On certain L-functions for deformations of knot group representations

Ryoto TANGE (Kyushu University)
(joint work with Takahiro KITAYAMA, Masanori MORISHITA and Yuji TERASHIMA)

Waseda University

December 25, 2015

Contents

What I am working on:

Arithmetic Topology

which studies

Interaction

Number theory $\longleftrightarrow$ Knot theory

What I am working on:

Arithmetic Topology

which studies

Interaction

Number theory $\longleftrightarrow$ Knot theory

Basic analogies

Number theory	Knot theory

Basic analogies

Number theory	Knot theory
prime ideal (p) $\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$	knot K $S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$

Basic analogies

Number theory	Knot theory
prime ideal (p) $\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$	knot K $S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$
p-adic integers $\mathrm{Spec}(\mathbb{Z}_p)$	tubular neighborhood V_K

Basic analogies

Number theory	Knot theory
prime ideal (p) $\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$	knot K $S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$
p-adic integers $\mathrm{Spec}(\mathbb{Z}_p)$	tubular neighborhood V_K
p-adic numbers $\mathrm{Spec}(\mathbb{Q}_p) = \mathrm{Spec}(\mathbb{Z}_p) \setminus \mathrm{Spec}(\mathbb{F}_p)$	boundary torus $\partial V_K = V_K \setminus K$

Basic analogies

Number theory	Knot theory
prime ideal (p) $\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$	knot K $S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$
p-adic integers $\mathrm{Spec}(\mathbb{Z}_p)$	tubular neighborhood V_K
p-adic numbers $\mathrm{Spec}(\mathbb{Q}_p) = \mathrm{Spec}(\mathbb{Z}_p) \setminus \mathrm{Spec}(\mathbb{F}_p)$	boundary torus $\partial V_K = V_K \setminus K$
prime complement $X_p = \mathrm{Spec}(\mathbb{Z}[1/p])$	knot complement $X_K = S^3 \setminus \mathrm{int}(V_K)$

Basic analogies

Number theory	Knot theory
prime ideal (p) $\mathrm{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$	knot K $S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$
p-adic integers Spec$(\mathbb{Z}_p)$	tubular neighborhood V_K
p-adic numbers $\mathrm{Spec}(\mathbb{Q}_p) = \mathrm{Spec}(\mathbb{Z}_p) \setminus \mathrm{Spec}(\mathbb{F}_p)$	boundary torus $\partial V_K = V_K \setminus K$
prime complement $X_p = \mathrm{Spec}(\mathbb{Z}[1/p])$	knot complement $X_K = S^3 \setminus \mathrm{int}(V_K)$
prime group $G_p = \pi_1^{\text{ét}}(X_p)$	knot group $G_K = \pi_1(X_K)$

§0 Introduction

In particular, there are close analogies between

Iwasawa theory	**Alexander theory**

In particular, there are close analogies between

Iwasawa theory	**Alexander theory**
$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty / X_p)$ X_p^∞: cyclotomic p-cover of X_p	$G_K^{\mathrm{ab}} = \mathrm{Gal}(X_K^\infty / X_K)$ X_K^∞: infinite cyclic cover of X_K

In particular, there are close analogies between

Iwasawa theory	**Alexander theory**
$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty / X_p)$ X_p^∞: cyclotomic p-cover of X_p	$G_K^{\mathrm{ab}} = \mathrm{Gal}(X_K^\infty / X_K)$ X_K^∞: infinite cyclic cover of X_K
1-dim. universal representation $\chi_p : G_p \to \mathbb{Z}_p[[G_p^{\mathrm{ab}}]]^\times \risingdotseq \mathbb{Z}_p[[T]]^\times$	1-dim. universal representation $\chi_K : G_K \to \mathbb{Z}[G_K^{\mathrm{ab}}]^\times = \mathbb{Z}[t^{\pm 1}]^\times$

In particular, there are close analogies between

Iwasawa theory	Alexander theory
$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty/X_p)$ X_p^∞: cyclotomic p-cover of X_p	$G_K^{\mathrm{ab}} = \mathrm{Gal}(X_K^\infty/X_K)$ X_K^∞: infinite cyclic cover of X_K
1-dim. universal representation $\chi_p : G_p \to \mathbb{Z}_p[[G_p^{\mathrm{ab}}]]^\times \fallingdotseq \mathbb{Z}_p[[T]]^\times$	1-dim. universal representation $\chi_K : G_K \to \mathbb{Z}[G_K^{\mathrm{ab}}]^\times = \mathbb{Z}[t^{\pm 1}]^\times$
Iwasawa module $H_1(X_p^\infty, \mathbb{Z}_p) = H_1(X_p, \chi_p)$	knot module $H_1(X_K^\infty, \mathbb{Z}) = H_1(X_K, \chi_K)$

In particular, there are close analogies between

Iwasawa theory	Alexander theory
$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty / X_p)$ X_p^∞: cyclotomic p-cover of X_p	$G_K^{\mathrm{ab}} = \mathrm{Gal}(X_K^\infty / X_K)$ X_K^∞: infinite cyclic cover of X_K
1-dim. universal representation $\chi_p : G_p \to \mathbb{Z}_p[[G_p^{\mathrm{ab}}]]^\times \fallingdotseq \mathbb{Z}_p[[T]]^\times$	1-dim. universal representation $\chi_K : G_K \to \mathbb{Z}[G_K^{\mathrm{ab}}]^\times = \mathbb{Z}[t^{\pm 1}]^\times$
Iwasawa module $H_1(X_p^\infty, \mathbb{Z}_p) = H_1(X_p, \chi_p)$	knot module $H_1(X_K^\infty, \mathbb{Z}) = H_1(X_K, \chi_K)$
Iwasawa polynomial (algebraic p-adic L-function) $\Delta_0(H_1(X_p, \chi_p))$	Alexander polynomial $\Delta_0(H_1(X_K, \chi_K))$

In particular, there are close analogies between

Iwasawa theory	Alexander theory
$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty / X_p)$ X_p^∞: cyclotomic p-cover of X_p	$G_K^{\mathrm{ab}} = \mathrm{Gal}(X_K^\infty / X_K)$ X_K^∞: infinite cyclic cover of X_K
1-dim. universal representation $\chi_p : G_p \to \mathbb{Z}_p[[G_p^{\mathrm{ab}}]]^\times \fallingdotseq \mathbb{Z}_p[[T]]^\times$	1-dim. universal representation $\chi_K : G_K \to \mathbb{Z}[G_K^{\mathrm{ab}}]^\times = \mathbb{Z}[t^{\pm 1}]^\times$
Iwasawa module $H_1(X_p^\infty, \mathbb{Z}_p) = H_1(X_p, \chi_p)$	knot module $H_1(X_K^\infty, \mathbb{Z}) = H_1(X_K, \chi_K)$
Iwasawa polynomial (algebraic p-adic L-function) $\Delta_0(H_1(X_p, \chi_p))$	Alexander polynomial $\Delta_0(H_1(X_K, \chi_K))$

cf. [Hillman-Matei-Morishita], [Kadokami-Mizusawa], [Sugiyama], [Ueki].

§0 Introduction

Motivated by the work of Hida, Mazur initiated to study the deformation theory for p-adic Galois representations:

For a given representation $\bar{\rho} : G_p \to \mathrm{GL}_2(\mathbb{F}_p)$, Mazur produced the universal deformation

$$\boldsymbol{\rho} : G_p \to \mathrm{GL}_2(\boldsymbol{R}_{\bar{\rho}}),$$

where $\boldsymbol{R}_{\bar{\rho}}$ is a "big" complete local ring (e.g. $\mathbb{Z}_p[[T]]$) and proposed a GL_2-analogue of Iwasawa polynomial (p-adic L-function)

$$\Delta_0(H_1(G_p, \boldsymbol{\rho})).$$

Motivated by the work of Hida, Mazur initiated to study the deformation theory for p-adic Galois representations:

For a given representation $\bar{\rho} : G_p \to \mathrm{GL}_2(\mathbb{F}_p)$,
Mazur produced the universal deformation

$$\boldsymbol{\rho} : G_p \to \mathrm{GL}_2(\boldsymbol{R}_{\bar{\rho}}),$$

where $\boldsymbol{R}_{\bar{\rho}}$ is a "big" complete local ring (e.g. $\mathbb{Z}_p[[T]]$)
and proposed a GL_2-analogue of
Iwasawa polynomial (p-adic L-function)

$$\Delta_0(H_1(G_p, \boldsymbol{\rho})).$$

Our project is to fill "?" in the following dictionary:

GL_1	Iwasawa theory	Alexander theory
GL_2	Hida-Mazur theory	?

We will introduce

a new invariant (L-function)

in knot theory, which may be a topological analogue of

the p-adic L-function

for Galois deformation in Number Theory.

