# Distances of complexes derived from spherical curves and their estimates

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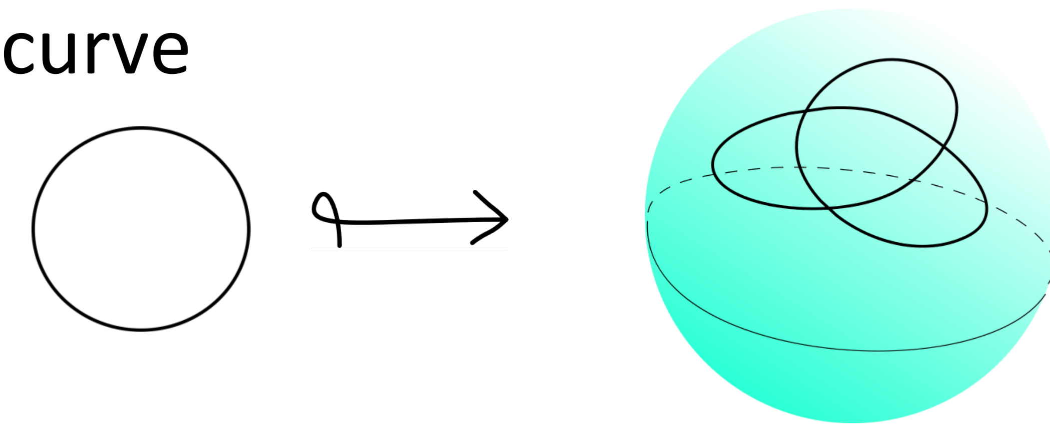
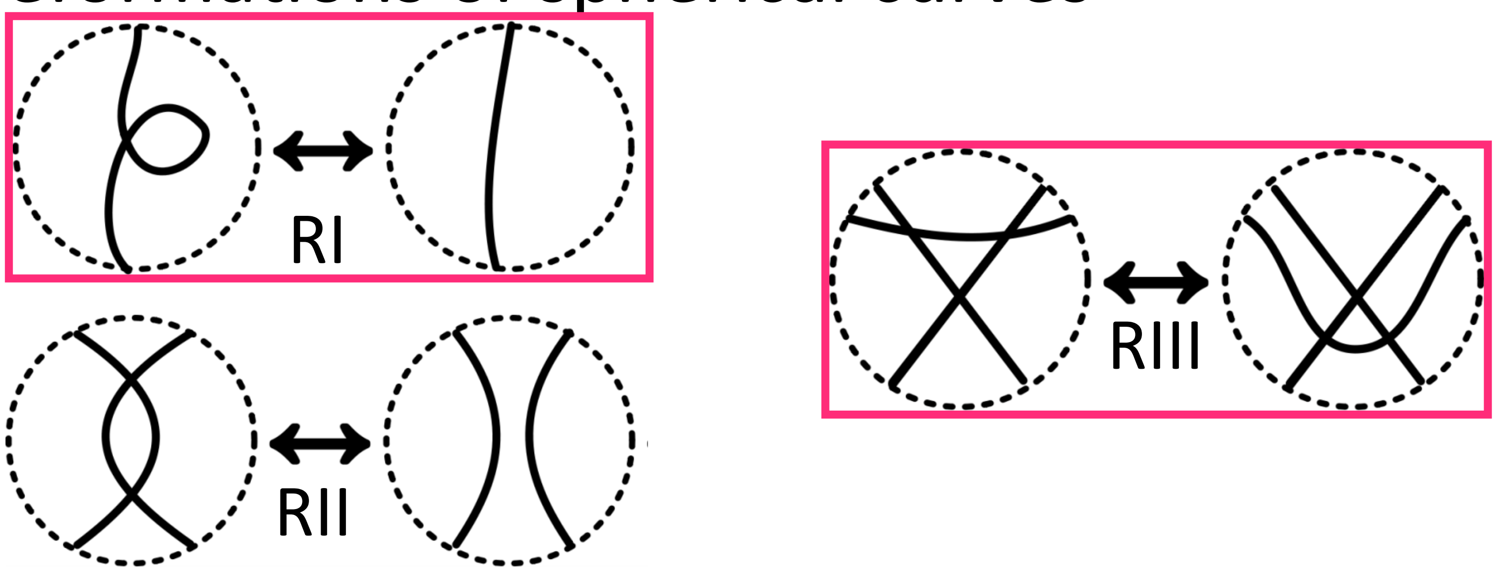
# Deformations of spherical curves

Spherical curve



Deformations of spherical curves




In this talk, we focus on RI and RIII.

