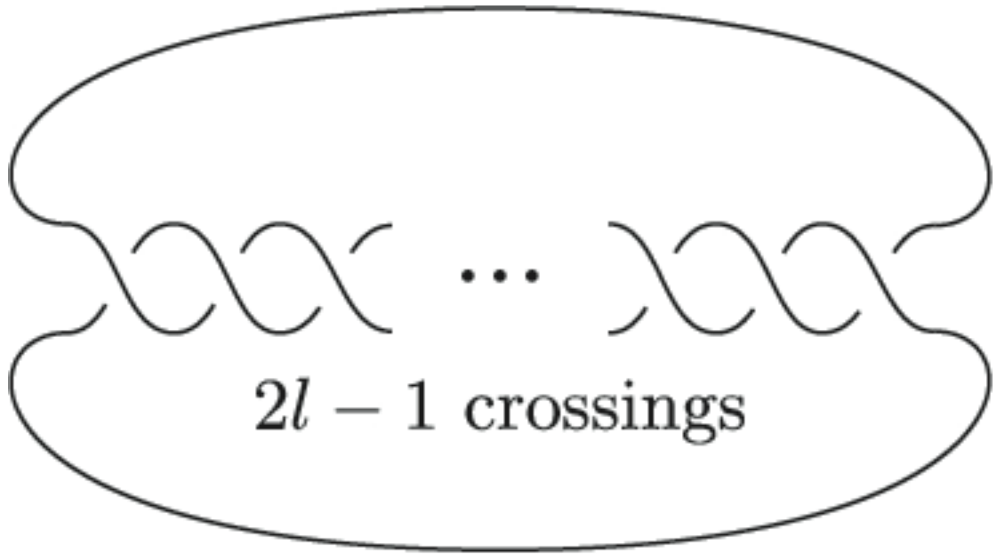
Proposition 1 (Takimura-I., IJM, 2018)

K: alternating knot. (A), (B), (C)は同値.

(A) K $\in \mathcal{T}$

(B) C(K) = 1

(C) u- (K) = 1



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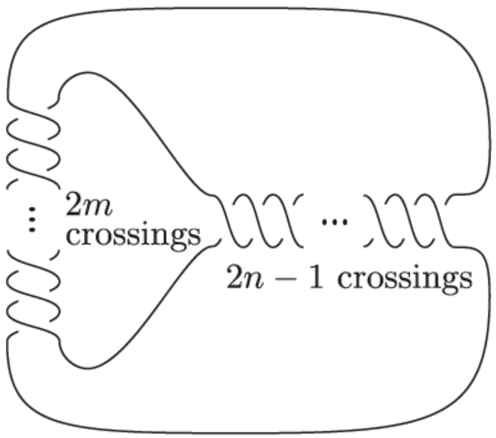
Theorem 2 (Takimura-I., IJM, 2018)

K: alternating knot. (A), (B), (C)は同値.

(A) $K \in \mathcal{T} \# \mathcal{T} \cup \mathcal{R} \cup \mathcal{P}$

(B) C(K) = 2

(C) u- (K) = 2



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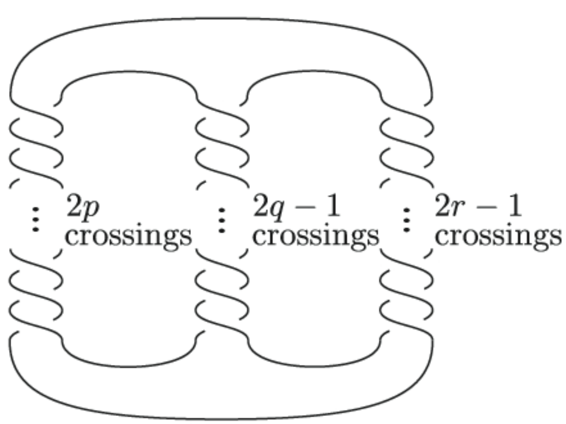

