The products of Alexander invariants and quandle cocycle invariants

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joint work with

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Recipe

L: a link

$\to G(L) := \pi_1(S^3 - L) \cong \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$

$\to A := \left((\alpha \circ \pi) \left(\frac{\partial r_i}{\partial x_j} \right) \right)$: the *Alexander matrix* of $G(L)$

$\to E_d(A) = (a_1, \ldots, a_l)$: the *$d$th elementary ideal* of A

$\to \Delta_\alpha^{(d)}(L) := \gcd(a_1, \ldots, a_l)$ the *dth Alexander invariant*

Def

- $\underset{\text{a free group}}{F(x_1, \ldots, x_n)} \overset{\pi}{\twoheadrightarrow} \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle \overset{\alpha}{\twoheadrightarrow} \langle t \rangle$

  (meridian) $\mapsto t$

- $\frac{\partial}{\partial x_i} : \mathbb{Z}[F(x_1, \ldots, x_n)] \to \mathbb{Z}[F(x_1, \ldots, x_n)]$: Fox derivative

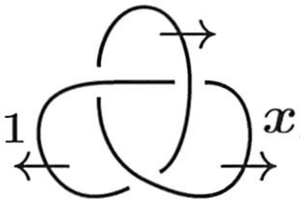
  $$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \qquad \frac{\partial 1}{\partial x_j} = 0, \qquad \frac{\partial (gh)}{\partial x_j} = \frac{\partial g}{\partial x_j} + g \frac{\partial h}{\partial x_j}$$

- $$E_d(A) := \begin{cases} 0 & (d < n - m) \\ I(\{(n-d)\text{-minors of } A\}) & (n - m \le d < n) \\ R & (n \le d) \end{cases}$$

  where a k-minor is $\det B$ (B: a $k \times k$ submatrix of A)

Example

K: trefoil



$$G(K) \cong \langle x_1, x_2, x_3 \mid x_2^{-1}x_1x_2x_3^{-1}, x_1^{-1}x_3x_1x_2^{-1}, x_3^{-1}x_2x_3x_1^{-1} \rangle$$

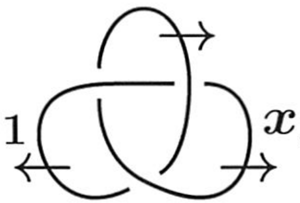
$$\begin{array}{lllllll}
x_2^{-1}x_1x_2x_3^{-1} & \xrightarrow{\frac{\partial}{\partial x_1}} & x_2^{-1} & \xrightarrow{\pi} & x_2^{-1} & \xrightarrow{\alpha} & t^{-1} \\
 & \xrightarrow{\frac{\partial}{\partial x_2}} & -x_2^{-1} + x_2^{-1}x_1 & \xrightarrow{\pi} & -x_2^{-1} + x_2^{-1}x_1 & \xrightarrow{\alpha} & -t^{-1} + 1 \\
 & \xrightarrow{\frac{\partial}{\partial x_3}} & -x_2^{-1}x_1x_2x_3^{-1} & \xrightarrow{\pi} & -1 & \xrightarrow{\alpha} & -1
\end{array}$$

$$\left((\alpha \circ \pi)\left(\frac{\partial r_i}{\partial x_j}\right)\right) = \begin{pmatrix} t^{-1} & -t^{-1}+1 & -1 \\ -t^{-1}+1 & -1 & t^{-1} \\ -1 & t^{-1} & -t^{-1}+1 \end{pmatrix}$$

- $E_0(A) = (\det A) = (0) \Rightarrow \Delta^{(0)}(K) \doteq 0$
- $E_1(A) = (\pm(-t^{-2} + t^{-1} - 1)) \Rightarrow \Delta^{(1)}(K) \doteq t - 1 + t^{-1}$
- $E_2(A) = (t^{-1}, -t^{-1} + 1, -1) = (1) \Rightarrow \Delta^{(2)}(K) \doteq 1$

