Enumeration Algorithm for Lattice Model

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State matrix recursion algorithm

State matrix recursion algorithm enumerates 2-dimensional lattice models such as

- Monomer-dimer coverings
- Multiple self-avoiding walks and polygons
- Independent vertex sets
- Quantum knot mosaics

These are famous problems in Combinatorics and Statistical Mechanics studied by topologists, combinatorialists and physicists alike.

State matrix recursion algorithm is divided into three stages:

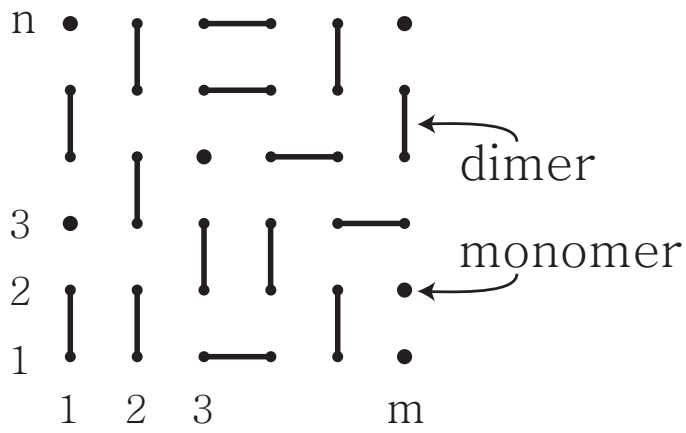
- Stage 1. Conversion to appropriate mosaics
- Stage 2. State matrix recursion formula
- Stage 3. State matrix analyzing

During this talk, the algorithm will be briefly demonstrated by solving the Monomer-Dimer Problem.

Contents

Monomer-dimer coverings

Monomer-dimer covering in $m \times n$ rectangle on the square lattice $\mathbb{Z}_{m \times n}$



Generating function

$$D_{m \times n}(z) = \sum k(t)\, z^t$$

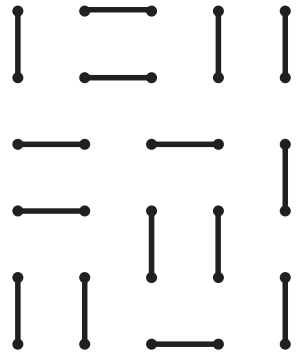
where $k(t)$ is the number of monomer-dimer coverings with t monomers.

- $D_{m \times n}(1)$ is the number of monomer-dimer coverings.
- $D_{m \times n}(0)$ is the number of pure dimer coverings (i.e., no monomers).

Breakthrough results

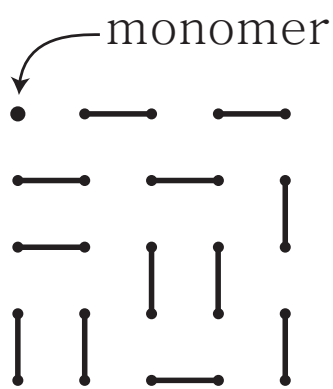
[Kasteleyn and Temperley-Fisher 1961]
Pure dimer problem for even mn

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\sqrt{2\cos\left(\frac{\pi j}{m+1}\right)+2i\cos\left(\frac{\pi k}{n+1}\right)}$$



[Tzeng-Wu 2003]
Single boundary monomer problem for odd mn
(it has a fixed single monomer on the boundary)

$$\prod_{j=1}^{\frac{m-1}{2}}\prod_{k=1}^{\frac{n-1}{2}}\left[4\cos^2\left(\frac{\pi j}{m+1}\right)+4\cos^2\left(\frac{\pi k}{n+1}\right)\right]$$



Question: How about if we allow many monomers? Generating function?

Monomer-Dimer Theorem

Theorem

$$D_{m\times n}(z) = (1,1)\text{-entry of } (A_m)^n$$

where A_m *is a* $2^m \times 2^m$ *matrix defined by the recurrence relation*

$$A_k = \begin{bmatrix} zA_{k-1} + \begin{bmatrix} A_{k-2} & \mathbb{O}_{k-2} \\ \mathbb{O}_{k-2} & \mathbb{O}_{k-2} \end{bmatrix} & A_{k-1} \\ A_{k-1} & \mathbb{O}_{k-1} \end{bmatrix}$$

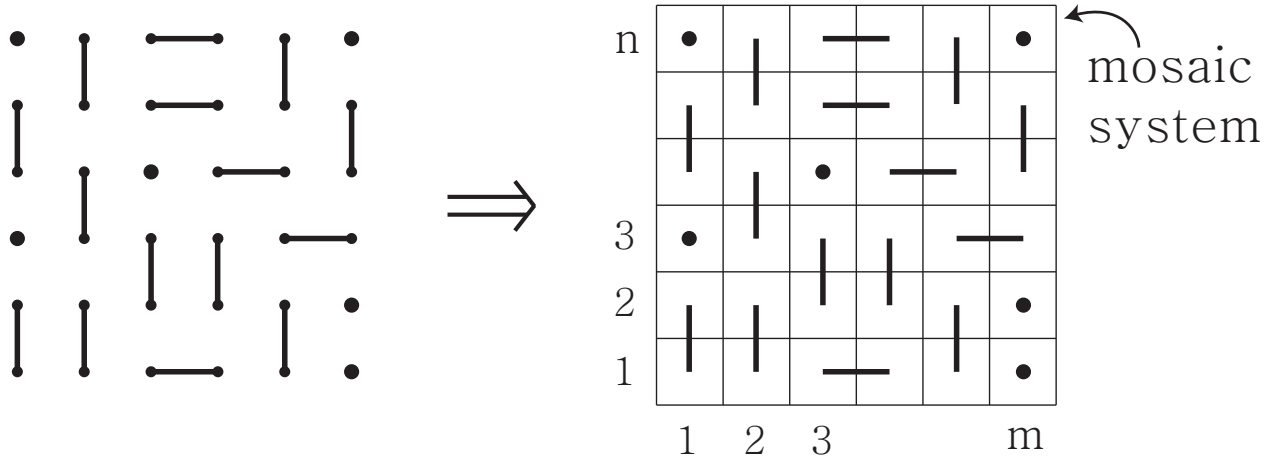
starting with $A_0 = \begin{bmatrix} 1 \end{bmatrix}$ *and* $A_1 = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$ *where* $\mathbb{O}_k$ *is the* $2^k \times 2^k$ *zero-matrix.*

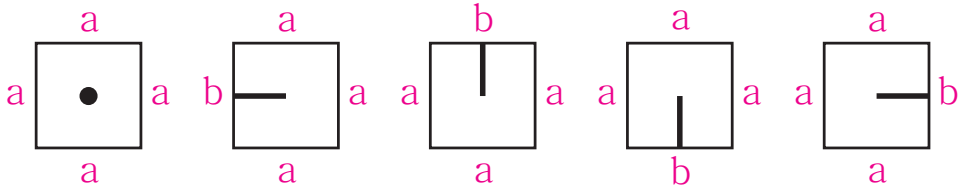
Note that it is not a closed form solution, but a sparse recurrence algorithm.