Our project is to fill "?" in the following dictionary:

GL_1	Iwasawa theory	Alexander theory
GL_2	Hida-Mazur theory	?

We will introduce

a new invariant (L-function)

in knot theory, which may be a topological analogue of

the p-adic L-function

for Galois deformation in Number Theory.

§0 Introduction

Number theory	Knot theory
Selmer module for deformation of Galois representation	
p-adic L-function associated to the Selmer module	

Number theory	Knot theory
Selmer module for deformation of Galois representation	twisted knot module for deformation of knot group representation
p-adic L-function associated to the Selmer module	L-function associated to the twisted knot module

Mazur proposed a number of problems related in his famous article

[Ma] B. Mazur, *The theme of p-adic variation*, Mathematics: frontiers and perspectives, American Mathematical Society, RI, (2000), 433–459.

In this talk, I will report
some results which solve Mazur's problem.

This talk is based on

[KMTT] T. Kitayama, M. Morishita, R. Tange, Y. Terashima, *On certain L-functions for deformations of knot group representations*, arxiv.math.1506.00351v2 (2015).

§1 Universal deformations

G: finitely generated group,
k: field with $\mathrm{char}(k) \neq 2$ (e.g. $\mathbb{F}_p$, $\mathbb{C}$),
$\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$ (e.g. $\mathbb{Z}_p$, $\mathbb{C}[[\hbar]]$),
$\overline{\rho} : G \to \mathrm{SL}_2(k)$: representation.

(R, ρ) is a *deformation* of $\overline{\rho}$

$$\overset{\text{def.}}{\Leftrightarrow} \begin{cases} \cdot\ R : \text{complete local } \mathcal{O}\text{-algebra with } R/\mathfrak{m}_R = k, \\ \cdot\ \rho : G \to \mathrm{SL}_2(R) : \text{ representation with } \rho \bmod \mathfrak{m}_R = \overline{\rho}. \end{cases}$$

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{SL}_2(R) \\ & \overset{\overline{\rho}}{\searrow} & \downarrow \bmod \mathfrak{m}_R \\ & & \mathrm{SL}_2(k) \end{array}$$

G: finitely generated group,
k: field with $\mathrm{char}(k) \neq 2$ (e.g. $\mathbb{F}_p$, $\mathbb{C}$),
$\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$ (e.g. $\mathbb{Z}_p$, $\mathbb{C}[[\hbar]]$),
$\overline{\rho} : G \to \mathrm{SL}_2(k)$: representation.

(R, ρ) is a *deformation* of $\overline{\rho}$

$$\overset{\text{def.}}{\Leftrightarrow} \begin{cases} \cdot\ R : \text{complete local } \mathcal{O}\text{-algebra with } R/\mathfrak{m}_R = k, \\ \cdot\ \rho : G \to \mathrm{SL}_2(R) : \text{ representation with } \rho \bmod \mathfrak{m}_R = \overline{\rho}. \end{cases}$$

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{SL}_2(R) \\ & \overline{\rho} \searrow & \downarrow \bmod \mathfrak{m}_R \\ & & \mathrm{SL}_2(k) \end{array}$$

§1 Universal deformations

G: finitely generated group,
k: field with $\mathrm{char}(k) \neq 2$ (e.g. $\mathbb{F}_p$, $\mathbb{C}$),
$\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$ (e.g. $\mathbb{Z}_p$, $\mathbb{C}[[\hbar]]$),
$\overline{\rho} : G \to \mathrm{SL}_2(k)$: representation.

(R, ρ) is a *deformation* of $\overline{\rho}$

$$\overset{\text{def.}}{\Leftrightarrow} \begin{cases} \cdot\ R : \text{complete local } \mathcal{O}\text{-algebra with } R/\mathfrak{m}_R = k, \\ \cdot\ \rho : G \to \mathrm{SL}_2(R) : \text{ representation with } \rho \bmod \mathfrak{m}_R = \overline{\rho}. \end{cases}$$

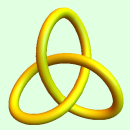
$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{SL}_2(R) \\ & \overset{\overline{\rho}}{\searrow} & \downarrow {\scriptstyle \mathrm{mod}\ \mathfrak{m}_R} \\ & & \mathrm{SL}_2(k) \end{array}$$

G: finitely generated group,
k: field with $\mathrm{char}(k) \neq 2$ (e.g. $\mathbb{F}_p$, $\mathbb{C}$),
$\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$ (e.g. $\mathbb{Z}_p$, $\mathbb{C}[[\hbar]]$),
$\overline{\rho} : G \to \mathrm{SL}_2(k)$: representation.

(R, ρ) is a *deformation* of $\overline{\rho}$

$$\overset{\text{def.}}{\Leftrightarrow} \begin{cases} \cdot\ R : \text{complete local } \mathcal{O}\text{-algebra with } R/\mathfrak{m}_R = k, \\ \cdot\ \rho : G \to \mathrm{SL}_2(R) : \text{ representation with } \rho \bmod \mathfrak{m}_R = \overline{\rho}. \end{cases}$$

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & \mathrm{SL}_2(R) \\ & \overline{\rho} \searrow & \downarrow \bmod \mathfrak{m}_R \\ & & \mathrm{SL}_2(k) \end{array}$$



Example (K: trefoil)

$G = G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$, $k = \mathbb{F}_3$, $\mathcal{O} = \mathbb{Z}_3$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

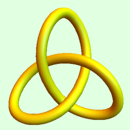
$\rho_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3)$;

$$\rho_1(g_1) = \begin{pmatrix} \frac{3}{2} & -1 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix},\ \rho_1(g_2) = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{4} & \frac{3}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3, \rho_1)$ is a deformation of $\overline{\rho}_1$.

$$\begin{array}{ccc} G_K & \xrightarrow{\rho_1} & \mathrm{SL}_2(\mathbb{Z}_3) \\ & \searrow^{\overline{\rho}_1} & \downarrow \text{mod } 3 \\ & & \mathrm{SL}_2(\mathbb{F}_3) \end{array}$$

§1 Universal deformations



Example (K: trefoil)

$G = G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$, $k = \mathbb{F}_3$, $\mathcal{O} = \mathbb{Z}_3$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

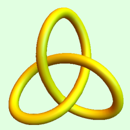
$\rho_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3)$;

$$\rho_1(g_1) = \begin{pmatrix} \frac{3}{2} & -1 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix},\ \rho_1(g_2) = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{4} & \frac{3}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3, \rho_1)$ is a deformation of $\overline{\rho}_1$.

$$\begin{array}{ccc} G_K & \xrightarrow{\rho_1} & \mathrm{SL}_2(\mathbb{Z}_3) \\ & \searrow^{\overline{\rho}_1} & \downarrow \bmod 3 \\ & & \mathrm{SL}_2(\mathbb{F}_3) \end{array}$$

§1 Universal deformations



Example (K: trefoil)

$G = G_K = \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle$, $k = \mathbb{F}_3$, $\mathcal{O} = \mathbb{Z}_3$,

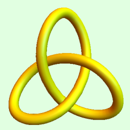
$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\rho_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3)$;

$$\rho_1(g_1) = \begin{pmatrix} \frac{3}{2} & -1 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix},\ \rho_1(g_2) = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{4} & \frac{3}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3, \rho_1)$ is a deformation of $\overline{\rho}_1$.

$$\begin{array}{ccc} G_K & \xrightarrow{\rho_1} & \mathrm{SL}_2(\mathbb{Z}_3) \\ & \overset{\overline{\rho}_1}{\searrow} & \downarrow \bmod 3 \\ & & \mathrm{SL}_2(\mathbb{F}_3) \end{array}$$



Example (K: trefoil)

$G = G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$, $k = \mathbb{F}_3$, $\mathcal{O} = \mathbb{Z}_3$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\rho_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3)$;

$$\rho_1(g_1) = \begin{pmatrix} \frac{3}{2} & -1 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix},\ \rho_1(g_2) = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{4} & \frac{3}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3, \rho_1)$ is a deformation of $\overline{\rho}_1$.

$$\begin{array}{ccc} G_K & \xrightarrow{\rho_1} & \mathrm{SL}_2(\mathbb{Z}_3) \\ & \searrow^{\overline{\rho}_1} & \downarrow \bmod 3 \\ & & \mathrm{SL}_2(\mathbb{F}_3) \end{array}$$

$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ is a *universal deformation* of $\overline{\rho}$

$$\overset{\text{def.}}{\Leftrightarrow} \begin{cases} \cdot\ (\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho}): \text{ deformation of } \overline{\rho}, \\ \cdot\ {}^{\forall}(R, \rho): \text{ deformation of } \overline{\rho}, {}^{\exists 1}\psi : \boldsymbol{R}_{\overline{\rho}} \to R \text{ s.t. } \psi \circ \boldsymbol{\rho} \approx \rho. \end{cases}$$

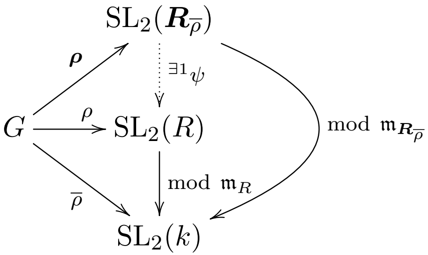
$(\rho_1 \approx \rho_2 \overset{\text{def.}}{\Leftrightarrow} {}^{\exists}\gamma \in I_2 + \mathrm{M}_2(\mathfrak{m}_R)$ s.t. $\rho_2(g) = \gamma \rho_1(g) \gamma^{-1}$ $({}^{\forall}g \in G).$)

$\boldsymbol{R}_{\overline{\rho}}$ is called the *universal deformation ring* of $\overline{\rho}$.

