Planar Legendrian Spatial Graphs

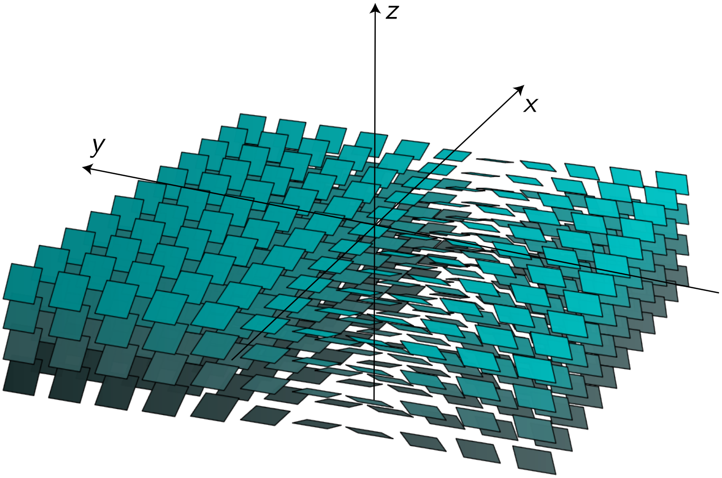
Danielle O'Donnol
Indiana University
Joint work with Peter Lambert-Cole

International Workshop on Spatial Graphs 2016
August 3, 2016

Outline:

- Contact structures
- Legendrian knots
- The invariants
  - the Thurston-Bennequin number
  - the rotation number
- Legendrian graphs
- Invariants of Legendrian graphs
- Legendrian Simplicity

Let M be an oriented 3-manifold and ξ a 2-plane field on M.

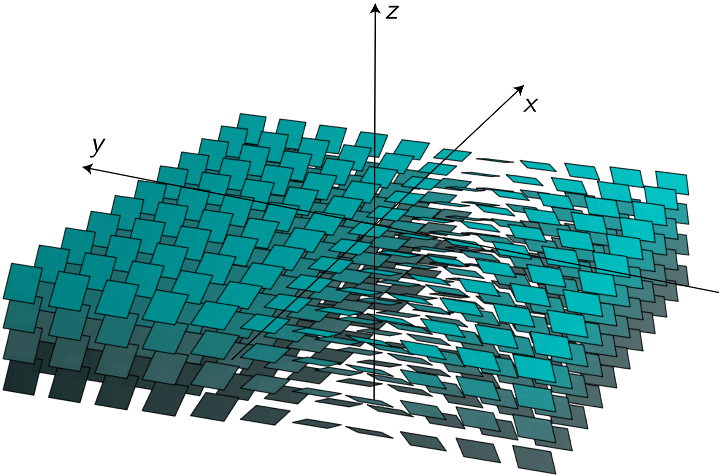


Let M be an oriented 3-manifold and ξ a 2-plane field on M. We say ξ is **a contact structure on M** if

$$\xi = \ker \alpha$$

for some $1-$form α satisfying $\alpha \wedge d\alpha > 0$.

Example: On $\mathbb{R}^3$, the $1-$form $\alpha = dz - ydx$ gives the standard contact structure on $\mathbb{R}^3$, ξ_{std}.



$$\xi_{std} = \ker(dz - ydx)$$

Let M be an oriented 3-manifold and ξ a 2-plane field on M.
We say ξ is **a contact structure on M** if

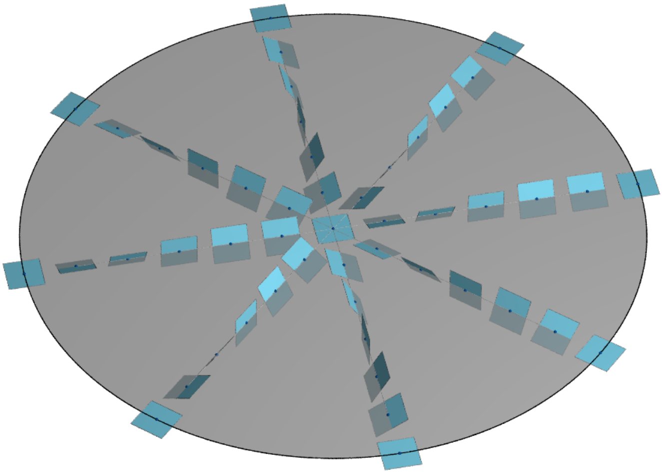
$$\xi = \ker \alpha$$

for some $1-$form α satisfying $\alpha \wedge d\alpha > 0$.