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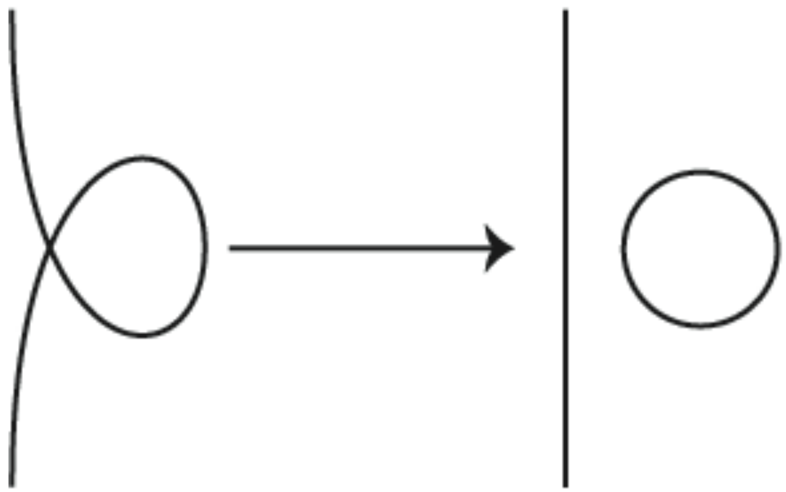
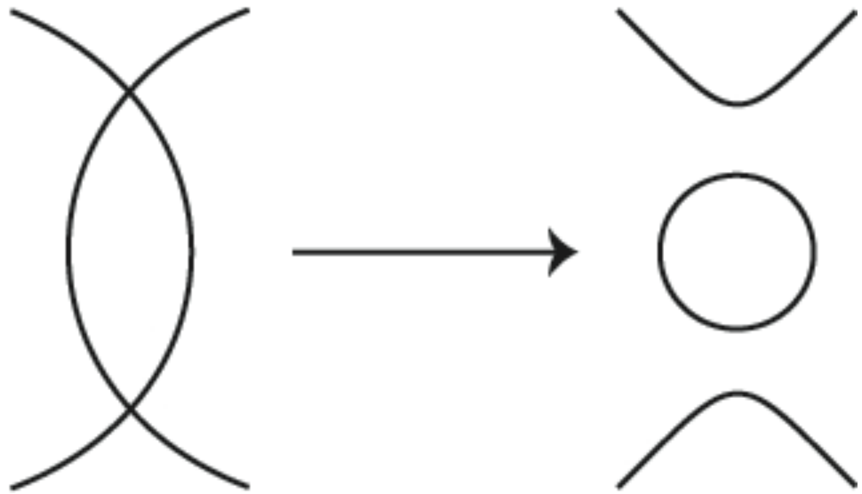
Adams–Kindred の最大オイラー数 surfaceの候補集合 (AGT, 2013)

最小 m 辺形を見つける度に splicesを行う (下図).
[1辺形がなければ, 2辺形, 1,2辺形がなければ
3辺形は必ずある]





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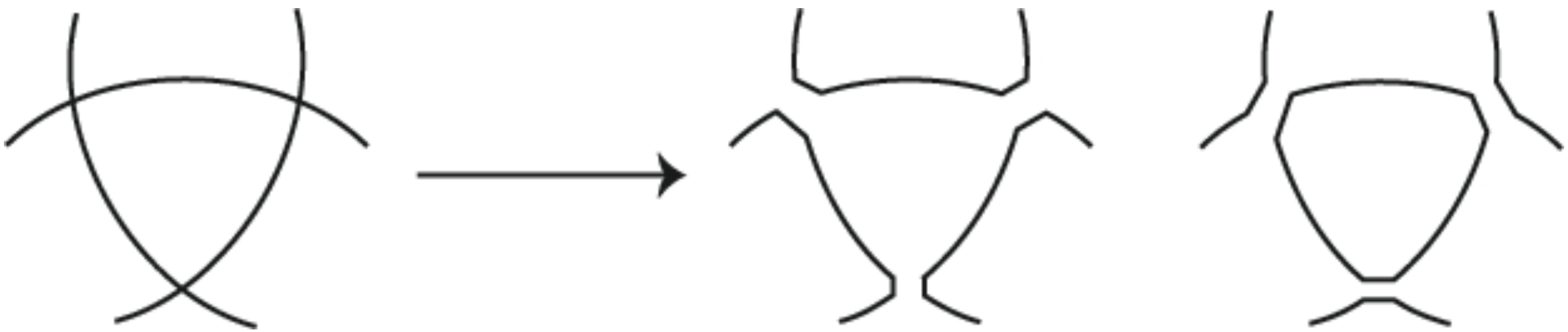
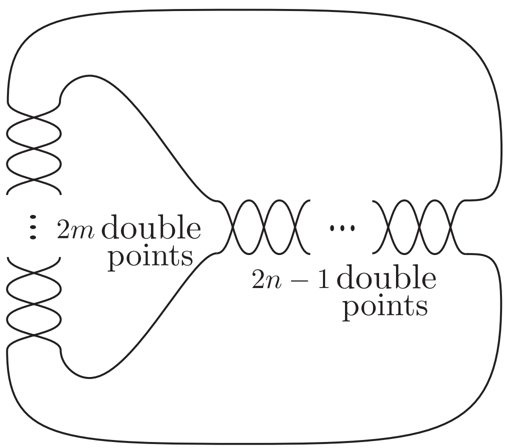
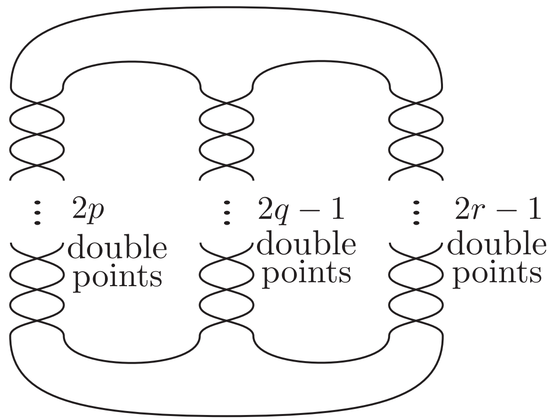


