A characterization of the Γ-polynomials of knots with clasp number at most two

Hideo Takioka, OCAMI

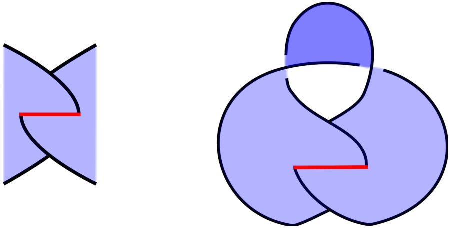
結び目の数学 VIII

早稲田大学

2015 年 12 月 26 日

Clasp number

Every knot in S^3 bounds a singular disk with only clasp singularities.



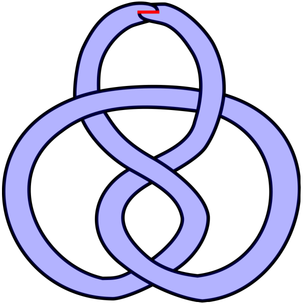
Such a singular disk is called a clasp disk.

The clasp number $\mathrm{clasp}(K)$ of a knot K is the minimum number of clasp singularities among all clasp disks of K.

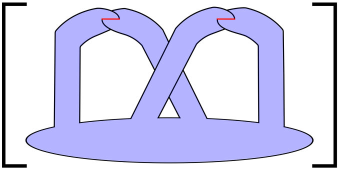
The clasp number of the unknot U is zero: $\mathrm{clasp}(U) = 0$.

$\mathcal{K}_1$: the set of knots which bound clasp disks with one clasp singularity.

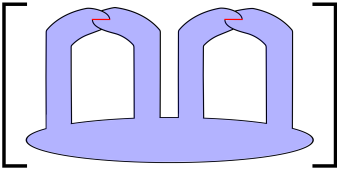
It is known that $\mathcal{K}_1 = \{\text{doubled knots}\}$.



There exist two homeomorphic classes of clasp disks with two clasp singularities:

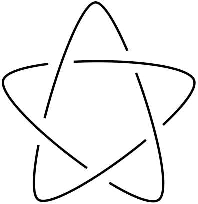


type 0

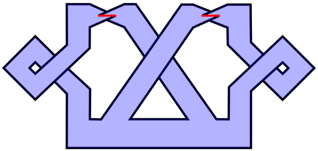


type 1

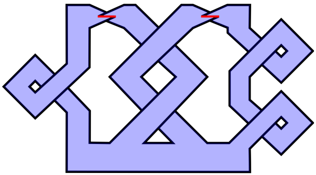
$\mathcal{K}_2^\delta$: the set of knots which bound clasp disks of type δ $(= 0, 1)$. We see that $5_1 \in \mathcal{K}_2^0 \cap \mathcal{K}_2^1$:



5_1

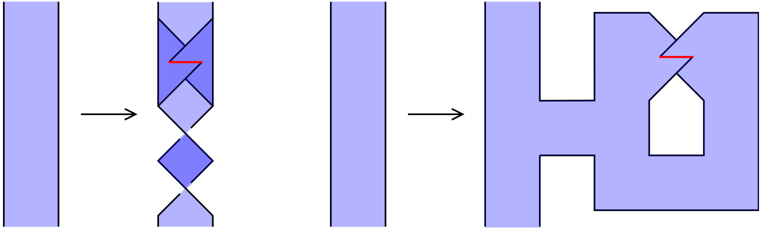


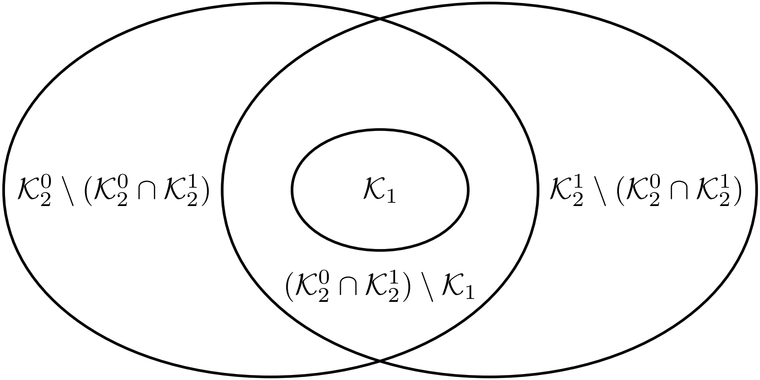
type 0



type 1

We see easily that $\mathcal{K}_1 \subset \mathcal{K}_2^0 \cap \mathcal{K}_2^1$.





