Cabling formulae of quandle cocycle invariants for surface knots

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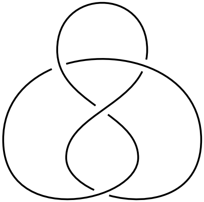
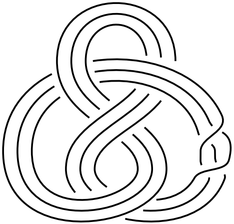


cabling $\longmapsto$



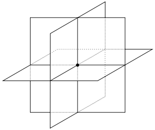
K $\qquad\qquad$ $K^{(m,n)}$

$$\text{inv. of } K^{(m,n)} \overset{\uparrow}{=} \text{inv. of } K$$

cabling formula

- (paral.) Jones polyn. (J. Murakami; 1989)
- Kontsevich inv.
  (D. Bar-Natan, T.T.Q. Le, D.P. Thurston; 2003)

Quandle cocycle invariants (CJKLS; 2003):



$$\text{3-cocycle} \\ \downarrow \\ \rightsquigarrow \quad \text{weight}$$

<u>Aim</u> QCI of $F^{(m,\nu)}$ $\;=\;$ inv. of F

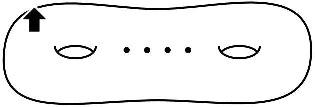
$\uparrow$ cabling formula

T. Naruse (2015): QCI of paral. $=$ RCI

$=$ QCI ?

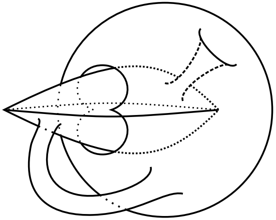
Contents

A **surface knot** F is the image of

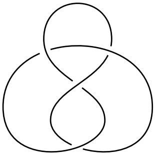


$$\xrightarrow[\text{smooth}]{} \mathbb{R}^4$$

$$\begin{array}{ccc} F & \subset & \mathbb{R}^4 \\ \downarrow & & \downarrow \end{array}$$



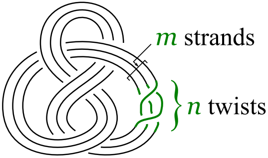
$\subset \quad \mathbb{R}^3$: a **diagram** of F



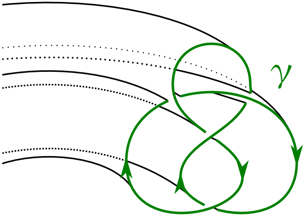
K

(m,n)-cabling

$\longmapsto$



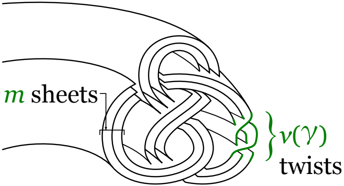
$K^{(m,n)}$



F

(m,ν)-cabling

$\longmapsto$

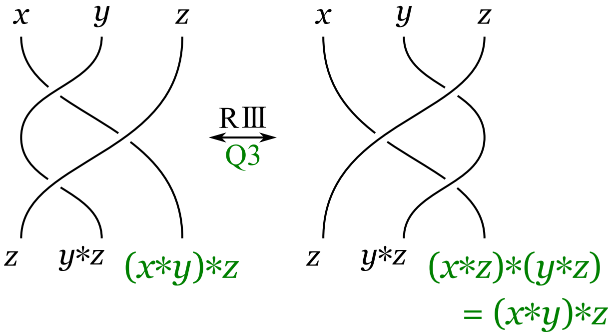


$F^{(m,\nu)}$

$(\nu \in H^1(F;\mathbb{Z}))$

$X = (X, *)$ is a **quandle**

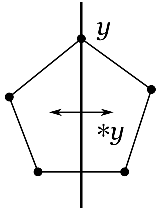
$$\overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{(Q1) } x * x = x, \\ \text{(Q2) the map } X \ni a \mapsto a * x \in X \text{ is a bijection,} \\ \text{(Q3) } (x * y) * z = (x * z) * (y * z), \\ \qquad\qquad\qquad\qquad \text{for } {}^{\forall} x, y, z \in X. \end{cases}$$



Ex.