$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ is a *universal deformation* of $\overline{\rho}$

$\overset{\text{def.}}{\Leftrightarrow}$
- $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: deformation of $\overline{\rho}$,
- ${}^{\forall}(R, \rho)$: deformation of $\overline{\rho}$, ${}^{\exists 1}\psi : \boldsymbol{R}_{\overline{\rho}} \to R$ s.t. $\psi \circ \boldsymbol{\rho} \approx \rho$.

($\rho_1 \approx \rho_2 \overset{\text{def.}}{\Leftrightarrow} {}^{\exists}\gamma \in I_2 + \mathrm{M}_2(\mathfrak{m}_R)$ s.t. $\rho_2(g) = \gamma \rho_1(g) \gamma^{-1}$ (${}^{\forall}g \in G$).)



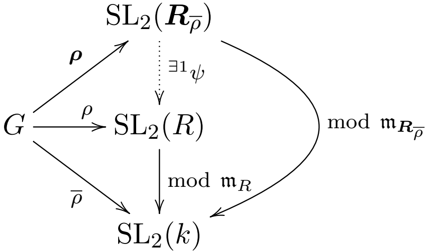
$\boldsymbol{R}_{\overline{\rho}}$ is called the *universal deformation ring* of $\overline{\rho}$.

§1 Universal deformations

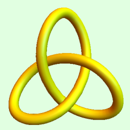
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$ is a *universal deformation* of $\overline{\rho}$

$\overset{\text{def.}}{\Leftrightarrow}$ $\begin{cases} \cdot\ (\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho}) : \text{ deformation of } \overline{\rho}, \\ \cdot\ {}^{\forall}(R, \rho) : \text{ deformation of } \overline{\rho}, {}^{\exists 1}\psi : \boldsymbol{R}_{\overline{\rho}} \to R \text{ s.t. } \psi \circ \boldsymbol{\rho} \approx \rho. \end{cases}$

$\left(\rho_1 \approx \rho_2 \overset{\text{def.}}{\Leftrightarrow} {}^{\exists}\gamma \in I_2 + \mathrm{M}_2(\mathfrak{m}_R) \text{ s.t. } \rho_2(g) = \gamma \rho_1(g) \gamma^{-1} \ ({}^{\forall} g \in G).\ \right)$



$\boldsymbol{R}_{\overline{\rho}}$ is called the *universal deformation ring* of $\overline{\rho}$.



Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

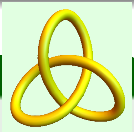
$\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$;

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is a universal deformation of $\overline{\rho}_1$.

§1 Universal deformations



Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

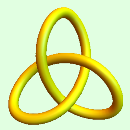
$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$;

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is a universal deformation of $\overline{\rho}_1$.



Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$;

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is a universal deformation of $\overline{\rho}_1$.

Theorem ([MTTU; Theorem 2.2.2.])

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation
$\Rightarrow {}^{\exists}(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$.

$\overline{\rho} : G \to \mathrm{SL}_2(k)$ is an *absolutely irreducible* representation
$\overset{\text{def.}}{\Leftrightarrow} G \overset{\overline{\rho}}{\to} \mathrm{SL}_2(k) \hookrightarrow \mathrm{SL}_2(\overline{k})$: irreducible ($\overline{k}$: algebraic closure of k).

[MTTU] M. Morishita, Y. Takakura, Y. Terashima, J. Ueki, *On the universal deformations for* SL_2*-representations of knot groups*, to appear in Tohoku Mathematical Journal.

Theorem ([MTTU; Theorem 2.2.2.])

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation
$\Rightarrow {}^{\exists}(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$.

$\overline{\rho} : G \to \mathrm{SL}_2(k)$ is an *absolutely irreducible* representation
$\overset{\text{def.}}{\Leftrightarrow} G \overset{\overline{\rho}}{\to} \mathrm{SL}_2(k) \hookrightarrow \mathrm{SL}_2(\overline{k})$: irreducible ($\overline{k}$: algebraic closure of k).

[MTTU] M. Morishita, Y. Takakura, Y. Terashima, J. Ueki,
On the universal deformations for SL_2-representations of knot groups, to appear in Tohoku Mathematical Journal.

$G \curvearrowright \mathrm{Ad}(\overline{\rho}) = \mathfrak{sl}_2(k);\ g \cdot X := \overline{\rho}(g) X \overline{\rho}(g)^{-1}$.

Theorem (Mazur)

$H^2(G, \mathrm{Ad}(\overline{\rho})) = 0$ (deformation problem is *unobstructed*)
$\Rightarrow \boldsymbol{R}_{\overline{\rho}} = \mathcal{O}[[X_1, \ldots, X_d]]$ ($d := \dim_k \mathfrak{t}_{\boldsymbol{R}_{\overline{\rho}}/\mathcal{O}}$).

Deformation problem is NOT unobstructed in general for a representation of a knot group.

Example

K: hyperbolic knot (e.g. figure-eight knot, knot 5_2, etc.),
$\rho_h : G_K \to \mathrm{SL}_2(\mathcal{O}_F) \subset \mathrm{SL}_2(\mathbb{C})$: holonomy representation,
$\overline{\rho_h} : G_K \to \mathrm{SL}_2(\mathcal{O}_F/\mathfrak{p})$ (F: number field, $\mathfrak{p}$: prime ideal of $\mathcal{O}_F$).
By using Thurston's theorem,

$$H^2(G_K, \mathrm{Ad}(\overline{\rho_h})) \neq 0.$$

$G \curvearrowright \mathrm{Ad}(\overline{\rho}) = \mathfrak{sl}_2(k)$; $g \cdot X := \overline{\rho}(g) X \overline{\rho}(g)^{-1}$.

Theorem (Mazur)

$H^2(G, \mathrm{Ad}(\overline{\rho})) = 0$ (deformation problem is *unobstructed*)
$\Rightarrow \boldsymbol{R}_{\overline{\rho}} = \mathcal{O}[[X_1, \ldots, X_d]]$ $(d := \dim_k \mathfrak{t}_{\boldsymbol{R}_{\overline{\rho}}/\mathcal{O}})$.

Deformation problem is NOT unobstructed in general for a representation of a knot group.

Example

K: hyperbolic knot (e.g. figure-eight knot, knot 5_2, etc.),
$\rho_h : G_K \to \mathrm{SL}_2(\mathcal{O}_F) \subset \mathrm{SL}_2(\mathbb{C})$: holonomy representation,
$\overline{\rho_h} : G_K \to \mathrm{SL}_2(\mathcal{O}_F/\mathfrak{p})$ (F: number field, $\mathfrak{p}$: prime ideal of $\mathcal{O}_F$).
By using Thurston's theorem,

$$H^2(G_K, \mathrm{Ad}(\overline{\rho_h})) \neq 0.$$

$G \curvearrowright \mathrm{Ad}(\overline{\rho}) = \mathfrak{sl}_2(k);\ g \cdot X := \overline{\rho}(g) X \overline{\rho}(g)^{-1}$.

Theorem (Mazur)

$H^2(G, \mathrm{Ad}(\overline{\rho})) = 0$ (deformation problem is *unobstructed*)
$\Rightarrow \boldsymbol{R}_{\overline{\rho}} = \mathcal{O}[[X_1, \ldots, X_d]]$ $(d := \dim_k \mathfrak{t}_{\boldsymbol{R}_{\overline{\rho}}/\mathcal{O}})$.

Deformation problem is NOT unobstructed in general for a representation of a knot group.