Exact enumeration

n	$D_{n\times n}(1)$	$(D_{n\times n}(1))^{\frac{1}{n^2}}$
1	1	1.000
2	7	1.627
3	131	1.719
4	10012	1.778
5	2810694	1.811
6	2989126727	1.833
7	11945257052321	1.849
8	1797883431019801 35	1.860
9	10185111919160666118608	1.869
10	217213878367309419393775001 5	1.877
11	17438298232401644946943864379706 40	1.882
12	52701379938160862669628743954502345348 87	1.887
13	59956919824257750508655631107474672284499736089	1.891

Stage 1. Conversion to monomer-dimer mosaics





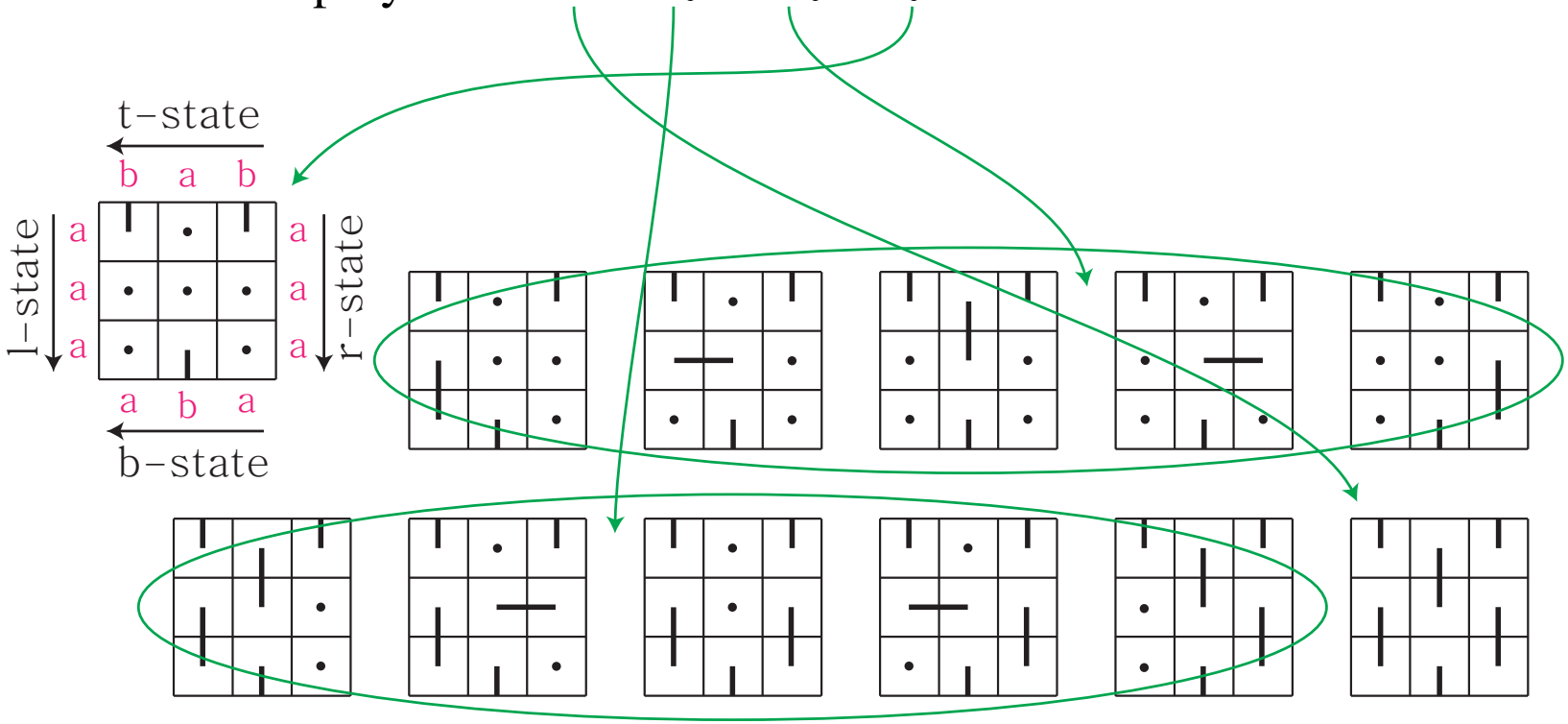
5 mosaic tiles labeled with two letters a, b

Adjacency Rule : Attaching edges of adjacent tiles have the same letter.

Boundary state requirement : All boundary edges are labeled with letter a.

Stage 2. State matrix recursion formula

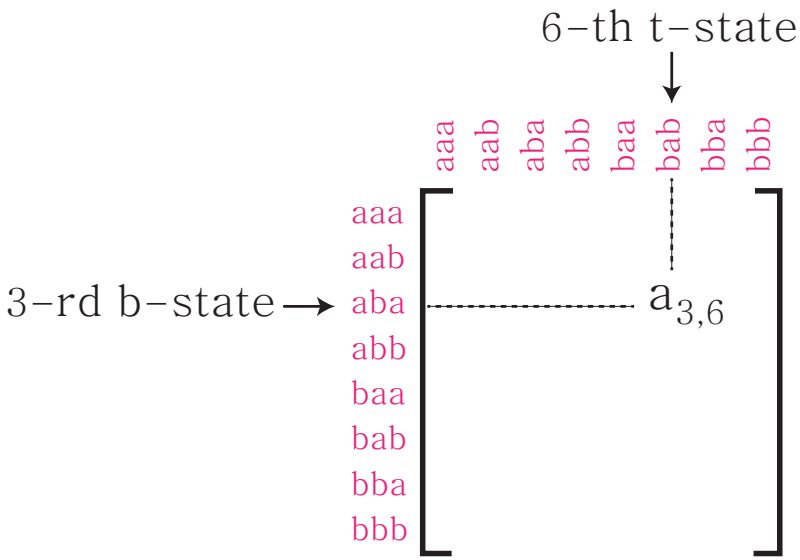
State polynomial : Twelve suitably adjacent 3×3-mosaics associated with b-state aba, t-state bab and the trivial l- and r-states aaa to produce the associated state polynomial $1 + 5z^2 + 5z^4 + z^6$.