Example

K: trefoil



$$G(K) \cong \langle x_1, x_2, x_3 \mid x_2^{-1}x_1x_2x_3^{-1}, x_1^{-1}x_3x_1x_2^{-1}, x_3^{-1}x_2x_3x_1^{-1} \rangle$$

$$\begin{array}{llllll} x_2^{-1}x_1x_2x_3^{-1} & \xrightarrow{\frac{\partial}{\partial x_1}} x_2^{-1} & \overset{\pi}{\to} & x_2^{-1} & \overset{\alpha}{\to} & t^{-1} \\ & \xrightarrow{\frac{\partial}{\partial x_2}} -x_2^{-1} + x_2^{-1}x_1 & \overset{\pi}{\to} & -x_2^{-1} + x_2^{-1}x_1 & \overset{\alpha}{\to} & -t^{-1} + 1 \\ & \xrightarrow{\frac{\partial}{\partial x_3}} -x_2^{-1}x_1x_2x_3^{-1} & \overset{\pi}{\to} & -1 & \overset{\alpha}{\to} & -1 \end{array}$$

$$\left((\alpha \circ \pi)\left(\frac{\partial r_i}{\partial x_j}\right)\right) = \begin{pmatrix} t^{-1} & -t^{-1}+1 & -1 & 0 \\ -t^{-1}+1 & -1 & t^{-1} & 0 \\ -1 & t^{-1} & -t^{-1}+1 & 0 \end{pmatrix}$$

- $E_0(A) = (\det A) = (0) \Rightarrow \Delta^{(0)}(K) \doteq 0$
- $E_1(A) = (\pm(-t^{-2} + t^{-1} - 1)) \Rightarrow \Delta^{(1)}(K) \doteq t - 1 + t^{-1}$
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Def [Joyce], [Matveev]

A *quandle* Q is a nonempty set with $*: Q \times Q \to Q$ satisfying

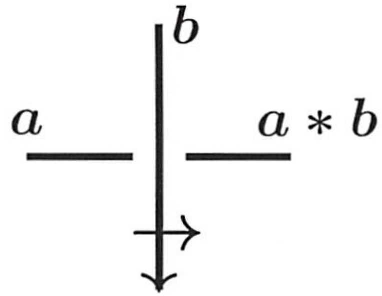
- $a * a = a$
- $*a : Q \to Q; x \mapsto x * a$ is bijective
- $(a * b) * c = (a * c) * (b * c)$

Def

D: a diagram of an oriented link L

$\mathcal{A}(D) := \{$arcs of $D\}$

$C : \mathcal{A}(D) \to Q$ is a *Q-coloring* of D, if



Prop

$\mathrm{Col}_Q(D) := \{Q$-colorings of $D\}$

$D_1 \overset{\text{R1-3}}{\longleftrightarrow} D_2 \Rightarrow \mathrm{Col}_Q(D_1) \overset{1:1}{\longleftrightarrow} \mathrm{Col}_Q(D_2)$

Def [Andruskiewitsch–Graña]

$(Q, \lhd)$: a quandle. X: a set, $*_a^b : X \times X \to X$ $(a, b \in Q)$.

$\phi = (*_a^b)_{a,b \in Q}$ is a *dynamical cocycle*, if

- $x *_a^a x = x$
- $*_a^b y : X \to X; x \mapsto x *_a^b y$ is bijective
- $(x *_a^b y) *_{a \lhd b}^c z = (x *_a^c z) *_{a \lhd c}^{b \lhd c} (y *_b^c z)$

We call $(X, (*_a^b)_{a,b \in Q})$ a *Q-twisted quandle*.

Prop

$(Q, \lhd)$: a quandle. X: a set, $\phi = (*_a^b : X \times X \to X)_{a,b \in Q}$.

$(a, x) *_\phi (b, y) := (a \lhd b, x *_a^b y)$, $Q \times_\phi X := (Q \times X, *_\phi)$.

Then, ϕ is a dynamical cocycle $\Leftrightarrow$ $Q \times_\phi X$ is a quandle

Def

We call $Q \times_\phi X$ an *extension* of Q, or the *associated quandle* of $(X, (*_a^b)_{a,b \in Q})$.

Def

$(Q, \triangleleft)$: a quandle, R: a ring, M: an R-module.

$f_1, f_2 : Q \times Q \to R$, $\phi : Q \times Q \to M$.

(f_1, f_2) is an *Alexander pair*, if (1)–(5) hold.

(f_1, f_2, ϕ) is an *Alexander triple*, if (1)–(7) hold.

Def (c.f. [Carter–Elhamdadi–Graña–Saito])

We call ϕ a *(f_1, f_2)-cocycle* or a *generalized quandle 2-cocycle*.