Example

K: hyperbolic knot (e.g. figure-eight knot, knot 5_2, etc.),
$\rho_h : G_K \to \mathrm{SL}_2(\mathcal{O}_F) \subset \mathrm{SL}_2(\mathbb{C})$: holonomy representation,
$\overline{\rho_h} : G_K \to \mathrm{SL}_2(\mathcal{O}_F/\mathfrak{p})$ (F: number field, $\mathfrak{p}$: prime ideal of $\mathcal{O}_F$).
By using Thurston's theorem,

$$H^2(G_K, \mathrm{Ad}(\overline{\rho_h})) \neq 0.$$

§2 Character schemes

G: finitely generated group

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity,} \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.,} \end{cases}$ satisfying the universal property:

$^\forall A$: comm. ring with identity, $^\forall \rho : G \to \mathrm{SL}_2(A)$: repr.,
$^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.

§2 Character schemes

G: finitely generated group

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity,} \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.,} \end{cases}$ satisfying the universal property:

${}^{\forall}A$: comm. ring with identity, ${}^{\forall}\rho : G \to \mathrm{SL}_2(A)$: repr.,

${}^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.

§2 Character schemes

G: finitely generated group

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity,} \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.,} \end{cases}$ satisfying the universal property:

$^\forall A$: comm. ring with identity, $^\forall \rho : G \to \mathrm{SL}_2(A)$: repr.,

$^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.

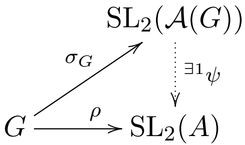
§2 Character schemes

G: finitely generated group

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity,} \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.,} \end{cases}$ satisfying the universal property:

$^\forall A$: comm. ring with identity, $^\forall \rho : G \to \mathrm{SL}_2(A)$: repr.,
$^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.



§2 Character schemes

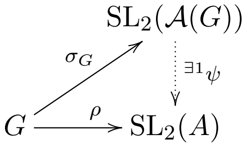
G: finitely generated group $= \langle g_1, \ldots, g_n \mid r_l \ (l \in L) \rangle$.

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity}, \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.}, \end{cases}$ satisfying the universal property:

${}^{\forall}A$: comm. ring with identity, ${}^{\forall}\rho : G \to \mathrm{SL}_2(A)$: repr.,

${}^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.



§2 Character schemes

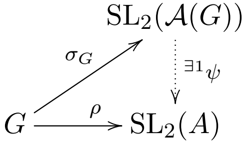
G: finitely generated group $= \langle g_1, \ldots, g_n \mid r_l \ (l \in L) \rangle$.

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity}, \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.}, \end{cases}$ satisfying the universal property:

${}^{\forall}A$: comm. ring with identity, ${}^{\forall}\rho : G \to \mathrm{SL}_2(A)$: repr.,

${}^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.



$$\mathcal{A}(G) = \frac{\mathbb{Z}[X_{ij}(g_1),\ \ldots,\ X_{ij}(g_n)\ (1 \leq i,\ j \leq 2)]}{\Big(r_l\big(X(g_1),\ \ldots,\ X(g_n)\big)_{ij},\ \det\big(X(g_h)\big) - 1\ (1 \leq h \leq n)\Big)},\ \sigma_G(g_h) = X(g_h).$$

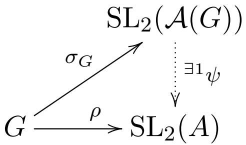
§2 Character schemes

G: finitely generated group $= \langle g_1, \ldots, g_n \mid r_l \ (l \in L)\rangle$.

$(\mathcal{A}(G), \sigma_G)$: *universal (tautological) representation* of G,

i.e. $\begin{cases} \cdot\ \mathcal{A}(G) : \text{comm. ring with identity}, \\ \cdot\ \sigma_G : G \to \mathrm{SL}_2(\mathcal{A}(G)) : \text{repr.}, \end{cases}$ satisfying the universal property:

${}^{\forall}A$: comm. ring with identity, ${}^{\forall}\rho : G \to \mathrm{SL}_2(A)$: repr.,
${}^{\exists 1}\psi : \mathcal{A}(G) \to A$ s.t. $\psi \circ \sigma_G = \rho$.



$$\mathcal{A}(G) = \frac{\mathbb{Z}[X_{ij}(g_1),\ \ldots,\ X_{ij}(g_n)\ (1 \le i,\ j \le 2)]}{\Big(r_l\big(X(g_1),\ \ldots,\ X(g_n)\big)_{ij},\ \det\big(X(g_h)\big) - 1\ (1 \le h \le n)\Big)},\ \sigma_G(g_h) = X(g_h).$$

The *representation scheme* of G over $\mathbb{Z}$:

$$\mathcal{R}(G) := \mathrm{Spec}(\mathcal{A}(G)).$$

§2 Character schemes

PGL_2: group scheme over $\mathbb{Z}$ with coordinate ring
$A(\mathrm{PGL}_2) =$ subring of $\mathbb{Z}[Y_{ij}\ (1 \le i, j \le 2)]_{\det(Y)}$ consisting of homogeneous elements of degree 0.

We have the adjoint action $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \to \mathcal{R}(G)$ given by

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2);\ \ X_{ij}(g) \mapsto (YX(g)Y^{-1})_{ij}.$$

The *character algebra* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \,|\, \mathrm{Ad}^*(x) = x \otimes 1\}.\end{aligned}$$

The *character scheme* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{X}(G) &:= \mathrm{Spec}(\mathcal{B}(G)) \\ &= \mathcal{R}(G)//\mathrm{PGL}_2.\end{aligned}$$

PGL_2: group scheme over $\mathbb{Z}$ with coordinate ring
$A(\mathrm{PGL}_2) =$ subring of $\mathbb{Z}[Y_{ij}\ (1 \leq i, j \leq 2)]_{\det(Y)}$ consisting of homogeneous elements of degree 0.

We have the adjoint action $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \to \mathcal{R}(G)$ given by

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2); \quad X_{ij}(g) \mapsto (YX(g)Y^{-1})_{ij}.$$

The *character algebra* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \mid \mathrm{Ad}^*(x) = x \otimes 1\}.\end{aligned}$$

The *character scheme* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{X}(G) &:= \mathrm{Spec}(\mathcal{B}(G)) \\ &= \mathcal{R}(G)//\mathrm{PGL}_2.\end{aligned}$$

PGL_2: group scheme over $\mathbb{Z}$ with coordinate ring
$A(\mathrm{PGL}_2) =$ subring of $\mathbb{Z}[Y_{ij}\ (1 \leq i, j \leq 2)]_{\det(Y)}$ consisting of homogeneous elements of degree 0.

We have the adjoint action $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \to \mathcal{R}(G)$ given by

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2); \quad X_{ij}(g) \mapsto (Y X(g) Y^{-1})_{ij}.$$

The *character algebra* of G over $\mathbb{Z}$:

$$\begin{aligned} \mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \mid \mathrm{Ad}^*(x) = x \otimes 1\}. \end{aligned}$$

The *character scheme* of G over $\mathbb{Z}$:

$$\begin{aligned} \mathcal{X}(G) &:= \mathrm{Spec}(\mathcal{B}(G)) \\ &= \mathcal{R}(G)//\mathrm{PGL}_2. \end{aligned}$$

PGL_2: group scheme over $\mathbb{Z}$ with coordinate ring
$A(\mathrm{PGL}_2)$ = subring of $\mathbb{Z}[Y_{ij}\ (1 \leq i, j \leq 2)]_{\det(Y)}$ consisting of homogeneous elements of degree 0.

We have the adjoint action $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \to \mathcal{R}(G)$ given by

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2); \quad X_{ij}(g) \mapsto (YX(g)Y^{-1})_{ij}.$$

The *character algebra* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \mid \mathrm{Ad}^*(x) = x \otimes 1\}.\end{aligned}$$

The *character scheme* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{X}(G) &:= \mathrm{Spec}(\mathcal{B}(G)) \\ &= \mathcal{R}(G)//\mathrm{PGL}_2.\end{aligned}$$

PGL_2: group scheme over $\mathbb{Z}$ with coordinate ring
$A(\mathrm{PGL}_2)$ = subring of $\mathbb{Z}[Y_{ij}\ (1 \le i,j \le 2)]_{\det(Y)}$ consisting of homogeneous elements of degree 0.

We have the adjoint action $\mathrm{Ad} : \mathcal{R}(G) \times \mathrm{PGL}_2 \to \mathcal{R}(G)$ given by

$$\mathrm{Ad}^* : \mathcal{A}(G) \longrightarrow \mathcal{A}(G) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2); \quad X_{ij}(g) \mapsto (YX(g)Y^{-1})_{ij}.$$

The *character algebra* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{B}(G) &:= \mathcal{A}(G)^{\mathrm{PGL}_2} \\ &:= \{x \in \mathcal{A}(G) \mid \mathrm{Ad}^*(x) = x \otimes 1\}.\end{aligned}$$

The *character scheme* of G over $\mathbb{Z}$:

$$\begin{aligned}\mathcal{X}(G) &:= \mathrm{Spec}(\mathcal{B}(G)) \\ &= \mathcal{R}(G)//\mathrm{PGL}_2.\end{aligned}$$

§2 Character schemes

Example

K: 2-bridge knot $B(m,n)$ (e.g. trefoil, figure-eight knot, knot 5_2),

$G_K = \langle g_1, g_2 \mid wg_1 = g_2 w\rangle$,

$w := g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_1^{\epsilon_{m-2}} g_2^{\epsilon_{m-1}}$, $\epsilon_i := (-1)^{[\frac{in}{m}]} = \epsilon_{m-i}$.