Fact [Morimoto 1998, Kadokami and Kawamura 2014]

The Conway polynomials of knots with clasp number at most two are characterized:

(i) $\nabla(\mathcal{K}_1) = \{1 + bz^2 \mid b \in \mathbb{Z}\}$.

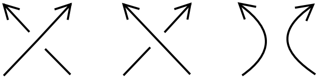
(ii) $\nabla(\mathcal{K}_2^\delta) =$
$\{1 + \big(b_1 + b_2 - \varepsilon(1-\delta)\big)z^2 + \big(b_1 b_2 + \varepsilon b_3(b_3 + 1 - \delta)\big)z^4$
$\mid \varepsilon = \pm 1, b_1, b_2, b_3 \in \mathbb{Z}\}$.

How about a characterization of the Γ-polynomials of knots with clasp number at most two?

The Γ-polynomial is the common zeroth coefficient polynomial of the HOMFLYPT and Kauffman polynomials.

The HOMFLYPT polynomial $P(L; y, z) \in \mathbb{Z}[y^{\pm 1}, z^{\pm 1}]$ **is an invariant of an oriented link** L **satisfying the following:**

$$P(U) = 1,\ yP(L_+) + y^{-1}P(L_-) = zP(L_0).$$

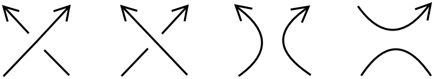


L_+ $\quad L_-$ $\quad L_0$

The Kauffman polynomial $F(L; a, b) \in \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$ **is an invariant of an oriented link** L **satisfying the following:**

$$F(U) = 1,\ aF(D_+) + a^{-1}F(D_-) = b\big(F(D_0) + a^{-2\nu}F(D_\infty)\big),$$

where $2\nu = w(D_+) - w(D_\infty) - 1.$



D_+ $\quad D_-$ $\quad D_0$ $\quad D_\infty$

Γ-polynomial

L: an r-component link.

$$P(L) = (yz)^{-r+1} \sum_{n \geq 0} p_n(L; y) z^{2n},$$

$$F(L) = (ab)^{-r+1} \sum_{n \geq 0} f_n(L; a) b^n,$$

where $p_n(L; y) \in \mathbb{Z}[y^{\pm 1}]$ and $f_n(L; a) \in \mathbb{Z}[a^{\pm 1}]$.

Fact [Lickorish 1988] $p_0(L; y) = f_0(L; y)$.

The polynomial $p_0(L; y) = f_0(L; y)$ is a Laurent polynomial of the variable $-y^2$. Putting $x = -y^2$, we call the polynomial the Γ-polynomial of L:

$$\Gamma(L; x) = \Gamma(L; -y^2) = p_0(L; y) = f_0(L; y).$$

Example H^-: the negative Hopf link.

$$\Gamma(H^-) = x - x^2.$$
$$P(H^-) = (yz)^{-1}((-y^2 - y^4) + y^2 z^2).$$
$$F(H^-) = (ab)^{-1}((-a^2 - a^4) + a^3 b + (a^2 + a^4)b^2).$$

$$\Gamma(H^-; -y^2) = p_0(H^-; y) = f_0(H^-; y) = -y^2 - y^4.$$

$$\Gamma(4_1) = x^{-1} - 1 + x.$$
$$P(4_1) = (-y^{-2} - 1 - y^2) + z^2.$$
$$F(4_1) = (-a^{-2} - 1 - a^2) + (-a^{-1} - a)b$$
$$+ (a^{-2} + 2 + a^2)b^2 + (a^{-1} + a)b^3.$$

$$\Gamma(4_1; -y^2) = p_0(4_1; y) = f_0(4_1; y) = -y^{-2} - 1 - y^2.$$

The Γ-polynomial has the following skein relation:

$$\Gamma(U)=1,\ -x\Gamma(L_+)+\Gamma(L_-)=\begin{cases}\Gamma(L_0) & \text{if } \mu=0,\\ 0 & \text{if } \mu=1,\end{cases}$$

where $\mu=(r_+-r_0+1)/2\ (=0,1)$ for the numbers r_+, r_0 of components of L_+, L_0, respectively.