# Complex induced by spherical curve and RI, RIII

**Notation**

$\mathcal{C}$ : the set of the ambient isotopy classes of the spherical curves

**Def (RI-equivalence)**

$v, v' \in \mathcal{C}$

$v \sim_{\mathrm{RI}} v' \underset{\mathrm{def}}{\Leftrightarrow} {}^{\exists}P, P'$: representatives of $v, v'$ s.t. $P \overset{\mathrm{RI's}}{\longleftrightarrow} P'$

**Notation**

$\tilde{\mathcal{C}} := \mathcal{C}/\sim_{\mathrm{RI}}$

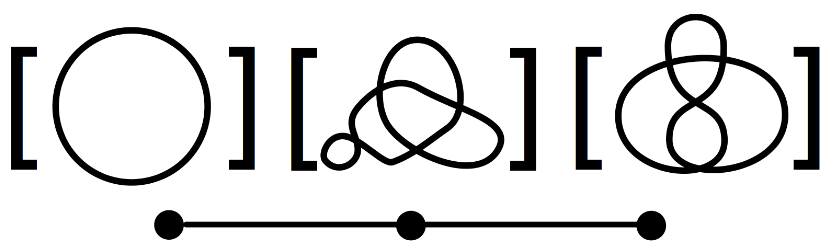
$[P] (\in \tilde{\mathcal{C}})$: the equivalence class containing $P$

# Complex induced by spherical curve and RI, RIII

$\tilde{\mathcal{K}}_3$: the 1-complex s.t.

- $\{v \mid v$: vertex of $\tilde{\mathcal{K}}_3\} \longleftrightarrow \tilde{\mathcal{C}}$
- $v, v'$ ($\in \tilde{\mathcal{C}}$) are joined by an edge

$\Leftrightarrow$ $P \xleftrightarrow{\text{some RI's and single RIII}} P'$



$d_3([P],[P'])$: the distance from $v$ to $v'$

# Result 1

$P$: a spherical curve

$D_P$: knot diag. obtained from $P$ by adding over/under information to each double pt. of $P$

$K^{alt}(P)$: an alternaing knot which possesses $D_P$ that is an alternating diag.