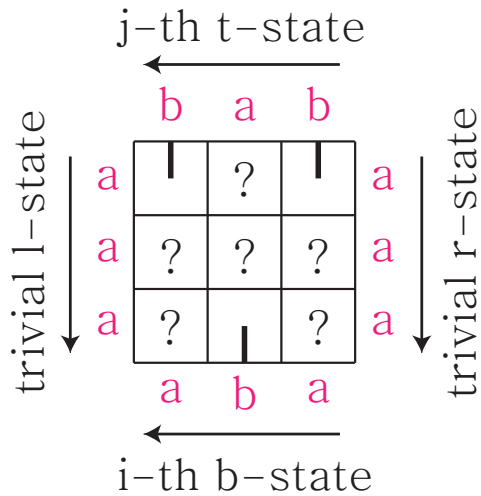


State matrix $A_{m\times n}$ for the set of suitably adjacent $m\times n$-mosaics is a $2^m\times 2^m$ matrix (a_{ij}) where a_{ij} is the state polynomial associated to i-th b-state, j-th t-state, and the trivial l- and r-states. (Trivial state condition is needed for the boundary state requirement)

We arrange 2^m states of length m in the lexicographic order.

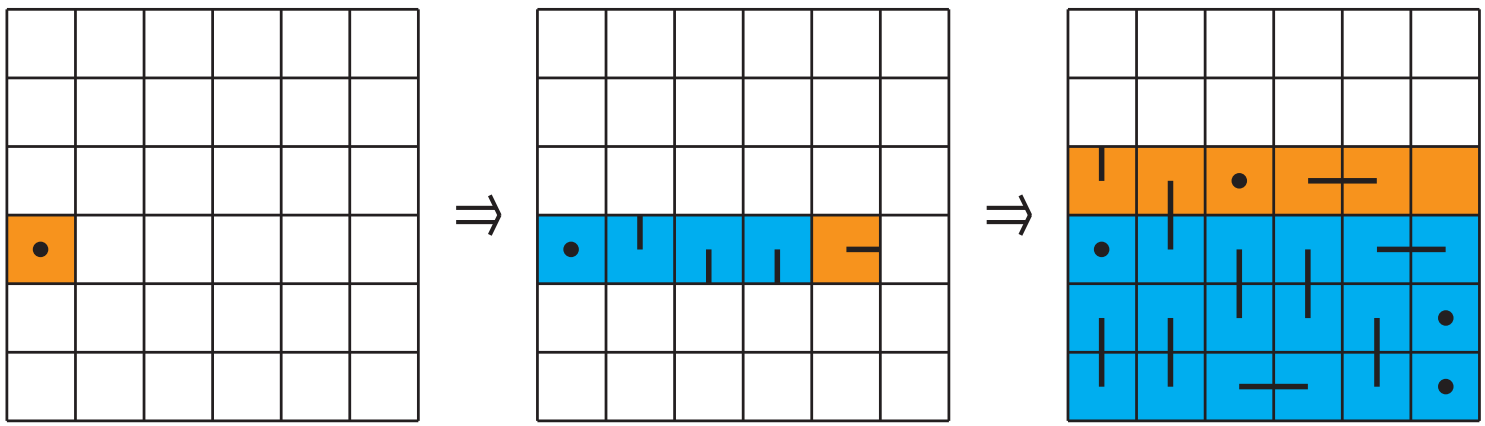
For example, (3,6)-entry of $A_{3\times 3}$ is $a_{3,6} = 1 + 5z^2 + 5z^4 + z^6$.





Recursion strategy to find the state matrix $A_{m\times n}$.

1. Find the starting state matrices A_1 and B_1 for 1×1-mosaics.
2. Find the bar state matrices A_k and B_k for suitably adjacent $k\times 1$-mosaics (or bar mosaics) by attaching a mosaic tile recursively on the right side.
3. Find the state matrix $A_{m\times k}$ for suitably adjacent $m\times k$-mosaics by attaching a bar mosaic of length m on the top side.



Summary

First, we get the recursive relation from the bar state matrix recursion lemma

$$A_k = \begin{bmatrix} zA_{k-1} + B_{k-1} & A_{k-1} \\ A_{k-1} & \mathbb{O}_{k-1} \end{bmatrix} \text{ and } B_k = \begin{bmatrix} A_{k-1} & \mathbb{O}_{k-1} \\ \mathbb{O}_{k-1} & \mathbb{O}_{k-1} \end{bmatrix}$$

starting with $A_0 = \begin{bmatrix} 1 \end{bmatrix}$ and $B_0 = \begin{bmatrix} 0 \end{bmatrix}$.

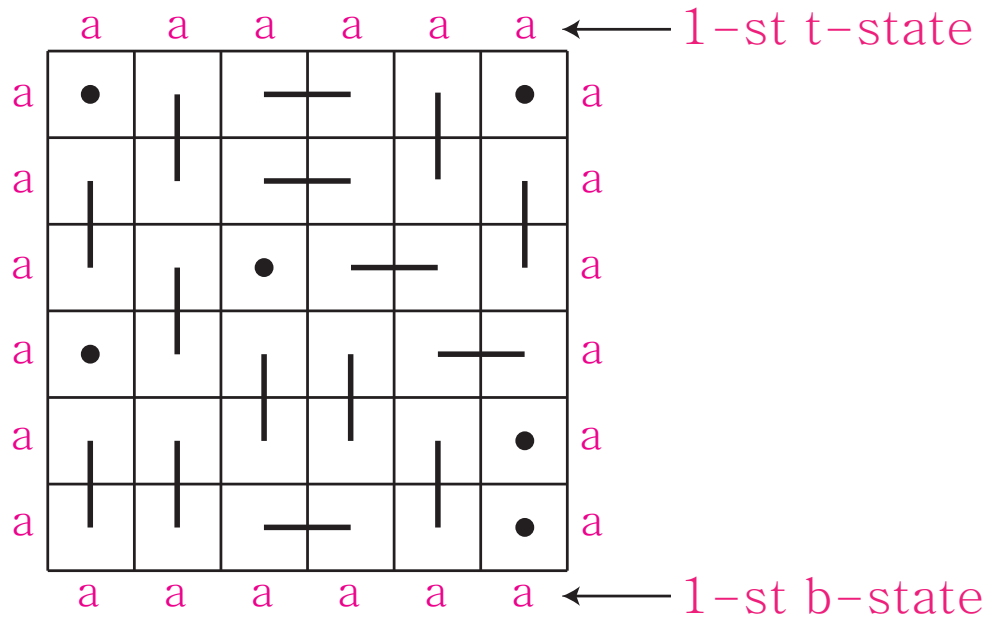
Then, we have the state matrix from the state matrix multiplication lemma