Example: On $\mathbb{R}^3$, the $1-$form $\alpha = dz - ydx$ gives the standard contact structure on $\mathbb{R}^3$, ξ_{std}.

Contact structures can be

- overtwisted - contains an overtwisted disk
- tight - does not contain an overtwisted disk



Overtwisted disk

Let M be an oriented 3-manifold and ξ a 2-plane field on M.
We say ξ is **a contact structure on M** if

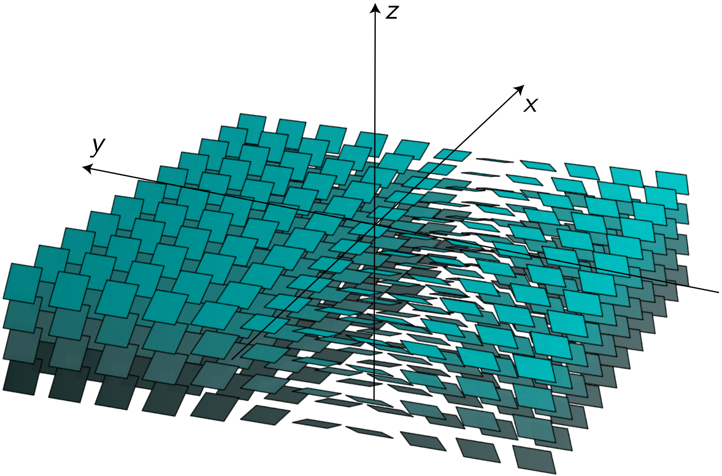
$$\xi = \ker \alpha$$

for some $1-$form α satisfying $\alpha \wedge d\alpha > 0$.

Example: On $\mathbb{R}^3$, the $1-$form $\alpha = dz - ydx$ gives the standard contact structure on $\mathbb{R}^3$, ξ_{std}.

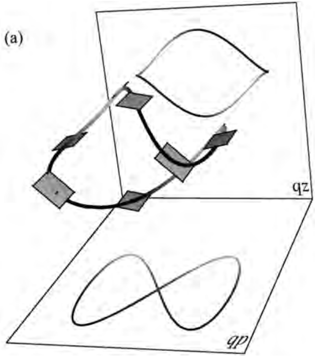
Contact structures can be

- overtwisted - contains an overtwisted disk (well understood)
- tight - does not contain an overtwisted disk (interesting)



$$\xi_{std} = \ker(dz - ydx) = \mathrm{Span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\} \text{ (tight)}$$

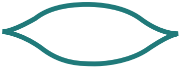
A knot $\gamma \subset (S^3, \xi_{std})$ is called **Legendrian** if for all $p \in \gamma$ and ξ_p the contact plane at p, $T_p\gamma \subset \xi_p$.



Legendrian unknot

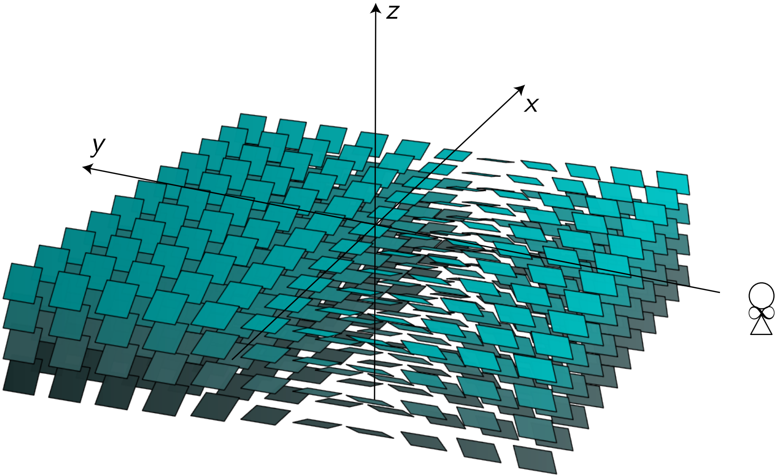
Front projections (on the $xz-$plane):

- Front projections of Legendrian knots do not have vertical tangencies (since $y = dz/dx$).
- At each crossing the overstrand is always the one with smaller slope (since the $y-$axis points away from the viewer).