Proposition Let $L=K_1\cup\cdots\cup K_r$ be an r-component link and $\mathrm{lk}(L)$ the total linking number of L. Then we have

$$\Gamma(L)=(1-x)^{r-1}x^{-\mathrm{lk}(L)}\Gamma(K_1)\cdots\Gamma(K_r).$$

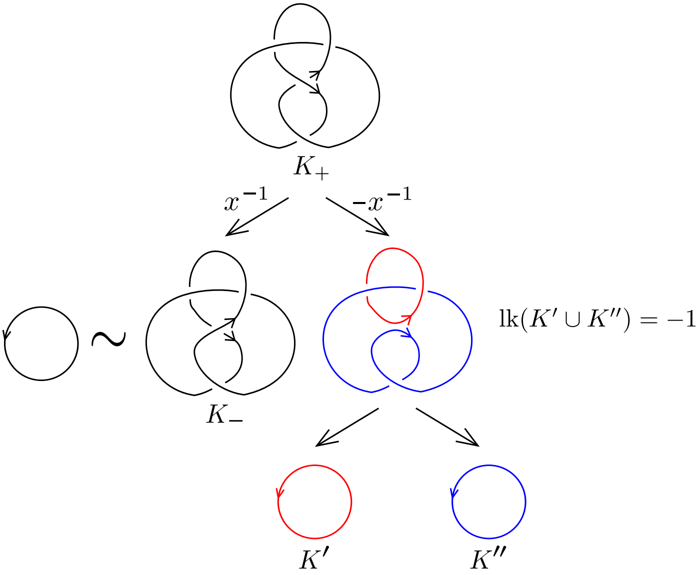
We obtain a skein relation for a knot as follows:

$$-x\Gamma(K_+)+\Gamma(K_-)=(1-x)x^{-\mathrm{lk}(K'\cup K'')}\Gamma(K')\Gamma(K''),$$

where $(K_+,K_-,K_0=K'\cup K'')$ is a skein triple such that K_+, K_-, K', and K'' are knots.

Example $\Gamma(K_+)$

$$= x^{-1}\Gamma(K_-) - x^{-1}(1-x)x^{-\operatorname{lk}(K'\cup K'')}\Gamma(K')\Gamma(K'')$$
$$= x^{-1} - 1 + x.$$



Fact [Kawauchi 1994] The Γ-polynomials of knots are characterized:

$\mathcal{K}$: the set of knots.

$$\Gamma(\mathcal{K}) = S,$$

where

$$S = \{1 + (1-x)^2 f(x) \mid f(x) \in \mathbb{Z}[x^{\pm 1}]\}.$$

In particular, for $K \in \mathcal{K}$,

$$\Gamma(K; -1) = 1, 5, 9, 13 \pmod{16}.$$

Theorem 1 The Γ-polynomials of knots with clasp number at most two are characterized:

(i) $\Gamma(\mathcal{K}_1) = S_1$, where

$$S_1 = \{x^{\varepsilon} + \varepsilon(1-x)x^{d}\big(1+(1-x)^2 f(x)\big)^2 \mid \varepsilon = \pm 1, d \in \mathbb{Z}, f(x) \in \mathbb{Z}[x^{\pm 1}]\}.$$