$$A_{m \times n} = (A_m)^n.$$

Stage 3. State matrix analyzing

Monomer-dimer generating function w.r.t. the number of monomers

$$D_{m\times n}(z) = (1,1)\text{-entry of } A_{m\times n}.$$



Monomer-Dimer Theorem

Theorem

$$D_{m\times n}(z) = (1,1)\text{-entry of } (A_m)^n$$

where A_m *is a* $2^m \times 2^m$ *matrix defined by the recurrence relation*

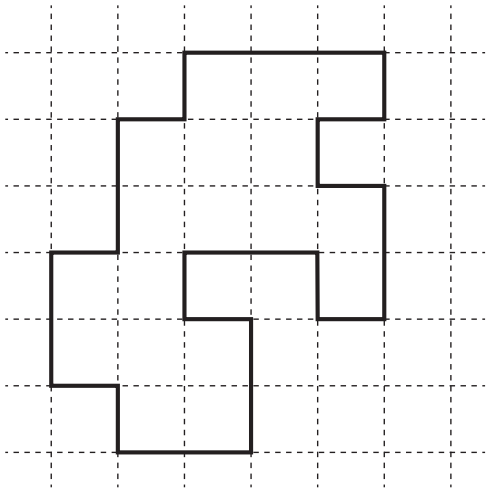
$$A_k = \begin{bmatrix} zA_{k-1} + \begin{bmatrix} A_{k-2} & \mathbb{O}_{k-2} \\ \mathbb{O}_{k-2} & \mathbb{O}_{k-2} \end{bmatrix} & A_{k-1} \\ A_{k-1} & \mathbb{O}_{k-1} \end{bmatrix}$$

starting with $A_0 = \begin{bmatrix} 1 \end{bmatrix}$ *and* $A_1 = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$ *where* $\mathbb{O}_k$ *is the* $2^k \times 2^k$ *zero-matrix.*

Contents

Self-avoiding polygons

Self-avoiding polygon (SAP) on the square lattice $\mathbb{Z}^2$



p_n = number of SAPs of length n up to translations

Finding p_n is the central unsolved problem during last 70 years in Combinatorics and Statistical Mechanics.

There are many numerical datas, but few mathematically proved results.

Breakthrough results

[Hammersley 1957]
The limit $\mu = \lim (p_n)^{\frac{1}{n}}$ exists.

$\mu = 2.638158530323 \pm 2\times 10^{-12}$: best estimate on $\mathbb{Z}^2$ during 50 years.

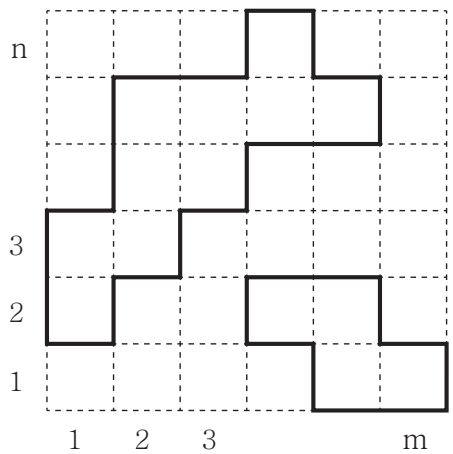
[Duminil-Copin and Smirnov 2012, Annals of Math.]
$\mu = \sqrt{2+\sqrt{2}}$ on the hexagonal lattice $\mathbb{H}^2$ (easier than on $\mathbb{Z}^2$).

Nobody expects that there will be a closed form of p_n.

Multiple self-avoiding polygons

Multiple self-avoiding polygon (MSAP) in $\mathbb{Z}_{m\times n}$

$p_{m\times n}$ = number of MSAPs in $\mathbb{Z}_{m\times n}$ (not up to translations)



Theorem

$$p_{m\times n} = (1,1)\text{-entry of } (A_m)^n - 1$$

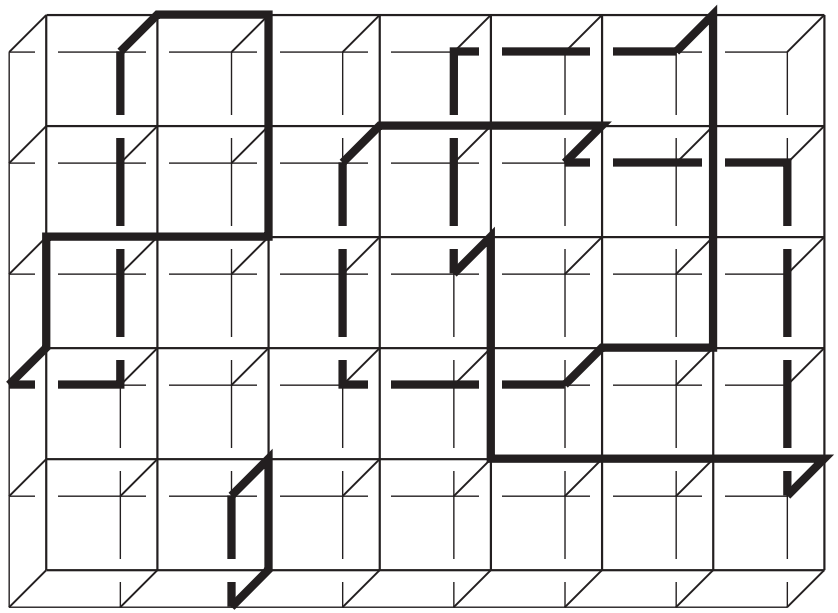
where the $2^m \times 2^m$ *matrix* A_m *is defined by*

$$A_{k+1} = \begin{bmatrix} A_k & B_k \\ B_k & A_k \end{bmatrix} \text{ and } B_{k+1} = \begin{bmatrix} B_k & A_k \\ A_k & \mathbb{O}_k \end{bmatrix}$$