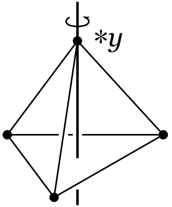
- **dihedral quandle** $R_k = \mathbb{Z}/k\mathbb{Z}$.

$$x * y = 2y - x.$$



- **tetrahedral quandle**
  $Q_4 = \mathbb{Z}[T]/(2, T^2 + T + 1)$.

$$x * y = Tx + (1 - T)y.$$



- **Alexander quandle**
  X : a $\mathbb{Z}[T^{\pm}]$-module

$$x * y = Tx + (1 - T)y.$$

$\phi : X \times X \to \underset{\text{ab.grp.}}{\underline{A}}$ is a **quandle 2-cocycle**

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{l} \phi(x,x) = 0. \\ \phi(x,y) - \phi(x,z) - \phi(x*y,z) + \phi(x*z,y*z) = 0. \end{array}$$

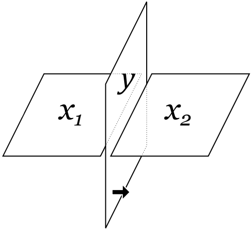
$\psi : X \times X \times X \to A$ is a **quandle 3-cocycle**

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{l} \psi(x,x,y) = \psi(x,y,y) = 0. \\ \psi(w,y,z) - \psi(w,x,z) + \psi(w,x,y) \\ -\psi(w*x,y,z) + \psi(w*y,x*y,z) - \psi(w*z,x*z,y*z) = 0. \end{array}$$

Ex. $\cdot H^2_Q(R_p;\mathbb{Z}/p\mathbb{Z}) = 0$
$H^3_Q(R_p;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ (Mochizuki 3-cocycle: a gen.)

$\cdot H^2_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$
$H^3_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$

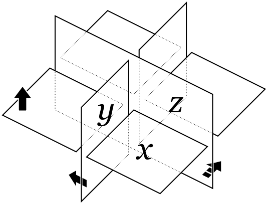
X-**coloring** of a diagram D: $\{\text{sheets}\} \to X$ such that



$$x_1 * y = x_2$$

quandle cocycle invariant:

ψ : a quandle 3-cocycle of $\underset{\text{fin.qdle.}}{\underline{X}}$



$$\Psi_\psi(F) = \sum_{\text{col.}} \prod_{\text{tri. pt.}} \psi(x,y,z)^{\pm} \in \mathbb{Z}[A].$$

Cabling formulae for R_p

$H^3_Q(R_p;\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$

Theorem A

p : an odd prime ψ : a 3-cocycle of R_p

$\Rightarrow$

$$\Psi_\psi(F^{(m,\nu)}) = \begin{cases} \Psi_{m\psi}(F) & \text{if } m, n \notin 2\mathbb{Z}, \\ p^{(m,n)-1}\left(\dfrac{m}{(m,n)}, p\right)\Psi_{m\psi}(F) & \text{if } m \notin 2\mathbb{Z},\ n \in 2\mathbb{Z}, \\ p^{(m,n)}\left(\dfrac{n}{(m,n)}, p\right) & \text{if } m \in 2\mathbb{Z},\ n \notin 2\mathbb{Z}, \\ p^{(m,n)-1}\left(\dfrac{mn}{(m,n)^2}, p\right) & \text{if } m, n \in 2\mathbb{Z}, \end{cases}$$

where $n \geq 0$ is the integer s.t. $n\mathbb{Z} = \nu(H_1(F)) \subset \mathbb{Z}$.

Cabling formulae for Q_4

$$H^2_Q(Q_4; \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$
$$H^3_Q(Q_4; \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$

Theorem B(i)

ψ : a quandle 3-cocycle of Q_4 of order 2
$\Rightarrow$

$$\Psi_\psi(F^{(m,\nu)}) = \begin{cases} \Psi_{m\psi}(F) & \text{if } m, n \notin 3\mathbb{Z}, \\ 4^{(m,n)-1}\left(\dfrac{m}{(m,n)}, 2\right)^2 \Psi_{m\psi}(F) & \text{if } m \notin 3\mathbb{Z},\ n \in 3\mathbb{Z}, \\ 4^{(m,n)}\left(\dfrac{n}{(m,n)}, 2\right)^2 & \text{if } m \in 3\mathbb{Z},\ n \notin 3\mathbb{Z}, \\ 4^{(m,n)-1}\left(\dfrac{mn}{(m,n)^2}, 2\right)^2 & \text{if } m, n \in 3\mathbb{Z}, \end{cases}$$

where $n \geq 0$ is the integer s.t. $n\mathbb{Z} = \nu(H_1(F)) \subset \mathbb{Z}$.