Sets R,   P,   T#T



$(2m, 2n-1)$-rational knot projection



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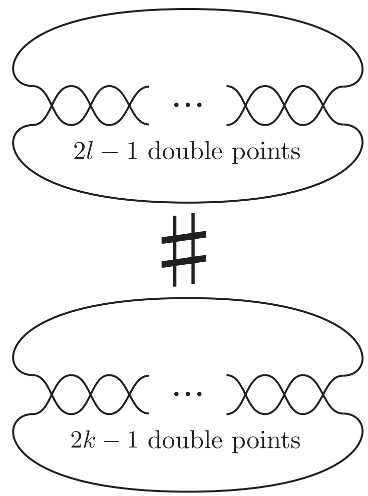

R **P** **T#T**

Key Lemma (Case u-(P) = 1)

P $\in$ **T**

$\Rightarrow$ C (K^{alt}(P)) = 1, u-(P) = 1.

∵ u-(P)のsplicesがAdams-Kindred algorithmのsplicesの一例だから.

Key Lemma (Case u-(P) = 2)

P $\in$ **R**, **P**, **T # T**

$\Rightarrow$ C (K^{alt}(P)) = 2, u-(P) = 2.

∵ u-(P)のsplicesがAdams-Kindred algorithmのsplicesの一例だから.

Proof of Prop. 1. [(A) $\Rightarrow$ (B)]

$\mathbf{T} \subseteq \{P \mid C(K^{alt}(P)) = u\text{-}(P) = 1\}$

$\subseteq \mathbf{T}.$

$\because$ **Key Lem.と $\{P \mid u\text{-}(P)=1\} = T.$**

<u>Proof of Prop. 1.</u> [(A) ⇔ (C)]

$K \in \mathcal{T}$

$\Leftrightarrow \exists P \in Z(K)$ s.t. $P \in \mathbf{T}$

$\Leftrightarrow u_-(K) = 1$

Proof of Prop. 1. [（B）⇒(A)]

$C(K) = 1 - \chi(\Sigma) \Rightarrow \chi(\Sigma) = 0$

⇒ (#crossings – 1) Seifert splices,

requested for P(K)

$\Rightarrow \exists \Sigma,\ 1 \leqq C(K) \leqq u\text{-}(P) \leqq 1$

⇒u-(P)からK(P)が決まる.

P(K) : knot Kのprojection

Proof of Theorem 2.

- (A) K $\in \mathcal{T} \# \mathcal{T} \cup \mathcal{R} \cup \mathcal{P}$ ⇒(B) C(K)=2
- (A) K $\in \mathcal{T} \# \mathcal{T} \cup \mathcal{R} \cup \mathcal{P}$ ⇔ (C) u-(K)=2

---ここまではC(K)=1と議論がパラレルなので省略し、
次を示す---

- (B) C(K)=2⇒(A) K $\in \mathcal{T} \# \mathcal{T} \cup \mathcal{R} \cup \mathcal{P}$

Proof of Theorem 2. [（B）⇒ (A)]

$C(K) = 1 - \chi(\Sigma) \Rightarrow \chi(\Sigma) = -1$

⇒ (#crossings – 2) Seifert splices, requested for P(K)

⇒ $\exists \Sigma$, $2 \leqq C(K) \leqq u_-(P) \leqq 2$

⇒ $u_-(P)$からK(P)が決まる.