Prop

∘ $(*_a^b)_{a,b \in Q}$: a dynamical cocycle ⇒ $*$: a quandle operation

$$(a, x) * (b, y) = (a \triangleleft b, x *_a^b y)$$

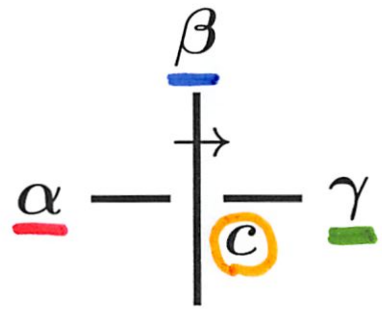
- (f_1, f_2): an Alexander pair ⇒ $*$: a quandle operation

$$(a, x) * (b, y) = (a \triangleleft b, f_1(a, b)x + f_2(a, b)y)$$

- (f_1, f_2, ϕ): an Alexander triple ⇒ $*$: a quandle operation

$$(a, x) * (b, y) = (a \triangleleft b, f_1(a, b)x + f_2(a, b)y + \phi(a, b))$$

L: an oriented link, D: Lの diagram. $C \in \mathrm{Col}_Q(D)$.



The (f_1, f_2)-Alexander matrix $A(D; f_1, f_2)$

$$
\begin{array}{c|ccccccc}
C(D) \diagdown \mathcal{A}(D) & & \alpha & & \beta & & \gamma & \\
\hline
 & * & * & * & * & * & * & * \\
c & 0 & f_1(\alpha, \beta) & 0 & f_2(\alpha, \beta) & 0 & -1 & 0 \\
 & * & * & * & * & * & * & *
\end{array}
$$

The ϕ-augmented (f_1, f_2)-Alexander matrix $A(D; f_1, f_2, \phi)$

$$
\begin{array}{c|ccccccc|c}
C(D) \diagdown \mathcal{A}(D) & & \alpha & & \beta & & \gamma & & \\
\hline
 & * & * & * & * & * & * & * & * \\
c & 0 & f_1(\alpha, \beta) & 0 & f_2(\alpha, \beta) & 0 & -1 & 0 & \phi(\alpha, \beta) \\
 & * & * & * & * & * & * & * & *
\end{array}
$$

where $f_i(\alpha, \beta) := f_i(C(\alpha), C(\beta))$, $\phi(\alpha, \beta) := \phi(C(\alpha), C(\beta))$.

Thm [I–Oshiro]

$$D \overset{\text{R1–3}}{\leftrightarrow} D' \Rightarrow A(D; f_1, f_2) \sim A(D'; f_1, f_2)$$
$$A(D; f_1, f_2, \phi) \sim A(D'; f_1, f_2, \phi)$$

Recipe

L: an oriented link, $\rho : Q(L) \to Q$

$\to$ D: a diagram of L, $C_\rho \in \mathrm{Col}_Q(D)$

$\to$ $A := A(D, f_1, f_2)$ or $A(D; f_1, f_2, \phi)$

$\to$ $E_d(A) = (a_1, \ldots, a_l)$: the dth elementary ideal of A

$\to$ $\Delta_\rho^{(d)}(L) := \gcd(a_1, \ldots, a_l)$ the dth Alexander invariant

- $f_1(a,a) + f_2(a,a) = 1$ • $f_1(a,b)$ is invertible
- $f_1(a \triangleleft b, c) f_1(a,b) = f_1(a \triangleleft c, b \triangleleft c) f_1(a,c)$
- $f_1(a \triangleleft b, c) f_2(a,b) = f_2(a \triangleleft c, b \triangleleft c) f_1(b,c)$
- $f_2(a \triangleleft b, c) = f_1(a \triangleleft c, b \triangleleft c) f_2(a,c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b,c)$

Def

We call $(1,0)$ the *trivial Alexander pair*.

Example

The following are Alexander pairs.

- $(Q, \triangleleft)$: a quandle, $R := \mathbb{Z}[t^{\pm 1}]$.
  - $f_1(a,b) = t$, $f_2(a,b) = 1 - t$
- $(Q, \triangleleft) := \mathrm{Conj}_n G$ $(a \triangleleft b = b^{-n} a b^n)$, $R := (\mathbb{Z}[t^{\pm 1}])[G]$.
  - $f_1(a,b) = b^{-n} a^n$, $f_2(a,b) = 0$
  - $f_1(a,b) = t b^{-n}$, $f_2(a,b) = 1 - t b^{-n}$
  - $f_1(a,b) = t b^{-n}$, $f_2(a,b) = b^{-n} a^n - t b^{-n}$

Prop

The (f_1, f_2)-Alexander matrix is

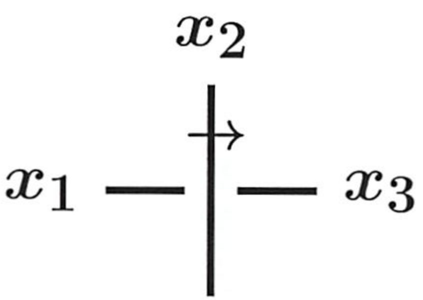
- the Alexander matrix

  if Q is a quandle, $(f_1, f_2) = (t^{-1}, 1 - t^{-1})$.