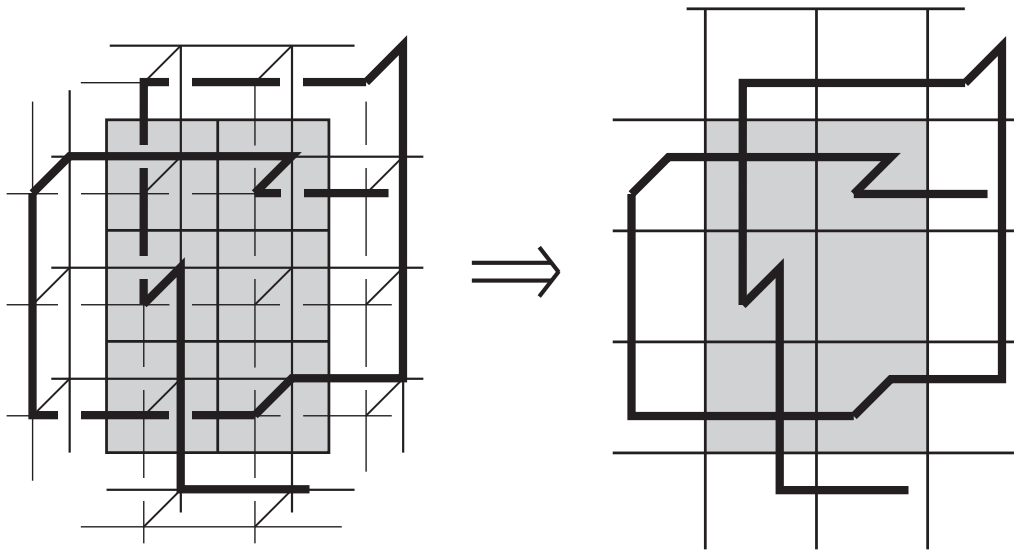
starting with $A_0 = \begin{bmatrix} 1 \end{bmatrix}$ *and* $B_0 = \begin{bmatrix} 0 \end{bmatrix}$.

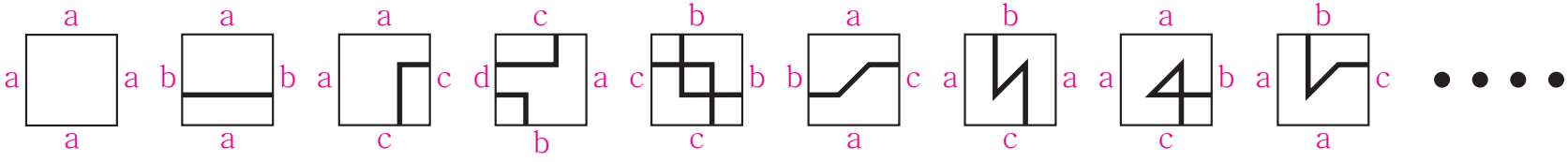
MSAPs in the 1-slab square lattice

Multiple self-avoiding polygons (links) in the 1-slab square lattice $\mathbb{Z}_{m\times n\times 2}$
(2 layers of the planes)



Conversion to 1-slab MSAP mosaics by using 65 mosaic tiles





MSAP enumeration in $\mathbb{Z}_{m\times n\times 2}$

Theorem

The number of MSAPs in the 1-slab square lattice $\mathbb{Z}_{m\times n\times 2}$ is

$$(1,1)\text{-entry of } (A_m)^n - 1$$

where the $4^m \times 4^m$ matrix A_m is defined by

$$A_{k+1} = \begin{bmatrix} A_k+D_k & B_k+C_k & B_k+C_k & A_k+D_k \\ B_k+C_k & A_k & A_k+D_k & C_k \\ B_k+C_k & A_k+D_k & A_k & B_k \\ A_k+D_k & C_k & B_k & A_k \end{bmatrix}, \; B_{k+1} = \begin{bmatrix} B_k+C_k & A_k & A_k+D_k & C_k \\ A_k & \mathbb{O}_k & C_k & \mathbb{O}_k \\ A_k+D_k & C_k & B_k & A_k \\ C_k & \mathbb{O}_k & A_k & \mathbb{O}_k \end{bmatrix},$$

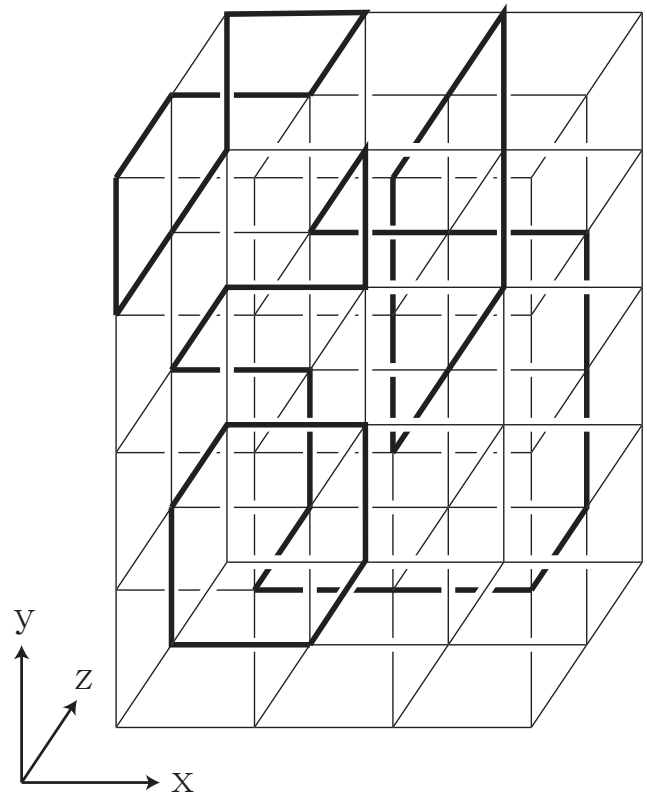
$$C_{k+1} = \begin{bmatrix} B_k+C_k & A_k+D_k & A_k & B_k \\ A_k+D_k & C_k & B_k & A_k \\ A_k & B_k & \mathbb{O}_k & \mathbb{O}_k \\ B_k & A_k & \mathbb{O}_k & \mathbb{O}_k \end{bmatrix} \text{ and } D_{k+1} = \begin{bmatrix} A_k+D_k & C_k & B_k & A_k \\ C_k & \mathbb{O}_k & A_k & \mathbb{O}_k \\ B_k & A_k & \mathbb{O}_k & \mathbb{O}_k \\ A_k & \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \end{bmatrix},$$

starting with $A_0 = \begin{bmatrix} 1 \end{bmatrix}$ and $B_0 = C_0 = D_0 = \begin{bmatrix} 0 \end{bmatrix}$.

- The number of MSAPs in $\mathbb{Z}_{7\times 60\times 2}$ is $5.345706\cdots\times 10^{261}$.