(ii) $\Gamma(\mathcal{K}_2^{\delta}) = S_2^{\delta}$ for $\delta = 0, 1$, where

$$S_2^{\delta} = \{x^{\varepsilon_1+\varepsilon_2} + \varepsilon_1(1-x)x^{d_1}\big(1+(1-x)^2 f_1(x)\big)^2 + \varepsilon_2(1-x)x^{d_2}\big(1+(1-x)^2 f_2(x)\big)^2 + \delta\varepsilon_1\varepsilon_2(1-x)^2 x^{d_3}\big(1+(1-x)^2 f_1(x)\big)\big(1+(1-x)^2 f_2(x)\big)\big(1+(1-x)^2 f_3(x)\big) \mid \varepsilon_1, \varepsilon_2 = \pm 1, d_1, d_2, d_3 \in \mathbb{Z}, f_1(x), f_2(x), f_3(x) \in \mathbb{Z}[x^{\pm 1}]\}.$$

Corollary

(i) If $K \in \mathcal{K}_1$, then we have

$$\Gamma(K;-1) = 1, 13 \pmod{16}.$$

(ii) If $K \in \mathcal{K}_2^{\delta}$, then we have

$$\Gamma(K;-1) = \begin{cases} 1, 5, 13 \pmod{16} & \text{if } \delta = 0, \\ 1, 5, 9, 13 \pmod{16} & \text{if } \delta = 1. \end{cases}$$

(iii) If $K\#K'$ is the connected sum of non-trivial knots K and K' with $\mathrm{clasp}(K\#K') = 2 \iff \mathrm{clasp}(K) = \mathrm{clasp}(K') = 1$ [Morimoto 1987], then we have

$$\Gamma(K\#K';-1) = 1, 9, 13 \pmod{16}.$$

Example It is known that $\mathrm{clasp}(3_1) = \mathrm{clasp}(4_1) = 1$, $\mathrm{clasp}(8_{15}) = \mathrm{clasp}(8_{20}) = \mathrm{clasp}(8_{21}) = 2$ in Kadokami-Kawamura's table.

We have $\Gamma(3_1\#3_1; -1) = \Gamma(3_1\#4_1; -1) = \Gamma(4_1\#4_1; -1) = \Gamma(8_{15}; -1) = \Gamma(8_{20}; -1) = \Gamma(8_{21}; -1) = 9 \pmod{16}$.

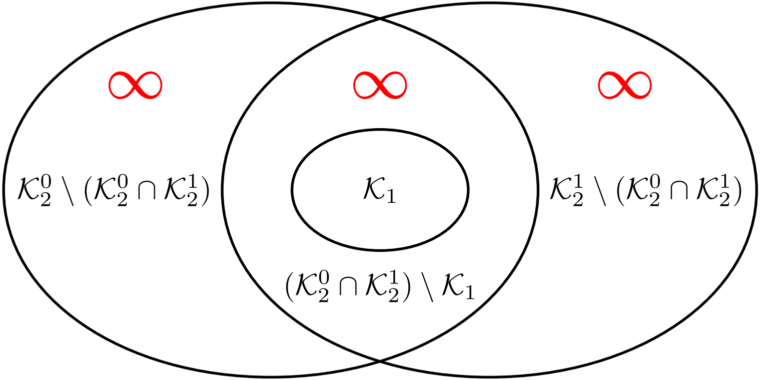
By Corollary, we see that $3_1\#3_1$, $3_1\#4_1$, $4_1\#4_1$, 8_{15}, 8_{20}, $8_{21} \in \mathcal{K}_2^1 \setminus (\mathcal{K}_2^0 \cap \mathcal{K}_2^1)$.

We see easily that the connected sum $K\#K' \in \mathcal{K}_2^1$ for knots K and K' with $\mathrm{clasp}(K) = \mathrm{clasp}(K') = 1$.

Question If $K \in \mathcal{K}_2^0$, then K is prime?

$\Gamma(\mathcal{K}) = \Gamma(\mathcal{K}_2^0 \cup \mathcal{K}_2^1)$?