$K$: a knot

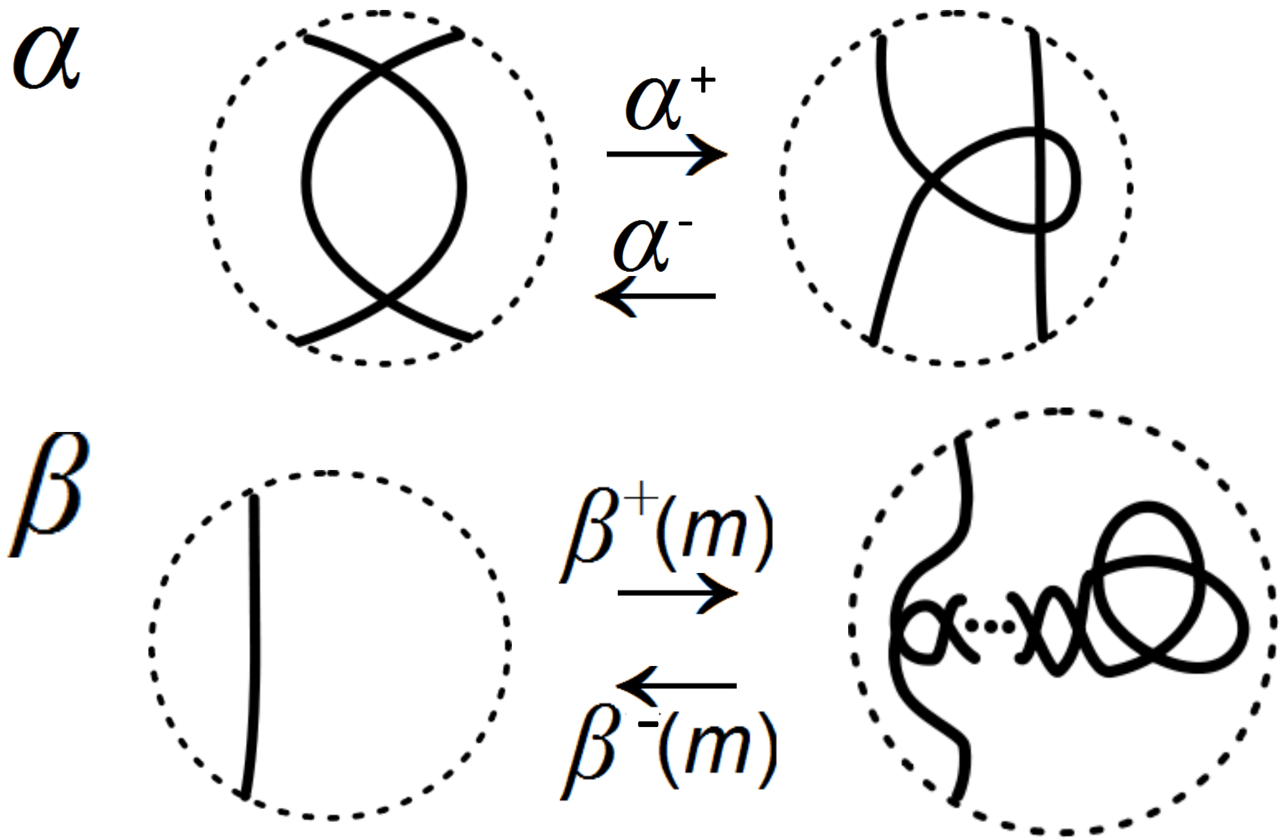
$g(K)$: the genus of $K$

Then, $d_3([P],[P']) \geqq |g(K^{alt}(P')) - g(K^{alt}(P))|$

N. Ito and Y. Takimura,
Crosscap number and knot projections, Intrnat J Math. 29, No. 12 pp21.

# Proof of Result 1

Key deformation



# Preliminaries

**Def (RI-minimal)**

A spherical curve *P* is called *RI-minimal* if *P* does not contain a monogon.

**Fact 1**[Ito-Takimura]
For any spherical curve *P*,
the RI-minimal spherical curve obtained from *P*
is unique up to ambi. iso.

$$P \xleftrightarrow[\text{some RI's}]{} \text{reduced}(P)$$

N. Ito and Y. Takimura,
(1, 2) and weak (1, 3) homotopies on knot projections,
J. Knot Theory Ramifications 22 (2013), 1350085, 14pp.

# Previous result

**Theorem**[Ito-H.]

*P*, *P'*: spherical curves

$$P \xleftrightarrow{\text{some RI's and single RIII}} P'$$

$$\Updownarrow$$

$$\text{reduced}(P) \xleftrightarrow[\text{single RIII, single } \alpha \text{ or single } \beta(m)]{} \text{reduced}(P')$$

# Complex induced by spherical curve and RIII, $\alpha$, $\beta$

$\mathcal{C}$ : the set of the ambient isotopy classes of the spherical curves

$\tilde{\mathcal{C}} := \mathcal{C}/\sim_{\mathrm{RI}}$

$[P](\in \tilde{\mathcal{C}})$: the equivalence class containing $P$

By Fact 1, reduced($P$) $\in [P]$.

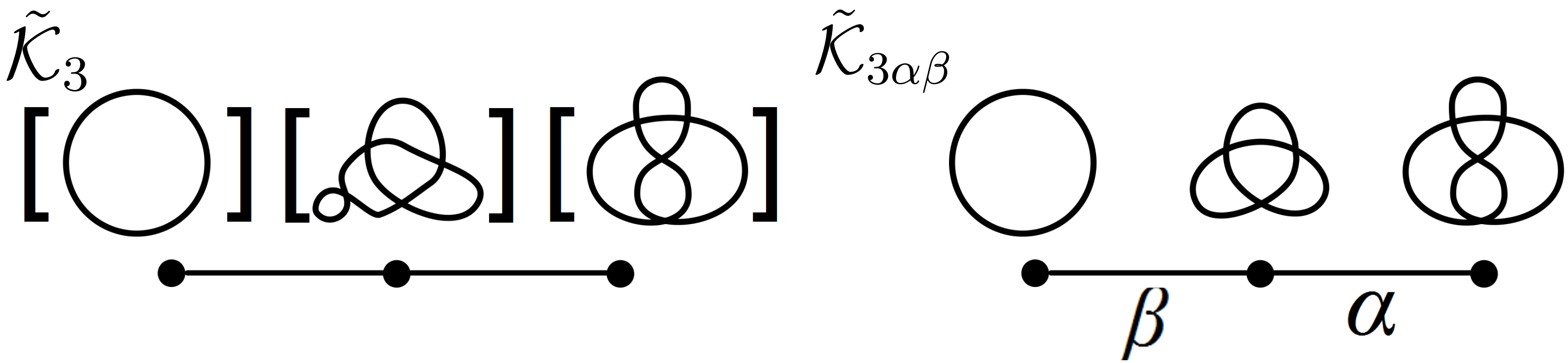
$\tilde{\mathcal{K}}_{3\alpha\beta}$: the 1-complex s.t.