Cabling formulae for Q_4

$H^2_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$
$H^3_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$

Theorem B(ii)

$\psi \in H^3_Q(Q_4;\mathbb{Z}/4\mathbb{Z})$ of order 4 $\quad 0 \neq \phi \in H^2_Q(Q_4;\mathbb{Z}/4\mathbb{Z}) \quad m \notin 3\mathbb{Z}$
$\Rightarrow$

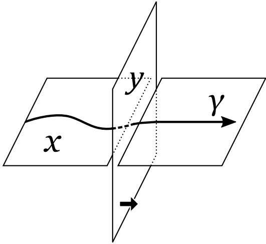
$$\Psi_\psi(F^{(m,\nu)}) = \begin{cases} \Psi_{m\psi}(F) & \text{if } n \notin 3\mathbb{Z}, \\ (2^{2(m,n)-3} + 2^{(m,n)-2})\Psi_{m\psi}(F) & \text{if } m \notin 2\mathbb{Z}, \\ \quad +(2^{2(m,n)-3} - 2^{(m,n)-2})((\nu \cup \Phi_\phi)\cdot\Psi_{m\psi})(F) & \quad n \in 3+6\mathbb{Z}, \\ 4^{(m,n)-1}\Psi_{2\psi}(F) & \text{if } m \in 2+4\mathbb{Z}, \\ \quad +3\cdot 4^{(m,n)-1}((\nu \cup \Phi_\phi)\cdot\Psi_{2\psi})(F) & \quad n \in 3+6\mathbb{Z}, \\ 4^{(m,n)-1}\left(\frac{m}{(m,n)}, 2\right)^2 \Psi_{m\psi}(F) & \text{otherwise}, \end{cases}$$

where $n \geq 0$ is the integer s.t. $n\mathbb{Z} = \nu(H_1(F)) \subset \mathbb{Z}$.

X : a finite quandle ϕ : a quandle 2-cocycle on X
F : a surface knot D : a diagram of F

$\mathcal{C}$: an X-coloring on D

$$\begin{array}{cccc} & F & & A \\ & \cup & & \Cup \\ \text{a loop} & \gamma & \overset{}{\underset{\Phi_\phi(D,\mathcal{C})}{\longmapsto}} & \prod \phi(x,y)^{\pm} \end{array}$$

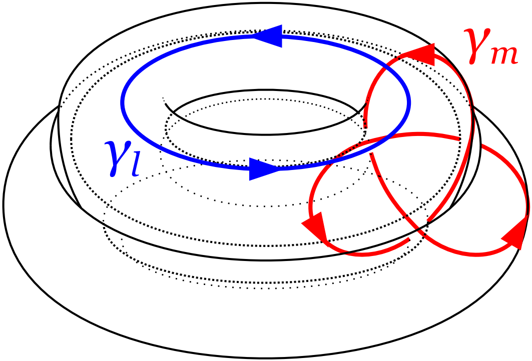


$$\rightsquigarrow \Phi_\phi(D,\mathcal{C}) \in Hom(H_1(F), A) \cong H^1(F;A)$$

The **quandle 2-cocycle invariant**:

$$\Phi_\phi(F) := \sum_{\mathcal{C}:\text{col.}} \Phi_\phi(D,\mathcal{C}) \in \mathbb{Z}[H^1(F;A)]$$

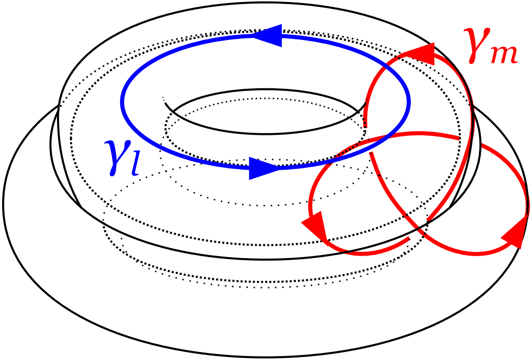
<u>Ex.</u> $F :=$ a spun T^2-knot of 3_1:



$$X = Q_4, \quad 0 \neq \phi \in H^2_Q(Q_4; \mathbb{Z}/4\mathbb{Z})$$

$$\Phi_\phi(F)(\gamma_m, \gamma_l) = 4(0,0) + 12(2,0).$$

Ex.