- the twisted Alexander matrix,

  if $Q = \mathrm{Conj}\, GL(n; F)$, $(f_1, f_2) = (t^{-1}b^{-1}, b^{-1}a - t^{-1}b^{-1})$

Proof



$$\begin{array}{llllll}
x_2^{-1}x_1x_2x_3^{-1} & \xrightarrow{\frac{\partial}{\partial x_1}} x_2^{-1} & \xrightarrow{\pi} & x_2^{-1} & \xrightarrow{\alpha} & t^{-1} \\
 & \xrightarrow{\frac{\partial}{\partial x_2}} -x_2^{-1} + x_2^{-1}x_1 & \xrightarrow{\pi} & -x_2^{-1} + x_2^{-1}x_1 & \xrightarrow{\alpha} & -t^{-1} + 1 \\
 & \xrightarrow{\frac{\partial}{\partial x_3}} -x_2^{-1}x_1x_2x_3^{-1} & \xrightarrow{\pi} & -1 & \xrightarrow{\alpha} & -1 \\
 & \xrightarrow{\frac{\partial}{\partial x_j}} 0 & \xrightarrow{\pi} & 0 & \xrightarrow{\alpha} & 0
\end{array}$$

- $\phi(a,a)=0$
- $\phi(a \lhd b, c) + f_1(a \lhd b, c)\phi(a,b)$
$= \phi(a \lhd c, b \lhd c) + f_1(a \lhd c, b \lhd c)\phi(a,c) + f_2(a \lhd c, b \lhd c)\phi(b,c)$

Remark

- (f_1, f_2): an Alexander pair $\Leftrightarrow$ $(f_1, f_2, 0)$: an Alexander triple
- ϕ: a quandle 2-cocycle $\Leftrightarrow$ $(1, 0, \phi)$: an Alexander triple

Example

$Q = R_3$, $R = M = \mathbb{Z}_2$.

- $f_1(a,b)=1$, $f_2(a,b)=0$, $\phi(a,b)=0$. Then $Q \times M \cong R_6$.
- $f_1(a,b)=1$, $f_2(a,b)=\begin{cases} 0 & (a=b) \\ 1 & (a \neq b) \end{cases}$, $\phi(a,b)=0$.
Then $Q \times M \cong QS6_1$.
- $f_1(a,b)=1$, $f_2(a,b)=\begin{cases} 0 & (a=b) \\ 1 & (a \neq b) \end{cases}$, $\phi(a,b)=\begin{cases} 1 & (b=a+1) \\ 0 & (b \neq a+1) \end{cases}$.
Then $Q \times M \cong QS6_2$.

Prop

$(f_1, f_2) := (1, 0)$, ϕ: a quandle 2-cocycle.

K: an oriented knot, $\rho : Q(K) \to Q$.

$\Phi(K; C_\rho)$: the quandle cocycle invariant of (K, ρ).

Then $\Delta_\rho^{(0)}(K) \doteq \Phi(K; C_\rho)$.

Proof

$$
\begin{array}{c}
\\
c \\
\\
\\
\end{array}
\left(\begin{array}{ccccccc|c}
 & \alpha & & \beta & & \gamma & & \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & \phi(C_\rho(\alpha), C_\rho(\beta)) \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{array}\right)
$$

$$
\sim
\begin{array}{c} c \\ \\ \\ \end{array}
\left(\begin{array}{ccccccc|c}
0 & \mathrm{sign}(c) & 0 & 0 & 0 & -\mathrm{sign}(c) & 0 & \mathrm{sign}(c)\phi(C_\rho(\alpha), C_\rho(\beta)) \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{array}\right)
$$

$$
\sim
\begin{array}{c} c \\ \\ \\ \end{array}
\left(\begin{array}{ccccccc|c}
0 & \mathrm{sign}(c) & 0 & 0 & 0 & -\mathrm{sign}(c) & 0 & \mathrm{sign}(c)\phi(C_\rho(\alpha), C_\rho(\beta)) \\
* & * & * & * & * & * & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Phi(K; C_\rho)
\end{array}\right)
$$