Links in the 3-dimensional cubic lattice

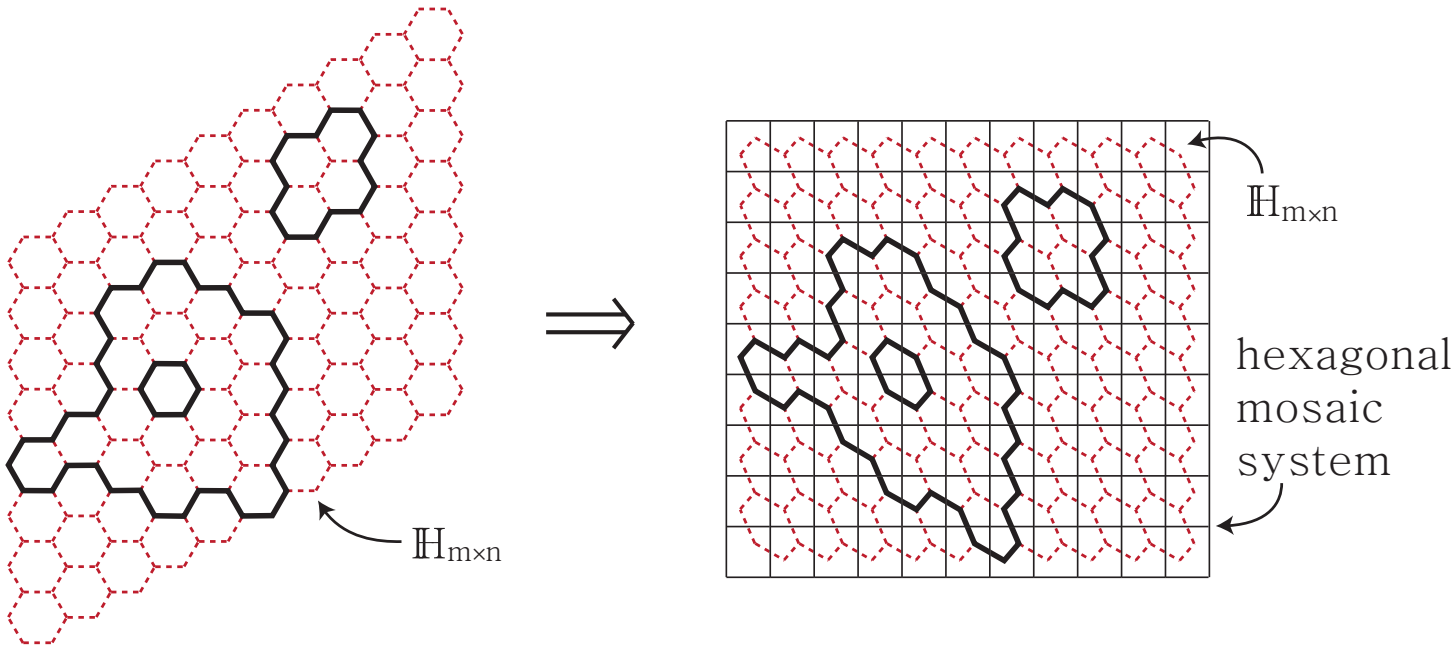
Links in the 3-dimensional cubic lattice $\mathbb{Z}_{l\times m\times n}$
(not up to translations and ambient isotopies)



Contents

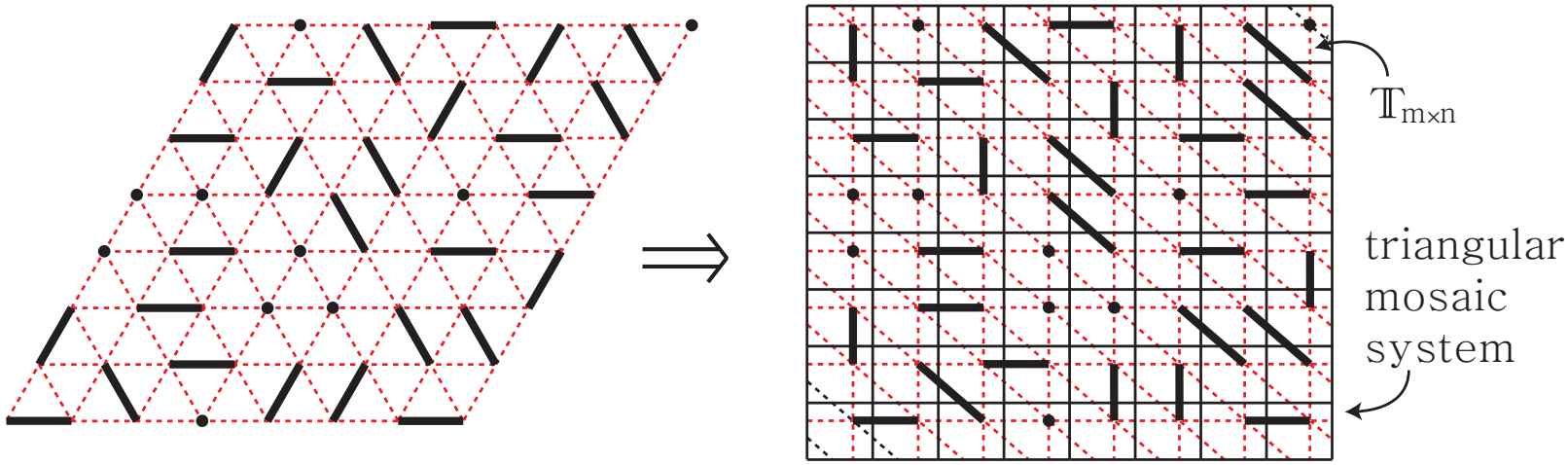
Different regular lattices

Hexagonal (honeycomb) lattice $\mathbb{H}_{m\times n}$ (MSAP model)



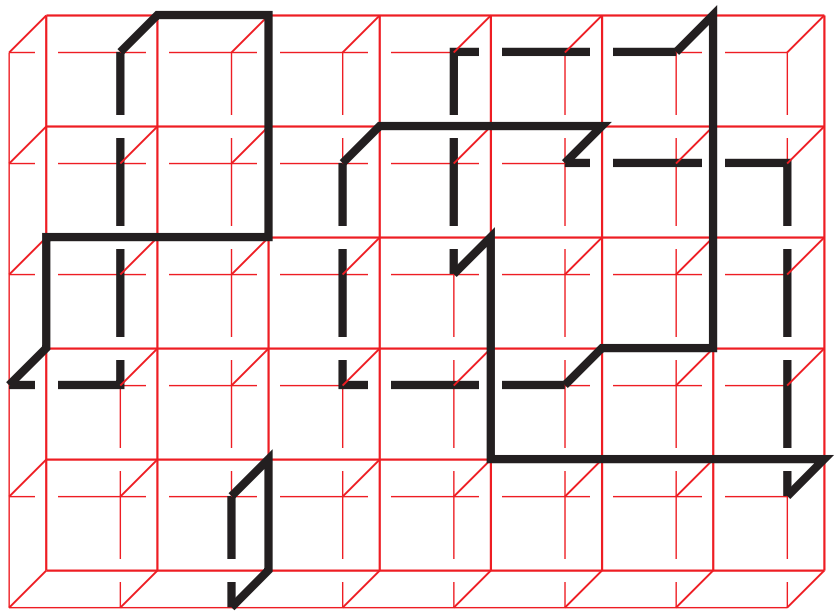
Different regular lattices

Triangular lattice $\mathbb{T}_{m\times n}$ (Monomer-dimer model)