$F =$ 

$X = Q_4 \quad \psi \in H^3_Q(Q_4; \mathbb{Z}/4\mathbb{Z})$

$(\Psi_\psi(3_1) = 16 + 48t \in \mathbb{Z}[\mathbb{Z}/4\mathbb{Z}] = \mathbb{Z}[t]/(t^4 - 1))$

$$\Psi_\psi(F^{(2,3\gamma_m^*)}) = 28 + 36t^2.$$

$$\Psi_\psi(F^{(2,3\gamma_l^*)}) = 64.$$

Cabling formulae

Theorem C

X: an Alexander quandle $\quad \psi$: a 3-cocycle on X

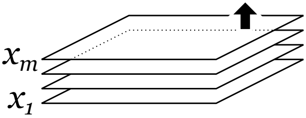
$\Rightarrow$

$$\exists \left\{ \begin{array}{rl} X_m : & \text{a conn. Alex. quandle} \\ \phi_{m,i} : & \text{a 2-cocycle on } X_m \\ \psi_{m,j} : & \text{a 3-cocycle on } X_m \end{array} \right\} \text{ such that}$$

$$\begin{aligned} \Psi_\psi(F^{(m,\nu)}) &= \sum_{i,j} (\text{const.})(((\nu/n) \cup \Phi_{\phi_{m,i}}) \cdot \Psi_{\psi_{m,j}})(F) \\ &= \sum_{i,j} (\text{const.}) \sum_{\mathcal{C}:\, X_m-\text{col.}} ((\nu/n) \cup \Phi_{\phi_{m,i}}(D,\mathcal{C})) \cdot \Psi_{\psi_{m,j}}(D,\mathcal{C}), \end{aligned}$$

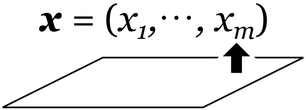
where $n \geq 0$ is the integer s.t. $n\mathbb{Z} = \nu(H_1(F)) \subset \mathbb{Z}$.

Idea of proof

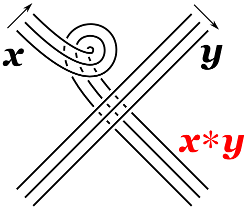


X-coloring on $D^{(m,\nu)}$

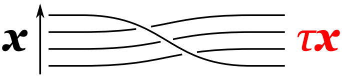
$\longleftrightarrow$



$\underline{X^m}$-coloring on D

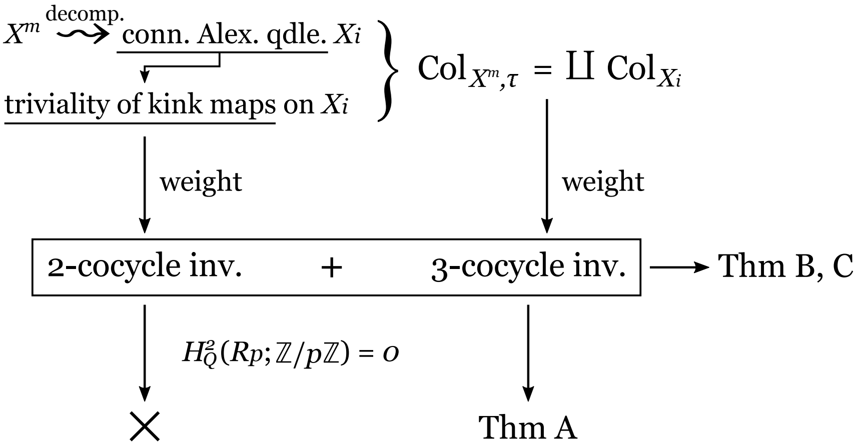


$\rightsquigarrow$ X^m is a quandle.

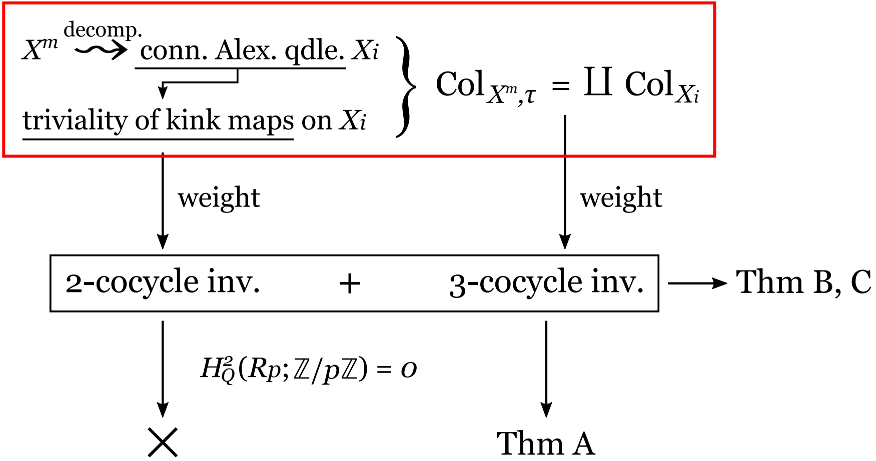


$\rightsquigarrow$ τ is a “kink map”.