- $v$: vertex of $\tilde{\mathcal{K}}_{3\alpha\beta}$ $\longleftrightarrow$ reduced($P$)
- $v, v'(\in \tilde{\mathcal{C}})$ are joined by an edge

  $\Leftrightarrow$ reduced($P$) $\longleftrightarrow$ reduced($P'$)

  single RIII, single $\alpha$ *or* single $\beta(m)$

# Complex induced by spherical curve and RIII, $\alpha$, $\beta$




$d_{3\alpha\beta}$(reduced($P$), reduced($P'$))

: the distance from $v$ to $v'$

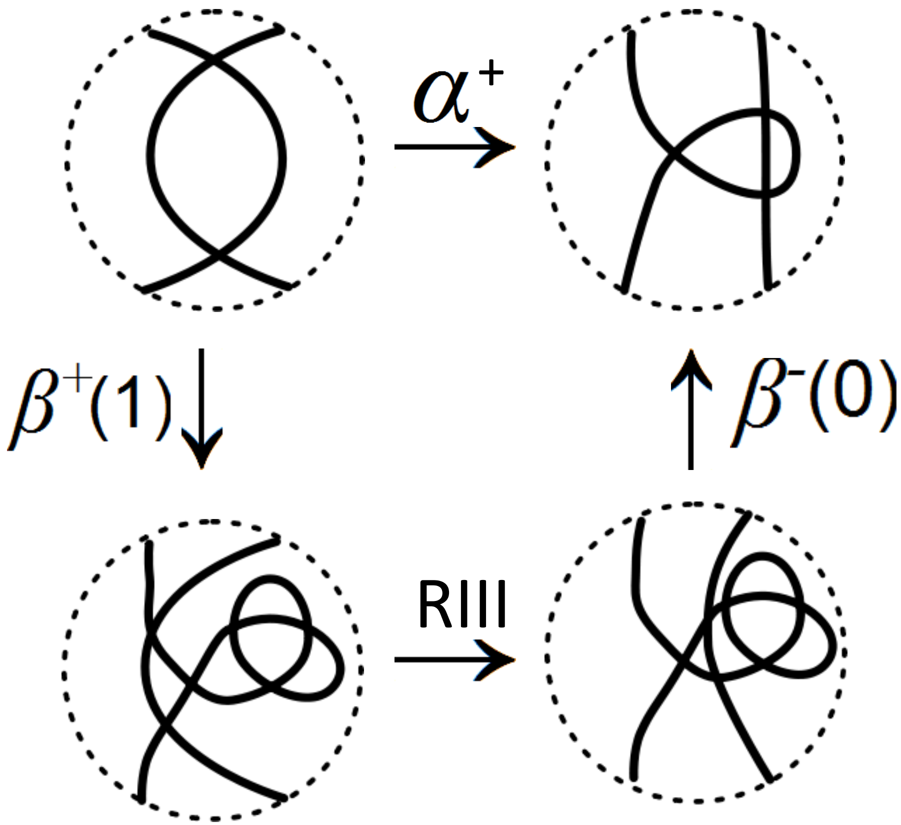
**Key fact**

$d_3([P],[P']) = d_{3\alpha\beta}$(reduced($P$), reduced($P'$))

# Lemmas

**Lemma 1**

$\alpha^+$ consists of $\beta^+(1)$, RIII, $\beta^-(0)$.