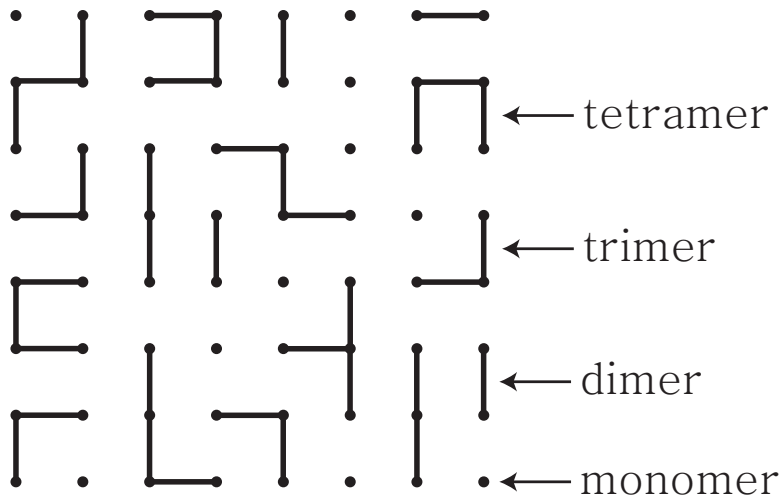
Different regular lattices

1-slab square lattice $\mathbb{Z}_{m\times n\times 2}$ (Multiple self-avoiding polygon (link) model)



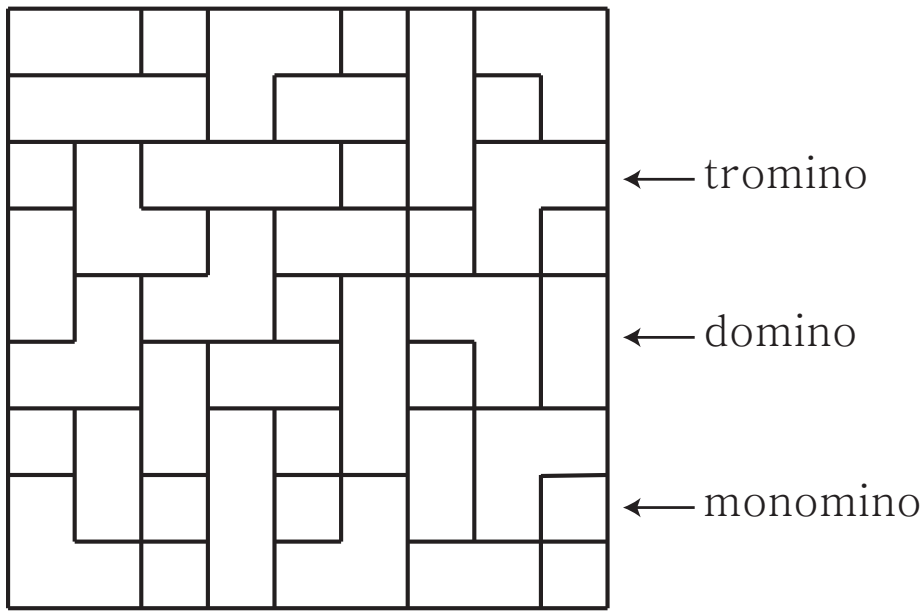
Polymer model

Monomer-dimer-trimer-tetramer covering



Polyomino model

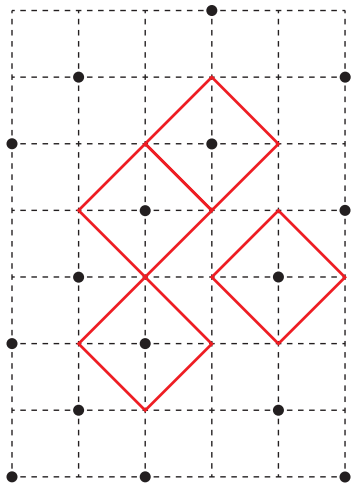
Monomino-domino-tromino tiling



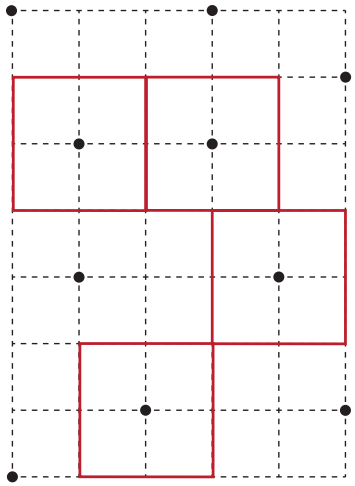
Independent vertex model

Independent vertex sets

independent vertex set
(Hard Square Problem)



independent vertex set
with 2-nb exclusion



independent vertex set
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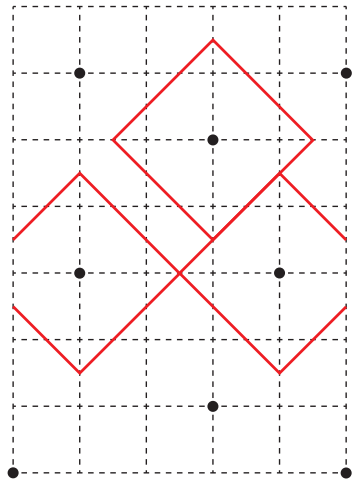
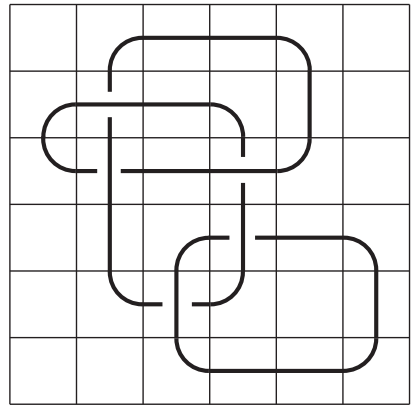
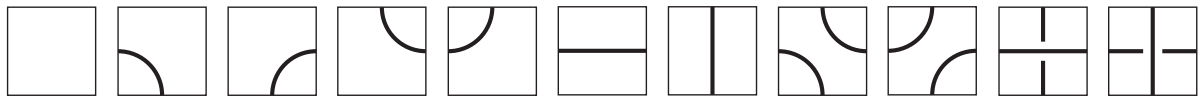


