Symmetries of Graphs in Homology Spheres

Song Yu
Advisor: Erica Flapan

Department of Mathematics, Pomona College

August 4, 2016

Notions of symmetry

Let M be an orientable 3-manifold and G be a graph.

Definition

An automorphism σ of G is said to be realizable in M if there is an embedding Γ of G into M and a homeomorphism $h : (M, \Gamma) \to (M, \Gamma)$ that induces σ on Γ. In this case, we say that h realizes σ in M.

Definition

We say that G is intrinsically chiral in M if for every embedding Γ of G in M, there is no orientation reversing homeomorphism of M that leaves Γ setwise invariant. An embedding Γ of G in M is achiral if there is an orientation reversing homeomorphism of M that setwise fixes Γ.

Motivating questions

How special is $\mathbb{S}^3$ among orientable 3-manifolds?

Question

Is it true that an automorphism of a graph is realizable in $\mathbb{S}^3$ if and only if it is realizable in every orientable 3-manifold?

Question

Is it true that a graph is intrinsically chiral in $\mathbb{S}^3$ if and only if it is intrinsically chiral in every orientable 3-manifold?

The easy direction

Proposition (Y.)

An automorphism of a graph that is realizable in $\mathbb{S}^3$ by an orientation preserving homeomorphism is realizable in every orientable 3-manifold. An automorphism of a graph that is realizable in $\mathbb{S}^3$ by an orientation reversing homeomorphism is realizable in every orientable 3-manifold that possesses an orientation reversing homeomorphism.

Corollary (Y.)

If a graph G is intrinsically chiral in an orientable 3-manifold M that possesses an orientation reversing homeomorphism, then G is also intrinsically chiral in $\mathbb{S}^3$.

The converse

The converses are not true!

Proposition (Y.)

For any automorphism σ of a graph G, there exists an orientable 3-manifold M, an embedding Γ of G in M, and an orientation preserving homeomorphism h of (M, Γ) such that h realizes σ.

Theorem (Flapan-Howards, 2015)

For any graph G, there are infinitely many orientable and irreducible 3-manifolds M such that some embedding of G is pointwise fixed by an orientation reversing involution of M.

No graph is intrinsically chiral in every 3-manifold.

Central objects: homology spheres

Need to restrict our attention
Natural generalization of $\mathbb{S}^3$:

Homology sphere

An integral homology 3-sphere (abbreviated as a homology sphere) is a 3-manifold whose homology groups with $\mathbb{Z}$ coefficients are the same as those of $\mathbb{S}^3$.

There are homology spheres which have no orientation reversing homeomorphisms, such as Poincaré's dodecahedron space.

Rigidity of symmetries in homology spheres

Rigidity Theorem (Flapan, 1995)

Let G be a 3-connected graph. Suppose σ is an automorphism of G that is realized in $\mathbb{S}^3$ by a homeomorphism h. Then σ is realizable in $\mathbb{S}^3$ by a homeomorphism f of finite order. Moreover, f can be chosen such that f is orientation reversing if and only if h is orientation reversing.

Rigidity Theorem (Y.)

Let G be a 3-connected graph and M be a homology sphere. Suppose σ is an automorphism of G that is realized in M by a homeomorphism h. Then σ is realizable in a homology sphere M' by a homeomorphism f of finite order. Moreover, f can be chosen such that f is orientation reversing if and only if h is orientation reversing.

Smith Theory

fix(h) - the fixed point set of a map h

Theorem (Smith, 1939)

Let M be a homology sphere and $h : M \to M$ be a homeomorphism of finite order. If h is orientation preserving, then fix(h) is homeomorphic to one of the following: M (in this case h is the identity map), $\mathbb{S}^1$, $\emptyset$; if h is orientation reversing, then fix(h) is homeomorphic to one of the following: $\mathbb{S}^2$, $\mathbb{S}^0$ (two points).

Realizable automorphisms of complete graphs

Proposition (Y.)

If an automorphism σ of the complete graph K_n is realizable in a homology sphere M by a homeomorphism h if and only if σ is also realizable in $\mathbb{S}^3$ by a homeomorphism g. Moreover, g can be chosen so that g is orientation reversing if and only if h is orientation reversing.

Intrinsic chirality

Recall that a graph G is intrinsic chiral in an orientable 3-manifold M if no embedding of G is setwise fixed by an orientation reversing homeomorphism of M.

Observation

Let M be an orientable 3-manifold that does not possess an orientation reversing homeomorphism. Then every graph is intrinsically chiral in M.

Chirality and planarity

Observation (Y.)

In an orientable 3-manifold that possesses an orientation reversing homeomorphism, planar graphs are achiral.

Proposition (Y.)

Any non-planar graph that has no order two automorphism is intrinsically chiral in any homology sphere.

With a slightly stronger requirement:

Proposition (Y.)

Let P be a connected simplicial complex embedded in a homology sphere M. If there is an orientation reversing homeomorphism h of M such that $P \subseteq \text{fix}(h)$, then P can be embedded into $\mathbb{S}^2$.

First established for $\mathbb{S}^3$ by Jiang and Wang in 2000

Intrinsic chirality of 3-connected graphs

Flapan-Weaver, 1996: Intrinsic chirality is related to not possessing certain types of automorphisms

Proposition (Y.)