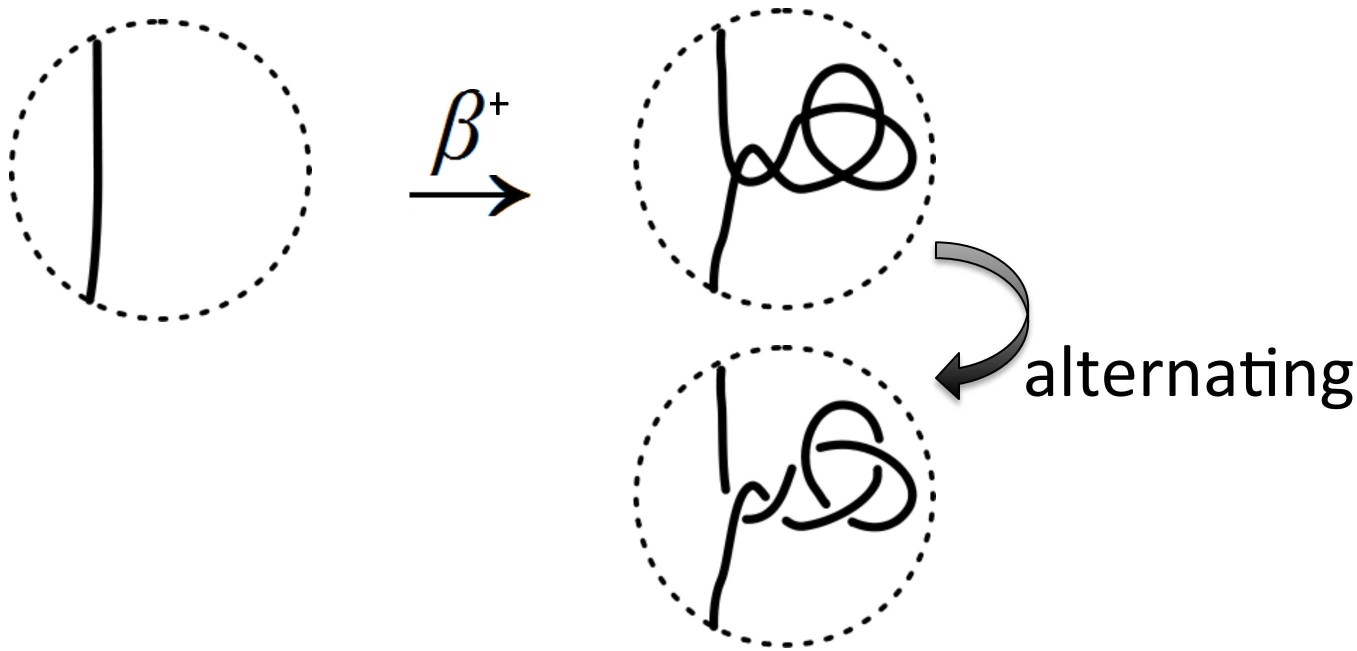


# Lemmas

**Lemma 2**

$P$, $P'$: spherical curves

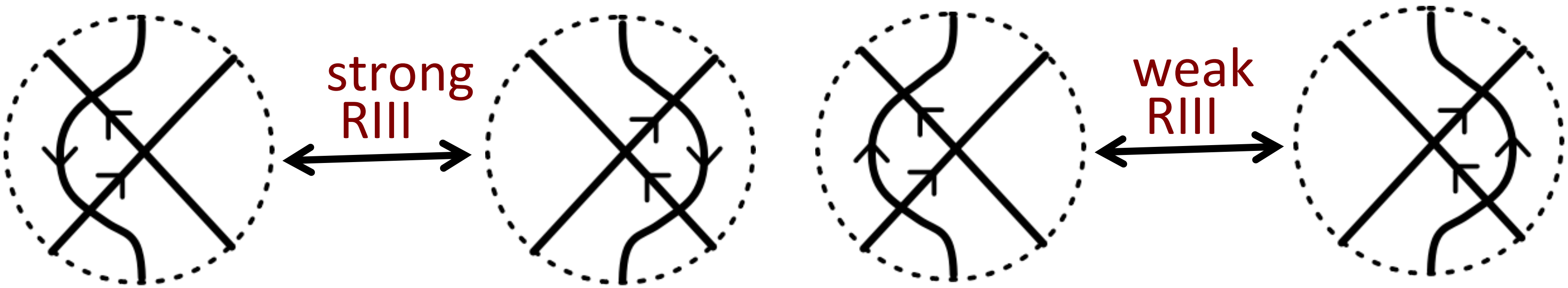
$$P \xrightarrow{\beta^+} P' \quad \Rightarrow \quad g(K^{alt}(P')) - g(K^{alt}(P))=1$$