Theorem 2 **There exist infinitely many knots in each set $(\mathcal{K}_2^0 \cap \mathcal{K}_2^1) \setminus \mathcal{K}_1$, $\mathcal{K}_2^0 \setminus (\mathcal{K}_2^0 \cap \mathcal{K}_2^1)$, $\mathcal{K}_2^1 \setminus (\mathcal{K}_2^0 \cap \mathcal{K}_2^1)$.**



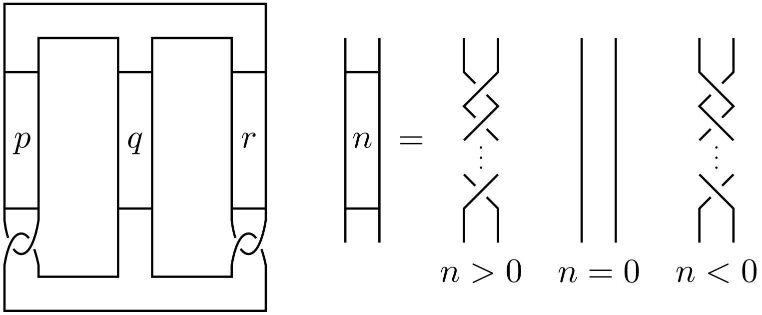
Proof of Theorem 2

<u>**Proposition**</u> **(i)** $K(p,q,r) = K(r,q,p)$.

(ii) $K(p,q,r) \in \mathcal{K}_2^0 \cup \mathcal{K}_2^1$.

(iii) If q **is odd, then** $K(p,q,r) \in \mathcal{K}_2^0$.

(iv) If q **is even, then** $K(p,q,r) \in \mathcal{K}_2^1$.



$K(p,q,r)$

By a characterization of the Conway polynomials of knots with clasp number at most two, we obtain the following:

<u>Lemma</u> K: a knot with $\nabla(K) = 1 + mz^2 + nz^4$ $(m, n \in \mathbb{Z})$.

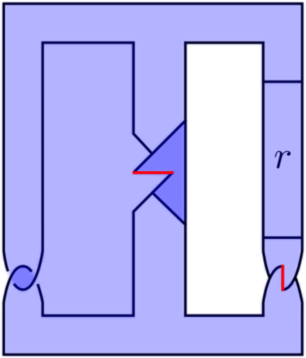
(i) If $4n - (m-1)^2 - 1 > 0$ and $\alpha + \beta \neq 2 \pmod 4$ for any integers α, β with $\alpha\beta = 4n - (m+1)^2 + 1$, then $K \notin \mathcal{K}_2^0$.

(ii) If $4n - m^2 > 0$ and $\alpha + \beta \neq 0 \pmod 4$ for any integers α, β with $\alpha\beta = 4n - m^2$, then $K \notin \mathcal{K}_2^1$.

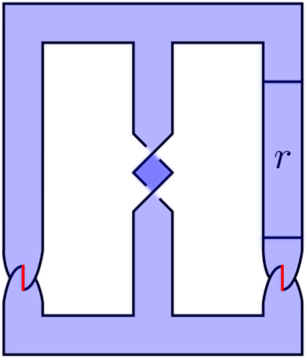
• **For any integer $r(\neq -1, 0)$,**

$K(0, 2, r) \in (\mathcal{K}_2^0 \cap \mathcal{K}_2^1) \setminus \mathcal{K}_1$ **and**

the knots $K(0, 2, r)$ are mutually distinct.



type 0



type 1

By Lemma and Corollary, we obtain the following:

- For any integer q,

$K(-4q-7, 2q+1, -4q+2) \in \mathcal{K}_2^0 \setminus (\mathcal{K}_2^0 \cap \mathcal{K}_2^1)$ and the knots $K(-4q-7, 2q+1, -4q+2)$ are mutually distinct.

$$\nabla(K(-4q-7, 2q+1, -4q+2)) = 1 + 4z^2 + 6z^4.$$

$$\Gamma(K(-4q-7, 2q+1, -4q+2)) = 1+(1-x)(x^{-q-3}-x^{-q+1}).$$

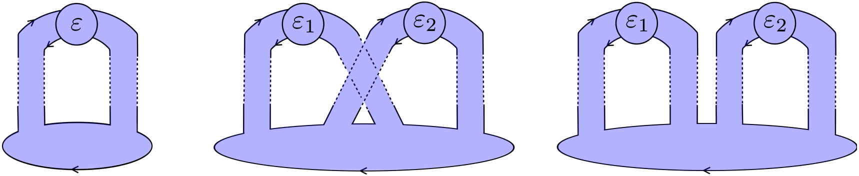
- For any integer q,

$K(4q-3, -4q+4, 4q-3) \in \mathcal{K}_2^1 \setminus (\mathcal{K}_2^0 \cap \mathcal{K}_2^1)$ and the knots $K(4q-3, -4q+4, 4q-3)$ are mutually distinct.