$x := \mathrm{tr}(\sigma_{G_K}(g_1))$, $y := \mathrm{tr}(\sigma_{G_K}(g_1 g_2)) \in \mathcal{B}(G_K)$.

Proposition (Le)

k: field with $\mathrm{char}(k) \neq 2$.
$\mathcal{X}(G_K)_k := \mathcal{X}(G_K) \otimes_{\mathbb{Z}} k$ is given by the algebraic curve

$$(y - x^2 + 2)\Phi_K(x, y - x^2 + 2) = 0$$

for some polynomial $\Phi_K(x, y - x^2 + 2) \in k[x, y]$.
$y - x^2 + 2 = 0$ is a locus of reducible repr.
$\Phi_K(x, y - x^2 + 2) = 0$ is a locus of absolutely irreducible repr.

Example

K: 2-bridge knot $B(m,n)$ (e.g. trefoil, figure-eight knot, knot 5_2),
$G_K = \langle g_1, g_2 \mid wg_1 = g_2 w \rangle$,
$w := g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_1^{\epsilon_{m-2}} g_2^{\epsilon_{m-1}}$, $\epsilon_i := (-1)^{[\frac{in}{m}]} = \epsilon_{m-i}$.

$x := \mathrm{tr}(\sigma_{G_K}(g_1))$, $y := \mathrm{tr}(\sigma_{G_K}(g_1 g_2)) \in \mathcal{B}(G_K)$.

Proposition (Le)

k: field with $\mathrm{char}(k) \neq 2$.
$\mathcal{X}(G_K)_k := \mathcal{X}(G_K) \otimes_{\mathbb{Z}} k$ is given by the algebraic curve

$$(y - x^2 + 2)\Phi_K(x, y - x^2 + 2) = 0$$

for some polynomial $\Phi_K(x, y - x^2 + 2) \in k[x, y]$.
$y - x^2 + 2 = 0$ is a locus of reducible repr.
$\Phi_K(x, y - x^2 + 2) = 0$ is a locus of absolutely irreducible repr.

§2 Character schemes

Example

K: 2-bridge knot $B(m,n)$ (e.g. trefoil, figure-eight knot, knot 5_2),
$G_K = \langle g_1, g_2 \mid wg_1 = g_2 w\rangle$,
$w := g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_1^{\epsilon_{m-2}} g_2^{\epsilon_{m-1}}$, $\epsilon_i := (-1)^{[\frac{in}{m}]} = \epsilon_{m-i}$.

$x := \mathrm{tr}(\sigma_{G_K}(g_1))$, $y := \mathrm{tr}(\sigma_{G_K}(g_1 g_2)) \in \mathcal{B}(G_K)$.

Proposition (Le)

k: field with $\mathrm{char}(k) \neq 2$.
$\mathcal{X}(G_K)_k := \mathcal{X}(G_K) \otimes_{\mathbb{Z}} k$ is given by the algebraic curve

$$(y - x^2 + 2)\Phi_K(x, y - x^2 + 2) = 0$$

for some polynomial $\Phi_K(x, y - x^2 + 2) \in k[x, y]$.
$y - x^2 + 2 = 0$ is a locus of reducible repr.
$\Phi_K(x, y - x^2 + 2) = 0$ is a locus of absolutely irreducible repr.

Example

K: 2-bridge knot $B(m, n)$ (e.g. trefoil, figure-eight knot, knot 5_2),
$G_K = \langle g_1, g_2 \mid wg_1 = g_2 w \rangle$,
$w := g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_1^{\epsilon_{m-2}} g_2^{\epsilon_{m-1}}$, $\epsilon_i := (-1)^{[\frac{in}{m}]} = \epsilon_{m-i}$.

$x := \mathrm{tr}(\sigma_{G_K}(g_1))$, $y := \mathrm{tr}(\sigma_{G_K}(g_1 g_2)) \in \mathcal{B}(G_K)$.

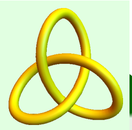
Proposition (Le)

k: field with $\mathrm{char}(k) \neq 2$.
$\mathcal{X}(G_K)_k := \mathcal{X}(G_K) \otimes_{\mathbb{Z}} k$ is given by the algebraic curve

$$(y - x^2 + 2)\Phi_K(x, y - x^2 + 2) = 0$$

for some polynomial $\Phi_K(x, y - x^2 + 2) \in k[x, y]$.
$y - x^2 + 2 = 0$ is a locus of reducible repr.
$\Phi_K(x, y - x^2 + 2) = 0$ is a locus of absolutely irreducible repr.

§2 Character schemes



Example (K: $B(3,1)$ (trefoil))

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \ \ w = g_1 g_2,$

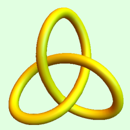
$$\Phi_K(x, y - x^2 + 2) = y - 1.$$

Example (K: $B(7,3)$ (knot 5_2))

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 = g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 \rangle,$
$w = g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2,$

$$\Phi_K(x, y - x^2 + 2) = y^3 - (x^2 + 1) y^2 + (3x^2 - 2) y - 2x^2 + 1.$$

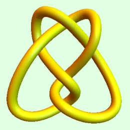
§2 Character schemes



Example (K: $B(3,1)$ (trefoil))

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$, $w = g_1 g_2$,

$$\Phi_K(x, y - x^2 + 2) = y - 1.$$



Example (K: $B(7,3)$ (knot 5_2))

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 = g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 \rangle$,
$w = g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2$,

$$\Phi_K(x, y - x^2 + 2) = y^3 - (x^2 + 1) y^2 + (3x^2 - 2) y - 2x^2 + 1.$$

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow}$ $\mathcal{B}(G)$.

§2 Character schemes

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow} \mathcal{B}(G)$.

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k,\ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq\ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow}$ $\mathcal{B}(G)$.

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow} \mathcal{B}(G)$.

§2 Character schemes

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow}$ $\mathcal{B}(G)$.

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k,\ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq\ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow} \mathcal{B}(G)$.

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $R_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow}$ $\mathcal{B}(G)$.

Relation between character algebras & universal deformation rings

k: field with $\mathrm{char}(k) \neq 2$, $\mathcal{O}$: complete DVR with $\mathcal{O}/\mathfrak{m}_{\mathcal{O}} = k$.

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_{\mathbb{Z}} k, \ \mathcal{X}(G)_k = \mathrm{Spec}(\mathcal{B}(G)_k).$$

$\overline{\rho} : G \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$\boldsymbol{R}_{\overline{\rho}}$: universal deformation ring of $\overline{\rho}$,
$[\overline{\rho}]$: maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$,
$(\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}$: $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

Theorem ([KMTT; Theorem 2.2.1.])

We have an isomorphism of k-algebras

$$\boldsymbol{R}_{\overline{\rho}} \otimes_{\mathcal{O}} k \ \simeq \ (\mathcal{B}(G)_k)^{\wedge}_{[\overline{\rho}]}.$$

$\boldsymbol{R}_{\overline{\rho}}$ is an infinitesimal deformation of $\mathcal{B}(G)_k$!

Sketch of proof) $\boldsymbol{R}_{\overline{\rho}} \longleftrightarrow$ skein algebra $\underset{\text{K. Saito}}{\longleftrightarrow}$ $\mathcal{B}(G)$.

Using the relation between $\mathcal{B}(G)$ and $\boldsymbol{R}_{\overline{\rho}}$,
we have the following useful theorem to determine $\boldsymbol{R}_{\overline{\rho}}$:

Theorem ([KMTT; Theorem 2.2.4.])

Suppose

- $[\overline{\rho}]$ is a regular point of $\mathcal{X}(G)_k$,
- $^{\exists}g \in G$ s.t. $\mathrm{tr}(\sigma_G(g)) - \mathrm{tr}(\overline{\rho}(g))$ is a local parameter at $[\overline{\rho}]$,
- $^{\exists}\beta \in \mathcal{O}$ s.t. $\beta \bmod \mathfrak{m}_{\mathcal{O}} = \mathrm{tr}(\overline{\rho}(g))$,
- $^{\exists}(\mathcal{O}[[x-\beta]], \rho)$: deformation of $\overline{\rho}$ s.t. $\mathrm{tr}(\rho(g)) = x$.