# Lemmas

**Lemma 3**

*P, P'*: spherical curves




$$P \xleftrightarrow{\text{single strong RIII}} P' \Rightarrow |g(K^{alt}(P')) - g(K^{alt}(P))| = 0 \text{ or } 1$$

$$P \xleftrightarrow{\text{single weak RIII}} P' \Rightarrow g(K^{alt}(P')) - g(K^{alt}(P)) = 0$$

# Proof of Lemma 3

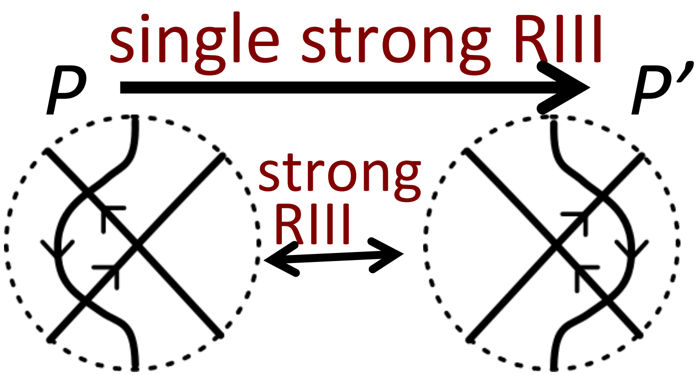
$s(P)$: the num. of the Seifert circles of $K^{alt}(P)$

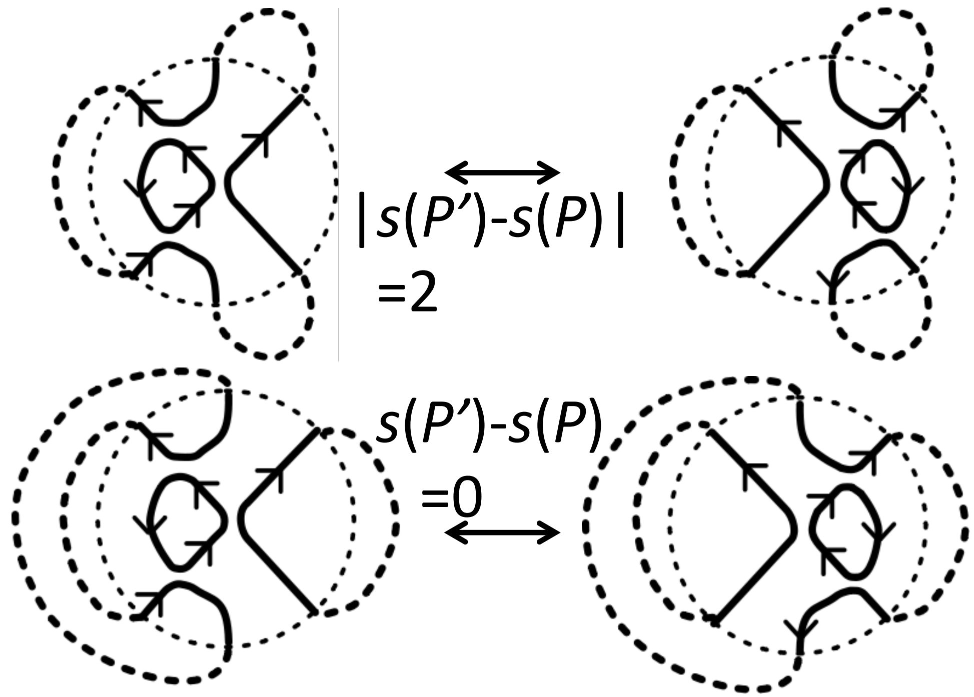
$n(P)$: the num. of the double points of $P$