Outline of proof



Outline of proof

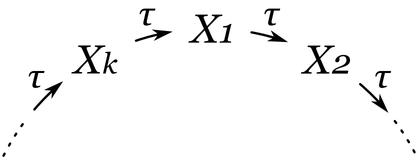


Sketch proof

Decomposition of X^m:

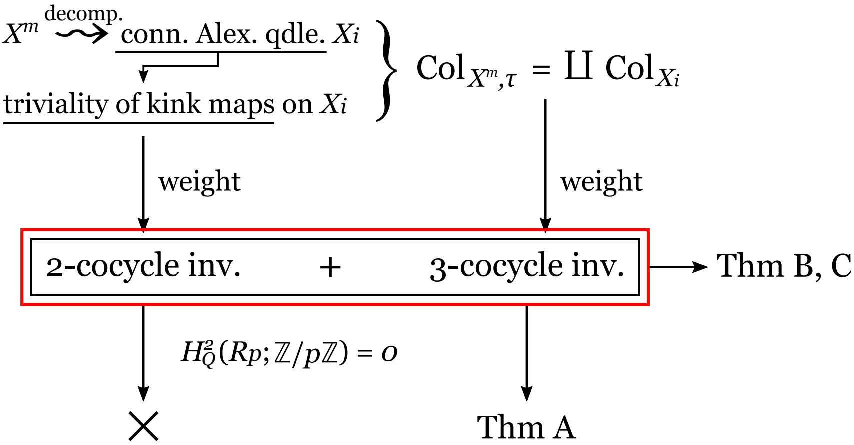
$$X^m = \coprod_i X_i \qquad X_i \cong X_j \left(\cong \begin{cases} R_p & m : \text{odd} \\ \text{triv.qdle.} & m : \text{even} \end{cases} \right)$$

Action of τ:



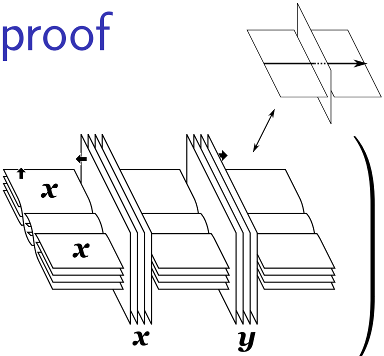
- $\tau^n(X_i) \neq X_j$
  $\Rightarrow \tau^n(X_i) \cap X_j = \phi.$
- $\tau^n(X_i) = X_i$
  $\Rightarrow \tau^n|_{X_i} = id_{X_i}$

Outline of proof



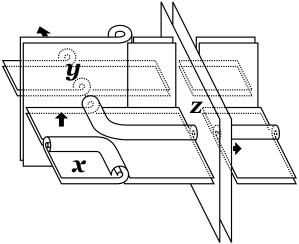
Sketch proof

Ψ_ψ (



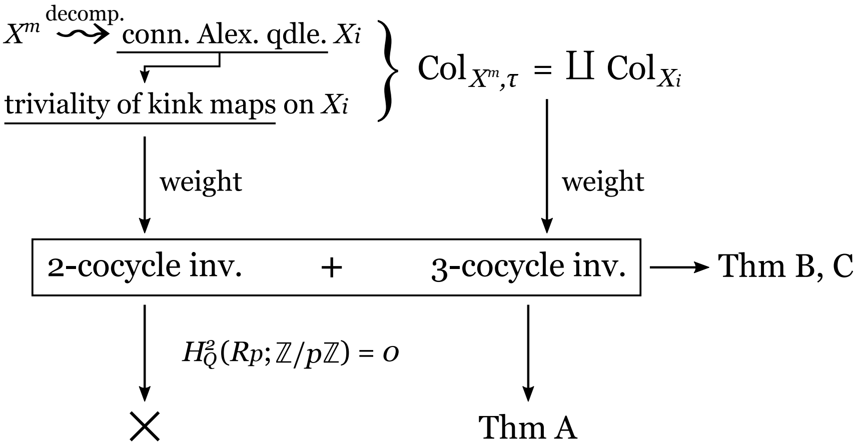
) : a 2-cocycle ϕ_i on X_i

Ψ_ψ (



) : a 3-cocycle ψ_i on X_i

Outline of proof



Future problems

Problems

- Decompose X^m for general X.
- What are QCI in the case of non-trivial kink maps?
  - can 2- and 3-cocycle invariants describe them?
  - or, does another invariant appear?
- Find cabling formulae of shadow cocycle invariants.