Then $(\mathcal{O}[[x-\beta]], \rho)$ is the universal deformation of $\overline{\rho}$.

Using the relation between $\mathcal{B}(G)$ and $\boldsymbol{R}_{\overline{\rho}}$,
we have the following useful theorem to determine $\boldsymbol{R}_{\overline{\rho}}$:

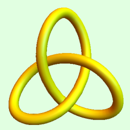
Theorem ([KMTT; Theorem 2.2.4.])

Suppose

- $[\overline{\rho}]$ is a regular point of $\mathcal{X}(G)_k$,
- ${}^{\exists}g \in G$ s.t. $\mathrm{tr}(\sigma_G(g)) - \mathrm{tr}(\overline{\rho}(g))$ is a local parameter at $[\overline{\rho}]$,
- ${}^{\exists}\beta \in \mathcal{O}$ s.t. $\beta \bmod \mathfrak{m}_{\mathcal{O}} = \mathrm{tr}(\overline{\rho}(g))$,
- ${}^{\exists}(\mathcal{O}[[x-\beta]], \rho)$: deformation of $\overline{\rho}$ s.t. $\mathrm{tr}(\rho(g)) = x$.

Then $(\mathcal{O}[[x-\beta]], \rho)$ is the universal deformation of $\overline{\rho}$.

§2 Character schemes



Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

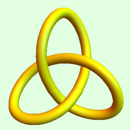
$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$;

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is a universal deformation of $\overline{\rho}_1$.



Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$;

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is a universal deformation of $\overline{\rho}_1$.

§3 Twisted knot modules

$K \subset S^3$: knot, $X_K := S^3 \setminus \mathrm{int}(V_K)$,
$G_K := \pi_1(X_K)$: knot group,
k: field with $\mathrm{char}(k) \neq 2$,
$\overline{\rho} : G_K \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$,
$V_{\boldsymbol{\rho}} := (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$: representation space of $\boldsymbol{\rho}$.

The *twisted knot module* for $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$:

$$H_1(\boldsymbol{\rho}) := H_1(X_K; V_{\boldsymbol{\rho}}).$$

Problem 1 (cf. [Ma])

What is the structure of $H_1(\boldsymbol{\rho})$ as an $\boldsymbol{R}_{\overline{\rho}}$-module?
Is $H_1(\boldsymbol{\rho})$ finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module?

We will give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\boldsymbol{\rho}; t)$.

$K \subset S^3$: knot, $X_K := S^3 \setminus \mathrm{int}(V_K)$,
$G_K := \pi_1(X_K)$: knot group,
k: field with $\mathrm{char}(k) \neq 2$,
$\overline{\rho} : G_K \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$,
$V_{\boldsymbol{\rho}} := (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$: representation space of $\boldsymbol{\rho}$.

The *twisted knot module* for $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$:

$$H_1(\boldsymbol{\rho}) := H_1(X_K; V_{\boldsymbol{\rho}}).$$

Problem 1 (cf. [Ma])

What is the structure of $H_1(\boldsymbol{\rho})$ as an $\boldsymbol{R}_{\overline{\rho}}$-module?
Is $H_1(\boldsymbol{\rho})$ finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module?

We will give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\boldsymbol{\rho}; t)$.

$K \subset S^3$: knot, $X_K := S^3 \setminus \mathrm{int}(V_K)$,
$G_K := \pi_1(X_K)$: knot group,
k: field with $\mathrm{char}(k) \neq 2$,
$\overline{\rho} : G_K \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$,
$V_{\boldsymbol{\rho}} := (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$: representation space of $\boldsymbol{\rho}$.

The *twisted knot module* for $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$:

$$H_1(\boldsymbol{\rho}) := H_1(X_K; V_{\boldsymbol{\rho}}).$$

Problem 1 (cf. [Ma])

What is the structure of $H_1(\boldsymbol{\rho})$ as an $\boldsymbol{R}_{\overline{\rho}}$-module?
Is $H_1(\boldsymbol{\rho})$ finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module?

We will give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\boldsymbol{\rho}; t)$.

$K \subset S^3$: knot, $X_K := S^3 \setminus \mathrm{int}(V_K)$,
$G_K := \pi_1(X_K)$: knot group,
k: field with $\mathrm{char}(k) \neq 2$,
$\overline{\rho} : G_K \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$,
$V_{\boldsymbol{\rho}} := (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$: representation space of $\boldsymbol{\rho}$.

The *twisted knot module* for $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$:

$$H_1(\boldsymbol{\rho}) := H_1(X_K; V_{\boldsymbol{\rho}}).$$

Problem 1 (cf. [Ma])

What is the structure of $H_1(\boldsymbol{\rho})$ as an $\boldsymbol{R}_{\overline{\rho}}$-module?
Is $H_1(\boldsymbol{\rho})$ finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module?

We will give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\rho; t)$.

§3 Twisted knot modules

$K \subset S^3$: knot, $X_K := S^3 \setminus \mathrm{int}(V_K)$,
$G_K := \pi_1(X_K)$: knot group,
k: field with $\mathrm{char}(k) \neq 2$,
$\overline{\rho} : G_K \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$,
$V_{\boldsymbol{\rho}} := (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$: representation space of $\boldsymbol{\rho}$.

The *twisted knot module* for $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$:

$$H_1(\boldsymbol{\rho}) := H_1(X_K; V_{\boldsymbol{\rho}}).$$

Problem 1 (cf. [Ma])

What is the structure of $H_1(\boldsymbol{\rho})$ as an $\boldsymbol{R}_{\overline{\rho}}$-module?
Is $H_1(\boldsymbol{\rho})$ finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module?

We will give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\rho; t)$.

§3 Twisted knot modules

$K \subset S^3$: knot, $X_K := S^3 \setminus \mathrm{int}(V_K)$,
$G_K := \pi_1(X_K)$: knot group,
k: field with $\mathrm{char}(k) \neq 2$,
$\overline{\rho} : G_K \to \mathrm{SL}_2(k)$: absolutely irreducible representation,
$(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$: universal deformation of $\overline{\rho}$,
$V_{\boldsymbol{\rho}} := (\boldsymbol{R}_{\overline{\rho}})^{\oplus 2}$: representation space of $\boldsymbol{\rho}$.

The *twisted knot module* for $(\boldsymbol{R}_{\overline{\rho}}, \boldsymbol{\rho})$:

$$H_1(\boldsymbol{\rho}) := H_1(X_K; V_{\boldsymbol{\rho}}).$$

Problem 1 (cf. [Ma])

What is the structure of $H_1(\boldsymbol{\rho})$ as an $\boldsymbol{R}_{\overline{\rho}}$-module?
Is $H_1(\boldsymbol{\rho})$ finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module?

We will give a criterion for $H_1(\boldsymbol{\rho})$ to be finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\rho; t)$.

§3 Twisted knot modules

Theorem ([KMTT; Theorem 3.2.4.])

Suppose that the following conditions are satisfied:
(1) $\boldsymbol{R}_{\overline{\rho}}$: Noetherian integral domain.
(2) $^{\exists}(R, \rho)$: deformation of $\overline{\rho}$, $^{\exists}g \in G_K$ s.t.
(2-1) $\det(\rho(g) - I) \neq 0$,
(2-2) $\Delta_K(\rho; 1) \neq 0$.
Then $H_1(\boldsymbol{\rho})$ is a finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module.

Theorem ([KMTT; Theorem 3.2.4.])

Suppose that the following conditions are satisfied:
(1) $\boldsymbol{R}_{\overline{\rho}}$: Noetherian integral domain.
(2) ${}^{\exists}(R, \rho)$: deformation of $\overline{\rho}$, ${}^{\exists}g \in G_K$ s.t.
(2-1) $\det(\rho(g) - I) \neq 0$,
(2-2) $\Delta_K(\rho; 1) \neq 0$.
Then $H_1(\boldsymbol{\rho})$ is a finitely generated torsion $\boldsymbol{R}_{\overline{\rho}}$-module.

§4 L-functions

Assume that
$\begin{cases} \cdot\ \boldsymbol{R}_{\overline{\rho}} \text{ is a Noetherian UFD,} \\ \cdot\ H_1(\boldsymbol{\rho}) \text{ is a finitely generated torsion } \boldsymbol{R}_{\overline{\rho}}\text{-module.} \end{cases}$

The *L-function* of K associated to $\boldsymbol{\rho}$:

$$L_K(\boldsymbol{\rho}) := \Delta_0(H_1(\boldsymbol{\rho})).$$

$L_K(\boldsymbol{\rho})$ is seen as a section of the coherent sheaf $\mathcal{H}_1(\boldsymbol{\rho})$ associated to $H_1(\boldsymbol{\rho})$ on the universal deformation space $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$.