$$\chi(P) = s(P) - n(P)$$

$$\parallel$$

$$1-2g(K^{alt}(P))$$

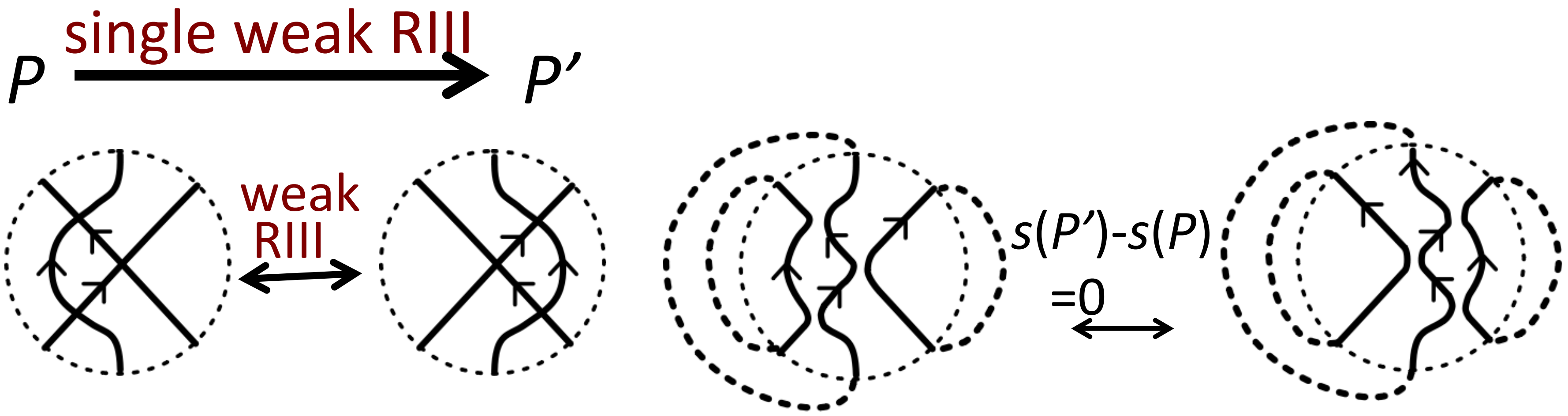



$$\Rightarrow |g(K^{alt}(P')) - g(K^{alt}(P))| = \frac{|s(P)-s(P')|}{2}$$

$$= 0 \text{ or } 1$$

# Proof of Observation 3




$\Rightarrow g(K^{alt}(P')) - g(K^{alt}(P)) = 0$

# Proof of Result 1

$$P \xrightarrow{\text{RI's}} P_0 \xrightarrow{\text{Op}_1} P_1 \xrightarrow{\text{Op}_2} \cdots \xrightarrow{\text{Op}_m} P_m \xrightarrow{\text{RI's}} P'$$

$P_0 = \text{reduced}(P)$, $P_m = \text{reduced}(P')$

$\text{Op}_i = \text{RIII}, \alpha$ or $\beta$

$$\begin{aligned}
&|g(K^{alt}(P')) - g(K^{alt}(P))| \\
&= |g(K^{alt}(P_m)) - g(K^{alt}(P_0))| \\
&= \left|\sum_{i=1}^{m} (g(K^{alt}(P_i)) - g(K^{alt}(P_{i-1})))\right| \\
&\leqq \sum_{i=1}^{m} |g(K^{alt}(P_i)) - g(K^{alt}(P_{i-1}))| \\
&\leqq d_{3\alpha\beta}(P_0, P_m) = d_3([P],[P'])
\end{aligned}$$

# Result 2

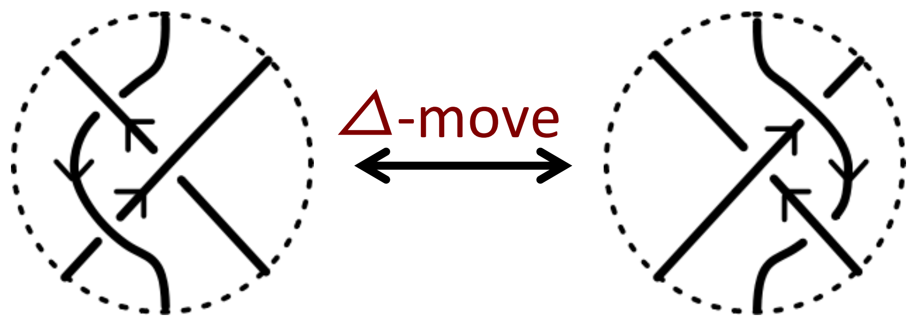
$K$: a knot

$a_2(K)$: the 2nd coefficient of Conway poly. of $K$

$P$: a spherical curve

$K^{pos}(P)$: a positive knot which possesses $D_P$

that is an positive diag.