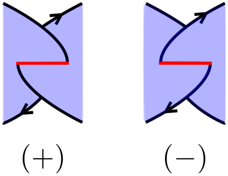
$$\Gamma(K(4q-3, -4q+4, 4q-3)) = x^2+2(1-x)x^2+(1-x)^2x^{2q}.$$

Proof of Theorem 1

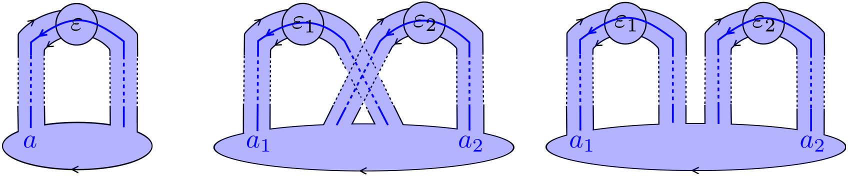
We consider the following three clasp disks:



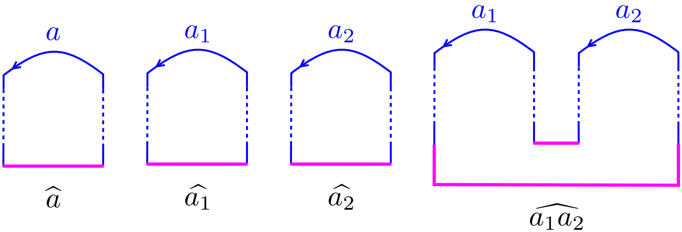
(ε)-, (ε_1)-, (ε_2)-clasp bands are curled bands with no twists and (ε)-, (ε_1)-, (ε_2)-clasp singularities, respectively.



a, a_1, a_2: core arcs of (ε)-, (ε_1)-, (ε_2)-clasp bands.



$\widehat{a}$, $\widehat{a_1}$, $\widehat{a_2}$, $\widehat{a_1 a_2}$: knot diagrams.



By applying the skein relation, we obtain the following:

- $K \in \mathcal{K}_1$.

$$\Gamma(K) = x^{-\varepsilon} + (-x)^{-\frac{\varepsilon+1}{2}}(1-x)x^{w(\widehat{a})}\Gamma(\widehat{a})^2,$$

where $w(\widehat{a})$ is the withe of $\widehat{a}$.

- $K \in \mathcal{K}_2^0$.

$$\begin{aligned}\Gamma(K) = x&^{-\varepsilon_1-\varepsilon_2} \\ &+ (-x)^{-\frac{\varepsilon_1+1}{2}}(1-x)x^{w(\widehat{a_1})-\varepsilon_2}\Gamma(\widehat{a_1})^2 \\ &+ (-x)^{-\frac{\varepsilon_2+1}{2}}(1-x)x^{w(\widehat{a_2})-\varepsilon_1}\Gamma(\widehat{a_2})^2,\end{aligned}$$

where $w(\widehat{a_1})$, $w(\widehat{a_2})$ are the writhes of $\widehat{a_1}$, $\widehat{a_2}$, respectively.

• $K \in \mathcal{K}_2^1$.

$$\begin{aligned}\Gamma(K) = x^{-\varepsilon_1-\varepsilon_2} \\ &+ (-x)^{-\frac{\varepsilon_1+1}{2}}(1-x)x^{w(\widehat{a_1})-\varepsilon_2}\Gamma(\widehat{a_1})^2 \\ &+ (-x)^{-\frac{\varepsilon_2+1}{2}}(1-x)x^{w(\widehat{a_2})-\varepsilon_1}\Gamma(\widehat{a_2})^2 \\ &+ (-x)^{-\frac{\varepsilon_1+\varepsilon_2+2}{2}}(1-x)^2x^{\mathrm{lk}(\widehat{a_1},\widehat{a_2})+w(\widehat{a_1})+w(\widehat{a_2})} \\ &\Gamma(\widehat{a_1})\Gamma(\widehat{a_2})\Gamma(\widehat{a_1a_2}),\end{aligned}$$