§4 L-functions

Assume that
$\begin{cases} \cdot\ \boldsymbol{R}_{\overline{\rho}} \text{ is a Noetherian UFD,} \\ \cdot\ H_1(\boldsymbol{\rho}) \text{ is a finitely generated torsion } \boldsymbol{R}_{\overline{\rho}}\text{-module.} \end{cases}$

The *L-function* of K associated to $\boldsymbol{\rho}$:

$$L_K(\boldsymbol{\rho}) := \Delta_0(H_1(\boldsymbol{\rho})).$$

$L_K(\boldsymbol{\rho})$ is seen as a section of the coherent sheaf $\mathcal{H}_1(\boldsymbol{\rho})$ associated to $H_1(\boldsymbol{\rho})$ on the universal deformation space $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$.

Assume that

$\begin{cases} \cdot\ \boldsymbol{R}_{\overline{\rho}} \text{ is a Noetherian UFD,} \\ \cdot\ H_1(\boldsymbol{\rho}) \text{ is a finitely generated torsion } \boldsymbol{R}_{\overline{\rho}}\text{-module.} \end{cases}$

The *L-function* of K associated to $\boldsymbol{\rho}$:

$$L_K(\boldsymbol{\rho}) := \Delta_0(H_1(\boldsymbol{\rho})).$$

$L_K(\boldsymbol{\rho})$ is seen as a section of the coherent sheaf $\mathcal{H}_1(\boldsymbol{\rho})$ associated to $H_1(\boldsymbol{\rho})$ on the universal deformation space $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$.

Assume that

$\begin{cases} \cdot\ \boldsymbol{R}_{\overline{\rho}} \text{ is a Noetherian UFD,} \\ \cdot\ H_1(\boldsymbol{\rho}) \text{ is a finitely generated torsion } \boldsymbol{R}_{\overline{\rho}}\text{-module.} \end{cases}$

The *L-function* of K associated to $\boldsymbol{\rho}$:

$$L_K(\boldsymbol{\rho}) := \Delta_0(H_1(\boldsymbol{\rho})).$$

$L_K(\boldsymbol{\rho})$ is seen as a section of the coherent sheaf $\mathcal{H}_1(\boldsymbol{\rho})$ associated to $H_1(\boldsymbol{\rho})$ on the universal deformation space $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$.

The L-function $L_K(\boldsymbol{\rho})$ is a computable invariant.

$$\partial_2 := \left(\boldsymbol{\rho} \circ \pi \left(\frac{\partial r_j}{\partial g_i}\right)\right) : (V_{\boldsymbol{\rho}})^{\oplus(n-1)} \to (V_{\boldsymbol{\rho}})^{\oplus n}.$$

Proposition ([KMTT; Proposition 3.2.7.])

$$L_K(\boldsymbol{\rho}) \doteq \Delta_2(\mathrm{Coker}(\partial_2)).$$

$\Delta_2(\mathrm{Coker}(\partial_2)) :=$ g.c.d. of generators of $E_2(\mathrm{Coker}(\partial_2))$.

The L-function $L_K(\boldsymbol{\rho})$ is a computable invariant.

$$\partial_2 := \left(\boldsymbol{\rho} \circ \pi\left(\frac{\partial r_j}{\partial g_i}\right)\right) : (V_{\boldsymbol{\rho}})^{\oplus(n-1)} \to (V_{\boldsymbol{\rho}})^{\oplus n}.$$

Proposition ([KMTT; Proposition 3.2.7.])

$$L_K(\boldsymbol{\rho}) \doteq \Delta_2(\mathrm{Coker}(\partial_2)).$$

$\Delta_2(\mathrm{Coker}(\partial_2)) :=$ g.c.d. of generators of $E_2(\mathrm{Coker}(\partial_2))$.

The L-function $L_K(\boldsymbol{\rho})$ is a computable invariant.

$$\partial_2 := \left(\boldsymbol{\rho} \circ \pi \left(\frac{\partial r_j}{\partial g_i}\right)\right) : (V_{\boldsymbol{\rho}})^{\oplus(n-1)} \to (V_{\boldsymbol{\rho}})^{\oplus n}.$$

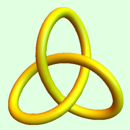
Proposition ([KMTT; Proposition 3.2.7.])

$$L_K(\boldsymbol{\rho}) \doteq \Delta_2(\mathrm{Coker}(\partial_2)).$$

$\Delta_2(\mathrm{Coker}(\partial_2)) :=$ g.c.d. of generators of $E_2(\mathrm{Coker}(\partial_2))$.

Problem 2 (cf. [Ma])

Is the order of $L_K(\boldsymbol{\rho})$ on $\mathrm{Spec}(\boldsymbol{R}_{\overline{\rho}})$ at any prime divisor zero or one?



Example (K: trefoil)

$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$,

$$\overline{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{F}_3);\ \overline{\rho}_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ \overline{\rho}_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix},$$

$\boldsymbol{\rho}_1 : G_K \to \mathrm{SL}_2(\mathbb{Z}_3[[x-2]])$;

$$\boldsymbol{\rho}_1(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x-\sqrt{x^2-3}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_1(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-3}}{2} & -1 \\ \frac{1}{4} & \frac{x+\sqrt{x^2-3}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_3[[x-2]], \boldsymbol{\rho}_1)$ is a universal deformation of $\overline{\rho}_1$.

$\cdot\Delta_0(H_0(\boldsymbol{\rho}_1)) \doteq 1$,
$\cdot\Delta_K(\overline{\rho}_1; t) = 1 + t^2$, hence, $\Delta_K(\overline{\rho}_1; 1) = 2 \neq 0$.
Therefore, we have

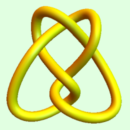
$$H_1(\boldsymbol{\rho}_1) = 0,\ L_K(\boldsymbol{\rho}_1) \doteq 1.$$

Lemma ([KMTT; Proposition 3.2.9.])

Assume that $\Delta_0(H_0(\boldsymbol{\rho})) \doteq 1$.

$$L_K(\boldsymbol{\rho}) \dot{\neq} 1 \Rightarrow \Delta_K(\overline{\rho}; 1) = 0.$$

Example (K: knot 5_2)



$$G_K = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 = g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 \rangle,$$

$$\overline{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{F}_{19});\ \overline{\rho}_2(g_1) = \begin{pmatrix} 14 & 1 \\ 1 & 11 \end{pmatrix},\ \overline{\rho}_2(g_2) = \begin{pmatrix} 11 & 1 \\ 1 & 14 \end{pmatrix}.$$

§5 Examples

$\beta := \frac{3+\sqrt{5}}{2} \in \mathbb{Z}_{19}$.

By Hensel's lemma, $^{\exists 1}y = y(x) \in \mathbb{Z}_{19}[[x-\beta]]$ s.t.

$$(\bigstar) \qquad y^3 - (x^2+1)y^2 + (3x^2-2)y - 2x^2 + 1 = 0$$

and

$$y(\beta) = \beta.$$

$\boldsymbol{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19}[[x-\beta]])$;

$$\boldsymbol{\rho}_2(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x-\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_2(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x+\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_{19}[[x-\beta]], \boldsymbol{\rho}_2)$ is the universal deformation of $\overline{\rho}_2$.

$\beta := \frac{3+\sqrt{5}}{2} \in \mathbb{Z}_{19}$.
By Hensel's lemma, ${}^{\exists 1}y = y(x) \in \mathbb{Z}_{19}[[x-\beta]]$ s.t.

$$(\bigstar) \qquad y^3 - (x^2+1)y^2 + (3x^2-2)y - 2x^2 + 1 = 0$$

and

$$y(\beta) = \beta.$$

$\boldsymbol{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19}[[x-\beta]])$;

$$\boldsymbol{\rho}_2(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x-\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_2(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x+\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_{19}[[x-\beta]], \boldsymbol{\rho}_2)$ is the universal deformation of $\overline{\rho}_2$.

$\beta := \frac{3+\sqrt{5}}{2} \in \mathbb{Z}_{19}$.
By Hensel's lemma, ${}^{\exists 1}y = y(x) \in \mathbb{Z}_{19}[[x-\beta]]$ s.t.

$$(\bigstar) \qquad y^3 - (x^2+1)y^2 + (3x^2-2)y - 2x^2 + 1 = 0$$

and

$$y(\beta) = \beta.$$

$\boldsymbol{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19}[[x-\beta]])$;

$$\boldsymbol{\rho}_2(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x-\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_2(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x+\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_{19}[[x-\beta]], \boldsymbol{\rho}_2)$ is the universal deformation of $\overline{\rho}_2$.