$K$, $K'$: knots

$d_{\triangle}(K,K')$: $\triangle$- Gordian distance from $K$ to $K'$


N. Ito and Y. Takimura,
(1, 2) and weak (1, 3) homotopies on knot projections,
J. Knot Theory Ramifications 22 (2013), 1350085, 14pp.

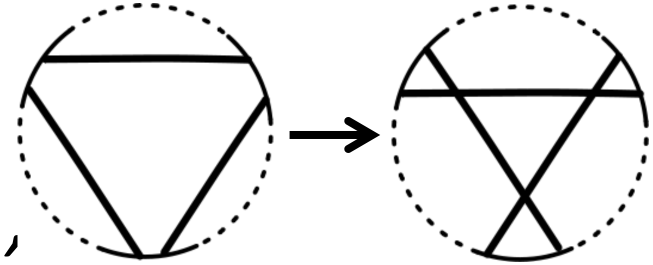
# Result 2

$P \overset{\text{RI, RIII}}{\longleftrightarrow} P'$

$\Rightarrow$ # of strong RIII's of a seq. of length $d_3([P],[P'])$ $\geqq d_\triangle(K^{pos}(P), K^{pos}(P'))$

$\geqq |a_2(K^{pos}(P'))-a_2(K^{pos}(P))|$

In particular,



$P \xrightarrow{\text{RI, weak RIII, negative strong RIII}} P'$

$\Rightarrow$ # of negative strong RIII's of a seq. of length $d_3([P],[P'])$ $= d_\triangle(K^{pos}(P), K^{pos}(P'))$

$= a_2(K^{pos}(P'))-a_2(K^{pos}(P))$

# Corollary of Result 2

$K$: a knot

$u_{\triangle}(K)$: $\triangle$-unknotting num. of $K$

# of negative strong RIII's
of a seq. of length $d_3([P],[\circ])$ $\geqq u_{\triangle}(K^{pos}(P))$
$\geqq a_2(K^{pos}(P))$

# Proof of Result 2

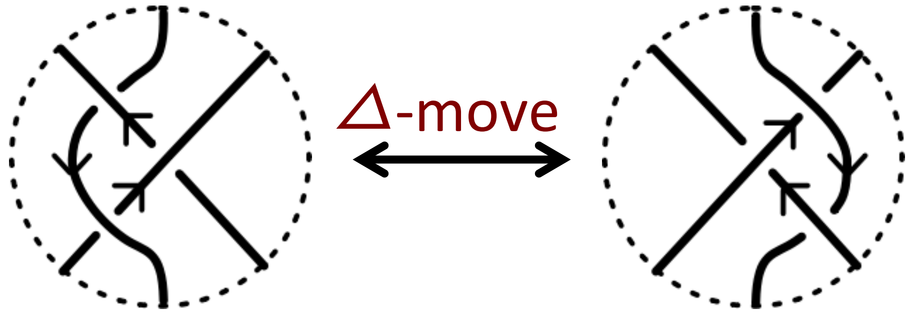
**Fact** [Okada]
$K, K'$: knots
If $K'$ is obtained from $K$ by a single $\triangle$-move,
then $|a_2(K') - a_2(K)| = 1$.

Then,

$$d_{\triangle}(K^{pos}(P), K^{pos}(P')) \geqq |a_2(K^{pos}(P')) - a_2(K^{pos}(P))|$$

M. Okada,
Delta-unknotting operation and the second coefficient of the Conway polynomial, J. Math. Soc. Japan Vol. 42, No. 4, 1990.

# Proof of Result 2




By [Polyak-Viro '94], $a_2(K)=\langle$  $, \dot{G}_K \rangle$.

Then, by negative strong RIII, $a_2$ is increased by 1.

Hence,

# of negative strong RIII's of a seq.

of length $d_3([P],[P'])$ is increased by

$a_2(K^{pos}(P'))-a_2(K^{pos}(P))$.

M. Polyak and O. Viro,
Gauss Diagram Formulas for Vassiliev Invariants,
International Math. Research Notices, No. 11, 1994.