where $\mathrm{lk}(\widehat{a_1},\widehat{a_2})$ is the linking number of $\widehat{a_1}$ and $\widehat{a_2}$, $w(\widehat{a_1})$, $w(\widehat{a_2})$ are the writhes of $\widehat{a_1}$, $\widehat{a_2}$, respectively.

By applying Kawauchi's result to $\Gamma(\widehat{a})$, $\Gamma(\widehat{a_1})$, $\Gamma(\widehat{a_2})$, $\Gamma(\widehat{a_1a_2})$, we obtain the desired presentation.

Conversely, we show that for any Laurent polynomial $\varphi(x) \in S_1$, S_2^0, S_2^1 there exists a knot $K \in \mathcal{K}_1$, $\mathcal{K}_2^0$, $\mathcal{K}_2^1$ satisfying $\Gamma(K) = \varphi(x)$, respectively.

If $\varphi(x) \in S_1$, S_2^0, then it is easy to construct the desired knot by Kawauchi's result.

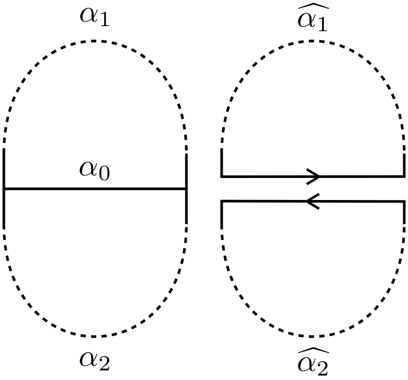
If $\varphi(x) \in S_2^1$, then there exist three knots K_1, K_2, K_3 such that

$$\Gamma(K_1) = 1 + (1-x)^2 f_1(x),$$
$$\Gamma(K_2) = 1 + (1-x)^2 f_2(x),$$
$$\Gamma(K_3) = 1 + (1-x)^2 f_3(x)$$

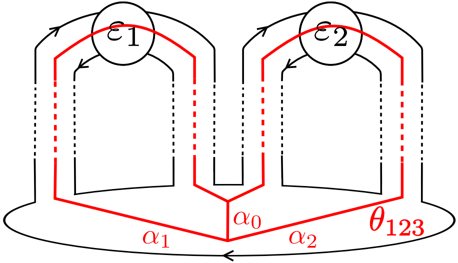
by Kawauchi's result.

Moreover, we apply the following fact to the knots K_1, K_2, K_3.

Fact [Kinoshita 1986] For any knots K', K'', K''' and any integer d, there exists a θ-curve $\alpha_0 \cup \alpha_1 \cup \alpha_2$ in S^3 with $\partial\alpha_0 = \partial\alpha_1 = \partial\alpha_2$ such that $\mathrm{lk}(\widehat{\alpha_1}, \widehat{\alpha_2}) = d$ and $\alpha_0 \cup \alpha_1$, $\alpha_0 \cup \alpha_2$, $\alpha_1 \cup \alpha_2$ are isotopic to K', K'', K''', respectively, where $\mathrm{lk}(\widehat{\alpha_1}, \widehat{\alpha_2})$ is the linking number of knots $\widehat{\alpha_1}$ and $\widehat{\alpha_2}$.



Therefore, for any integer d, there exists a θ-curve $\theta_{123} = \alpha_0 \cup \alpha_1 \cup \alpha_2$ with $\partial\alpha_0 = \partial\alpha_1 = \partial\alpha_2$ such that $\mathrm{lk}(\widehat{\alpha_1}, \widehat{\alpha_2}) = d$ and $\alpha_0 \cup \alpha_1$, $\alpha_0 \cup \alpha_2$, $\alpha_1 \cup \alpha_2$ are isotopic to K_1, K_2, K_3, respectively. By using θ_{123}, we can construct the desired knot.



Thank you.