Quantum knot model

Quantum knot mosaic

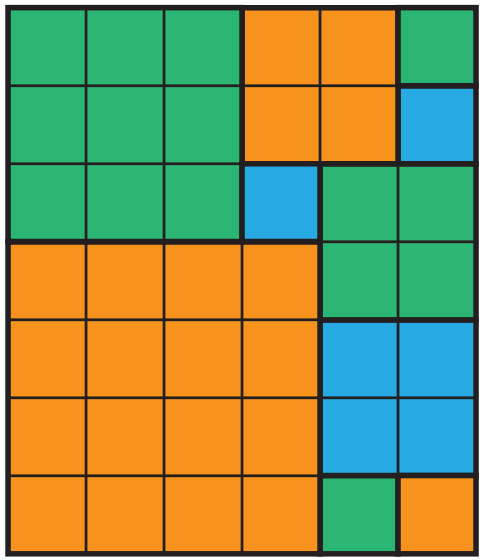


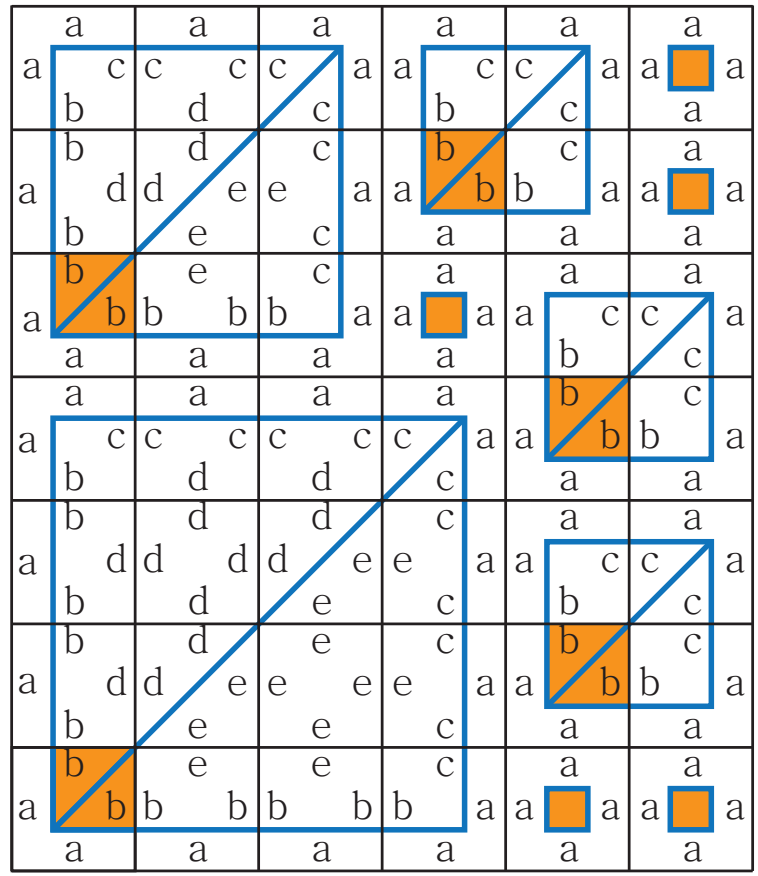
with 11 knot mosaic tiles as follows



Squared rectangle model

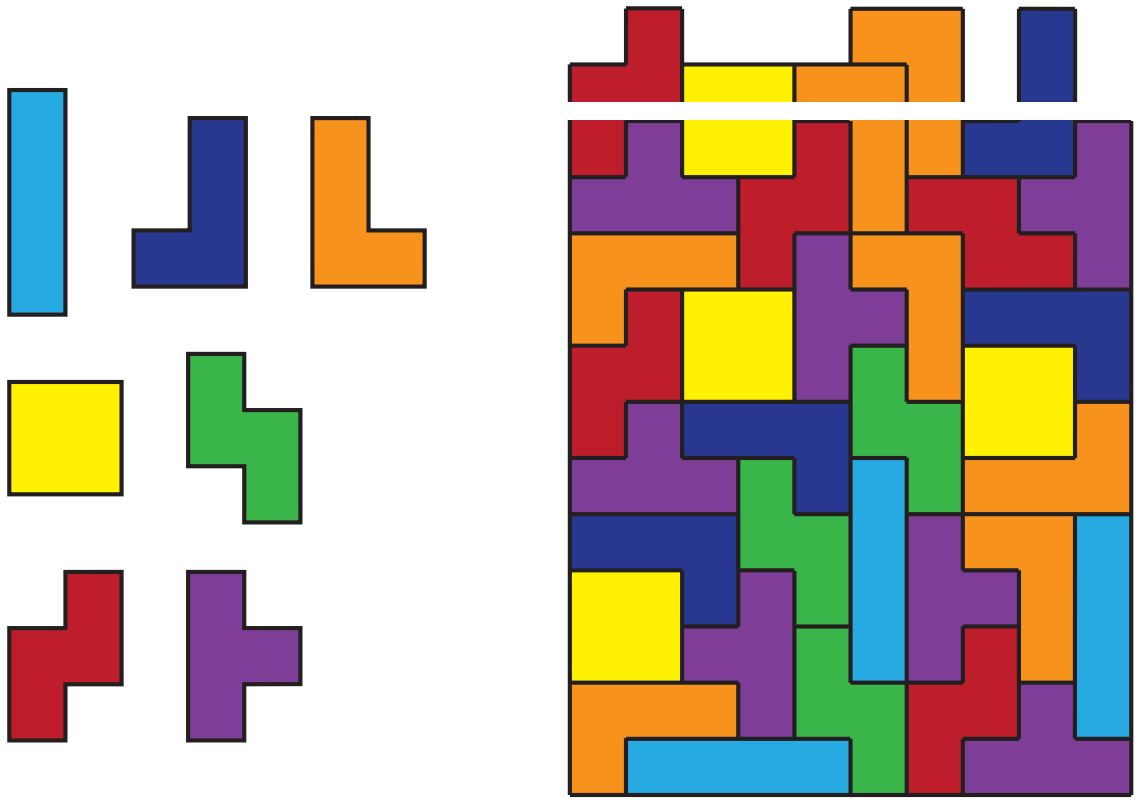
Tiling a rectangle by squares with various integer sizes





Tetris model

Tetris configuration by 7 tetrominoes



Thank you!