$\beta := \frac{3+\sqrt{5}}{2} \in \mathbb{Z}_{19}$.
By Hensel's lemma, $^{\exists 1} y = y(x) \in \mathbb{Z}_{19}[[x-\beta]]$ s.t.

$$(\bigstar) \qquad y^3 - (x^2+1)y^2 + (3x^2-2)y - 2x^2 + 1 = 0$$

and

$$y(\beta) = \beta.$$

$\boldsymbol{\rho}_2 : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19}[[x-\beta]])$;

$$\boldsymbol{\rho}_2(g_1) = \begin{pmatrix} \frac{x+\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x-\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

$$\boldsymbol{\rho}_2(g_2) = \begin{pmatrix} \frac{x-\sqrt{x^2-y(x)-2}}{2} & 1 \\ \frac{y(x)-2}{4} & \frac{x+\sqrt{x^2-y(x)-2}}{2} \end{pmatrix},$$

then $(\mathbb{Z}_{19}[[x-\beta]], \boldsymbol{\rho}_2)$ is the universal deformation of $\overline{\rho}_2$.

§5 Examples

$\rho_2 := \boldsymbol{\rho}_2|_{x=6} : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19})$;

$$\rho_2(g_1) = \begin{pmatrix} \frac{6+\sqrt{34-\gamma}}{2} & 1 \\ \frac{\gamma-2}{4} & \frac{6-\sqrt{34-\gamma}}{2} \end{pmatrix},$$

$$\rho_2(g_2) = \begin{pmatrix} \frac{6-\sqrt{34-\gamma}}{2} & 1 \\ \frac{\gamma-2}{4} & \frac{6+\sqrt{34-\gamma}}{2} \end{pmatrix},$$

where γ is the unique solution in $\mathbb{Z}_{19}$ satisfying $(\bigstar)$ with $x = 6$ and γ mod $19 = 6$.

$\cdot \det(\rho_2(g_2) - I) = -4 \neq 0$,

$\cdot \Delta_K(\rho_2; t) = \gamma^2 - 36\gamma + 72 - 12t + (\gamma^2 - 36\gamma + 72)t^2$, hence,
$\Delta_K(\rho_2; 1) = 2(\gamma^2 - 36\gamma + 66) \neq 0$.

Thus, $H_1(\boldsymbol{\rho}_2)$ is a finitely generated torsion $\mathbb{Z}_{19}[[x-\beta]]$-module.

$\rho_2 := \boldsymbol{\rho}_2|_{x=6} : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19})$;

$$\rho_2(g_1) = \begin{pmatrix} \frac{6+\sqrt{34-\gamma}}{2} & 1 \\ \frac{\gamma-2}{4} & \frac{6-\sqrt{34-\gamma}}{2} \end{pmatrix},$$

$$\rho_2(g_2) = \begin{pmatrix} \frac{6-\sqrt{34-\gamma}}{2} & 1 \\ \frac{\gamma-2}{4} & \frac{6+\sqrt{34-\gamma}}{2} \end{pmatrix},$$

where γ is the unique solution in $\mathbb{Z}_{19}$ satisfying ($\bigstar$) with $x = 6$ and γ mod $19 = 6$.

$\cdot \det(\rho_2(g_2) - I) = -4 \neq 0$,

$\cdot \Delta_K(\rho_2; t) = \gamma^2 - 36\gamma + 72 - 12t + (\gamma^2 - 36\gamma + 72)t^2$, hence,

$\Delta_K(\rho_2; 1) = 2(\gamma^2 - 36\gamma + 66) \neq 0$.

Thus, $H_1(\rho_2)$ is a finitely generated torsion $\mathbb{Z}_{19}[[x-\beta]]$-module.

$\rho_2 := \boldsymbol{\rho}_2|_{x=6} : G_K \to \mathrm{SL}_2(\mathbb{Z}_{19})$;

$$\rho_2(g_1) = \begin{pmatrix} \frac{6+\sqrt{34-\gamma}}{2} & 1 \\ \frac{\gamma-2}{4} & \frac{6-\sqrt{34-\gamma}}{2} \end{pmatrix},$$

$$\rho_2(g_2) = \begin{pmatrix} \frac{6-\sqrt{34-\gamma}}{2} & 1 \\ \frac{\gamma-2}{4} & \frac{6+\sqrt{34-\gamma}}{2} \end{pmatrix},$$

where γ is the unique solution in $\mathbb{Z}_{19}$ satisfying (★) with $x = 6$ and γ mod $19 = 6$.

$\cdot \det(\rho_2(g_2) - I) = -4 \neq 0$,

$\cdot \Delta_K(\rho_2; t) = \gamma^2 - 36\gamma + 72 - 12t + (\gamma^2 - 36\gamma + 72)t^2$, hence, $\Delta_K(\rho_2; 1) = 2(\gamma^2 - 36\gamma + 66) \neq 0$.

Thus, $H_1(\boldsymbol{\rho}_2)$ is a finitely generated torsion $\mathbb{Z}_{19}[[x - \beta]]$-module.

$$r := g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 (g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2)^{-1},$$

$$\partial_2 = \left(\boldsymbol{\rho}_2 \circ \pi \left(\frac{\partial r}{\partial g_1} \right), \boldsymbol{\rho}_2 \circ \pi \left(\frac{\partial r}{\partial g_2} \right) \right) =: (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4).$$

By the computer calculation, all 2-minors of ∂_2 are given by

$$\begin{aligned}
\det(\boldsymbol{a}_1, \boldsymbol{a}_2) &= 2(x-2)\{(y-2)x^2 + x - y^2\},\\
\det(\boldsymbol{a}_1, \boldsymbol{a}_3) &= -\tfrac{1}{2}\{(y-2)x^2 - (y-2)x - (y-1)^2\}\sqrt{x^2-y-2},\\
\det(\boldsymbol{a}_1, \boldsymbol{a}_4) &= (y-2)x^4 - (2y-5)x^3 - (y^2-y+4)x^2\\
&\quad +4(2y^2-y+2)x - (y-1)^2\\
&\quad -(x-2)\{(y-2)x^2 + x - y^2\}\sqrt{x^2-y-2},\\
\det(\boldsymbol{a}_2, \boldsymbol{a}_3) &= -\{(y-2)x^4 - (2y-5)x^3 - (y^2-y+4)x^2\\
&\quad +4(2y^2-y+2)x - (y-1)^2\}\\
&\quad -(x-2)\{(y-2)x^2 + x - y^2\}\sqrt{x^2-y-2},\\
\det(\boldsymbol{a}_2, \boldsymbol{a}_4) &= 2\{(y-2)x^2 + x - y^2\}\sqrt{x^2-y-2},\\
\det(\boldsymbol{a}_3, \boldsymbol{a}_4) &= 2(x-2)\{(y-2)x^2 + x - y^2\},
\end{aligned}$$

$$r := g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2 g_1 (g_2 g_1 g_2 g_1^{-1} g_2^{-1} g_1 g_2)^{-1},$$

$$\partial_2 = \left(\boldsymbol{\rho}_2 \circ \pi \left(\frac{\partial r}{\partial g_1}\right), \boldsymbol{\rho}_2 \circ \pi \left(\frac{\partial r}{\partial g_2}\right)\right) =: (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4).$$

By the computer calculation, all 2-minors of ∂_2 are given by

$$\begin{aligned}
\det(\boldsymbol{a}_1, \boldsymbol{a}_2) &= 2(x-2)\{(y-2)x^2 + x - y^2\}, \\
\det(\boldsymbol{a}_1, \boldsymbol{a}_3) &= -\tfrac{1}{2}\{(y-2)x^2 - (y-2)x - (y-1)^2\}\sqrt{x^2 - y - 2}, \\
\det(\boldsymbol{a}_1, \boldsymbol{a}_4) &= (y-2)x^4 - (2y-5)x^3 - (y^2 - y + 4)x^2 \\
&\quad +4(2y^2 - y + 2)x - (y-1)^2 \\
&\quad -(x-2)\{(y-2)x^2 + x - y^2\}\sqrt{x^2 - y - 2}, \\
\det(\boldsymbol{a}_2, \boldsymbol{a}_3) &= -\{(y-2)x^4 - (2y-5)x^3 - (y^2 - y + 4)x^2 \\
&\quad +4(2y^2 - y + 2)x - (y-1)^2\} \\
&\quad -(x-2)\{(y-2)x^2 + x - y^2\}\sqrt{x^2 - y - 2}, \\
\det(\boldsymbol{a}_2, \boldsymbol{a}_4) &= 2\{(y-2)x^2 + x - y^2\}\sqrt{x^2 - y - 2}, \\
\det(\boldsymbol{a}_3, \boldsymbol{a}_4) &= 2(x-2)\{(y-2)x^2 + x - y^2\},
\end{aligned}$$

and we have

$$H_1(\boldsymbol{\rho}_2) \simeq \mathbb{Z}_{19},\ L_K(\boldsymbol{\rho}_2) \doteq x - \beta.$$

Thank you for your attention!