P(K) : knot Kのprojection

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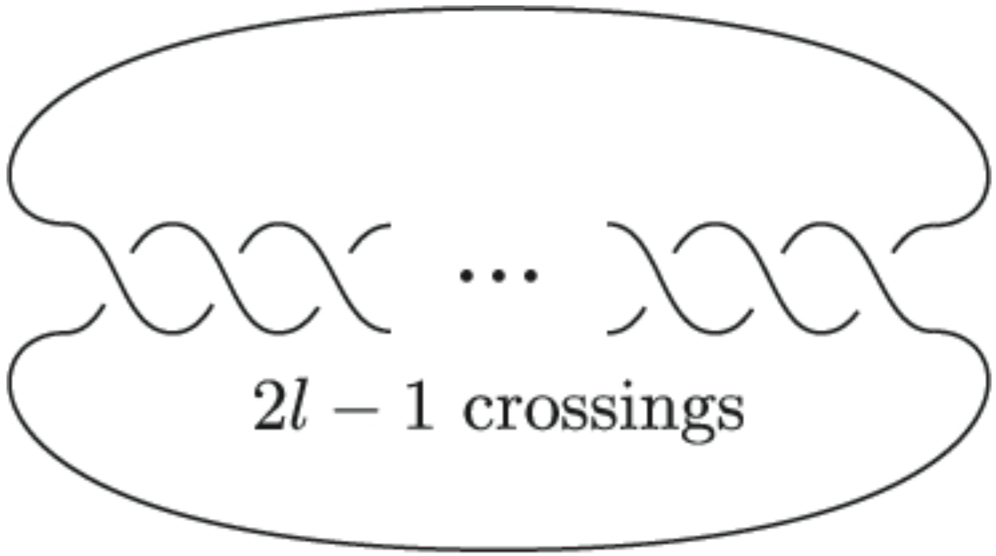

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Theorem 2 (Takimura-I., IJM, 2018)

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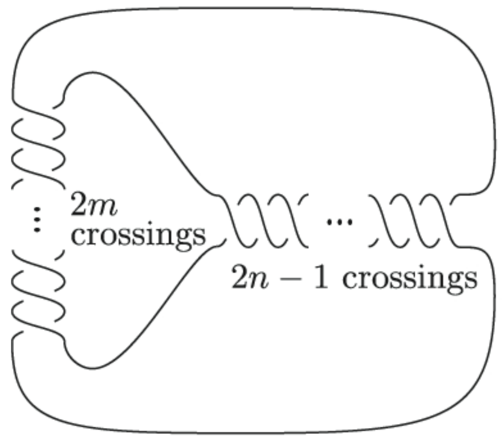

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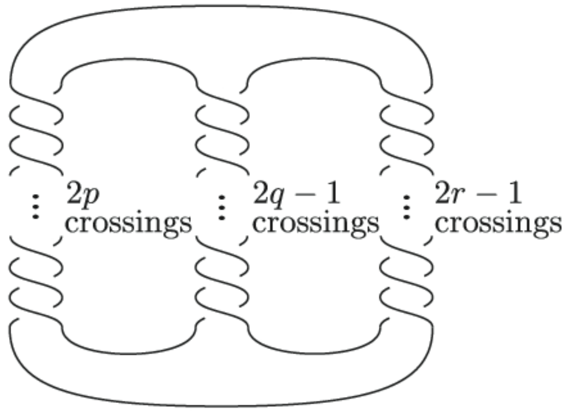

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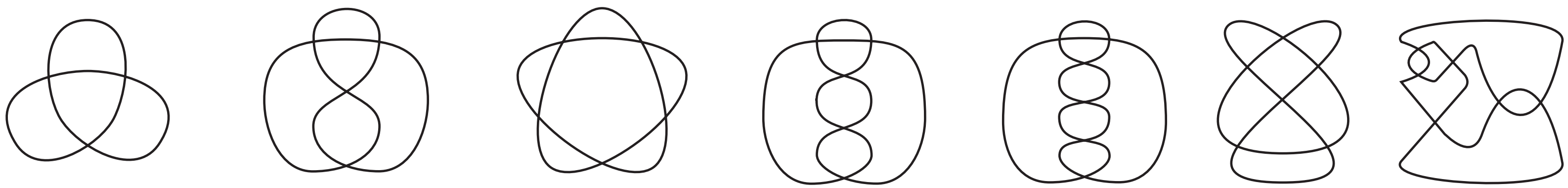
ご静聴ありがとうございました。



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