An automorphism of a 3-connected graph is realizable in $\mathbb{S}^3$ by an orientation reversing homeomorphism if and only if it is realizable in every homology sphere that possesses an orientation reversing homeomorphism by an orientation reversing homeomorphism.

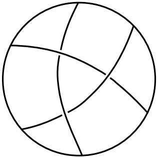
Example

Let M be a homology sphere that possesses an orientation reversing homeomorphism. Then the complete graph K_n is intrinsically chiral in M if and only if $n \equiv 3 \mod 4$ and $n \neq 3$.

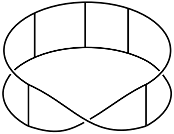
First established for $\mathbb{S}^3$ by Flapan and Weaver in 1992

Intrinsic chirality of Möbius ladders

The Möbius ladder M_n consists of a loop of $2n$ vertices and n rungs connecting antipodal vertices on the loop.



M_3



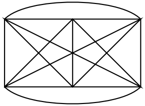
M_5

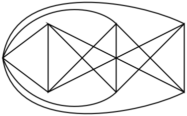
Example

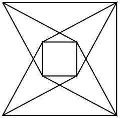
Let M be a homology sphere that possesses an orientation reversing homeomorphism. Then M_n is intrinsically chiral in M if and only if n is odd and $n > 3$. Moreover, no orientation reversing homeomorphism of M can setwise fix an embedded M_3 and its loop.

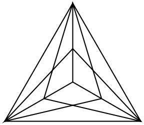
First established for $\mathbb{S}^3$ by Flapan in 1989

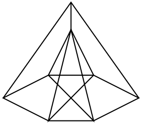
Petersen graphs

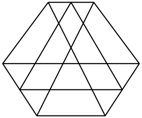

K_6

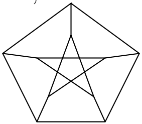

$K_{3,3,1}$


$K_{4,4}^-$


G_7


G_8


G_9


PG

$$\begin{tikzcd}
K_6 \arrow[r, "\Delta Y"] & G_7 \arrow[r, "\Delta Y"] \arrow[d, "\Delta Y"] & K_{4,4}^- & \\
K_{3,3,1} \arrow[r, "\Delta Y"] & G_8 \arrow[r, "\Delta Y"] & G_9 \arrow[r, "\Delta Y"] & PG
\end{tikzcd}$$

Linking of Petersen graphs

The only graphs that are intrinsically linked and minor minimal with respect to this property

ω - modulo 2 sum of linking numbers of all disjoint pairs of loops of an embedded graph in a homology sphere

Theorem (Sachs, 1984; Flapan-Howards-Lawrence-Mellor, 2006; Nikkuni-Taniyama, 2012)

Let Γ be an embedding of a Petersen graph in $\mathbb{S}^3$. Then $\omega(\Gamma) = 1$.

Proposition (Y.)

Let Γ be an embedding of a graph in the Petersen family in a homology sphere M. Then $\omega(\Gamma) = 1$.

Unrealizable automorphisms of K_6, G_8 and PG

Proposition (Y.)

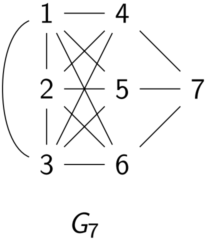
The automorphism (1234) of K_6 is not realizable in any homology sphere.

First established for $\mathbb{S}^3$ by Flapan in 1989
Orbits of pairs of loops under the action of (1234):

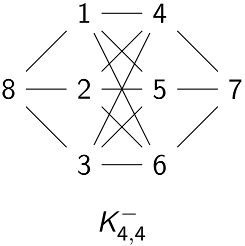
$$\{156, 256, 356, 456\}, \{125, 235, 345, 145\}, \{135, 245\}$$

Same method applies to find unrealizable automorphisms of G_8 and PG

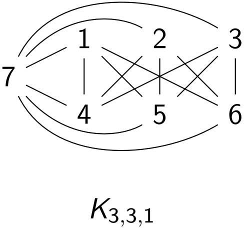
Unrealizable automorphism of $G_7, K_{4,4}^-$ and $K_{3,3,1}$



G_7



$K_{4,4}^-$



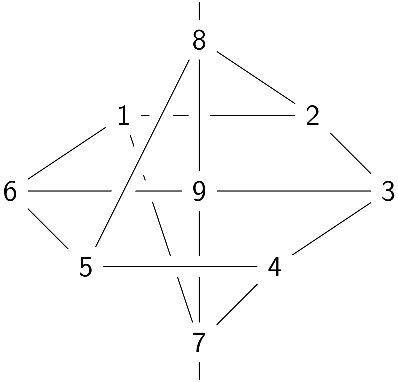
$K_{3,3,1}$

Proposition (Y.)

With the labeling above, the automorphism (123) of each of $G_7, K_{4,4}^-$ and $K_{3,3,1}$ is not realizable in any homology sphere.

Any homeomorphism h realizing (123) must fix edges $\overline{47}, \overline{57}, \overline{67} \Rightarrow \text{fix}(h)$ is $\mathbb{S}^2 \Rightarrow h^2$ is the identity map

Automorphisms of G_9



Proposition (Y.)

Every automorphism of G_9 is realizable in $\mathbb{S}^3$.

ありがとうございます!

Thank you!