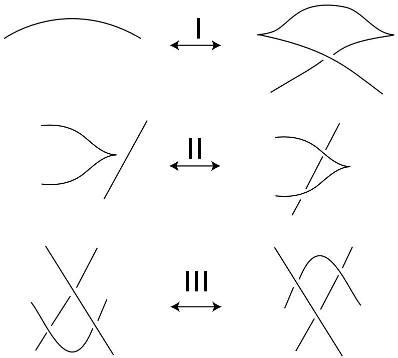
unknot

rh-trefoil



$$\xi_{std} = \ker(dz - ydx)$$

Two generic front projections of a Legendrian knot are related by:



Reidemeister moves

Classical invariants of Legendrian knots:

- the Thurston-Bennequin number: tb
- the rotation number: rot

The Thurston-Bennequin number measures the amount of twisting of the contact planes along the knot and does not depend on the chosen orientation of K.

To compute $tb(K)$:

- take non-zero vector field v transverse to ξ
- take K', the push-off of K in the direction of v.

$$\mathrm{tb}(K) = lk(K, K').$$

The Thurston-Bennequin number measures the amount of twisting of the contact planes along the knot and does not depend on the chosen orientation of K.

To compute $tb(K)$:

- take non-zero vector field v transverse to ξ
- take K', the push-off of K in the direction of v.

$$\mathrm{tb}(K) = lk(K, K').$$

In the front projection:

$$\mathrm{tb}(K) = writhe - \frac{1}{2}\#cusps$$

writhe = signed count of crossings in the projection

The rotation number does depend on orientation. Let T be a non-zero vector field tangent to K pointing in the direction of the orientation on K.

Then, $\text{rot}(K) =$ the winding number of T about the origin.

(in the trivialization $\xi|_K = K \times \mathbb{R}^2$ induced by the trivialization of $\xi|_\Sigma$, $K = \partial\Sigma$)

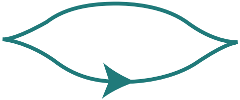
The rotation number does depend on orientation. Let T be a non-zero vector field tangent to K pointing in the direction of the orientation on K.

Then, $\mathrm{rot}(K) =$ the winding number of T about the origin.

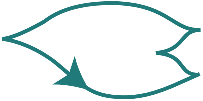
(in the trivialization $\xi|_K = K \times \mathbb{R}^2$ induced by the trivialization of $\xi|_\Sigma$, $K = \partial\Sigma$)

In the front projection:

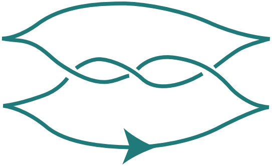
$$\mathrm{rot}(K) = \frac{1}{2}(\#\textit{down cusps} - \#\textit{up cusps})$$



unknot
tb $= -1$
rot $= 0$



stabilized unknot
tb $= -2$
rot $= -1$



rh-trefoil
tb $= 1$
rot $= 0$

$\mathrm{tb}(K) = \mathit{writhe} - \frac{1}{2}\#\mathit{cusps}$ $\quad$ $\mathrm{rot}(K) = \frac{1}{2}(\#\mathit{down\ cusps} - \#\mathit{up\ cusps})$

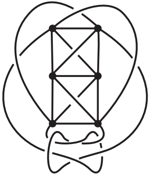
Theorem (Eliashberg)

If K is a Legendrian knot in $(\mathbb{R}^3, \xi_{std})$ and Σ a Seifert surface for K, then

$$tb(K) + |rot(K)| \leq -\chi(\Sigma).$$

- if K is the unknot then $\text{tb}(K) \leq -1$ (sharp bound)
- tb it can be made arbitrary small within the same topological knot class by adding stabilizations

A spatial graph is an embedding of a graph in S^3.

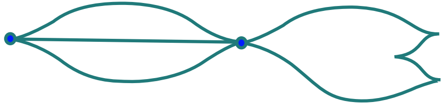


A Legendrian graph in $(\mathbb{R}^3, \xi_{std})$ is an embedded graph that is everywhere tangent to the contact planes.

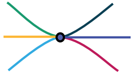
- have been used in proofs of important theorems:
  - Giroux: correspondence between open book decompositions and contact structures of a 3-manifold
  - Eliashberg and Fraser: tb and rot determine the Legendrian unknot

Theorem (O'D, Pavelescu)

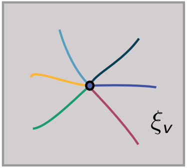
Any spatial graph in $\mathbb{R}^3$ can be Legendrian realized in (S^3, ξ_{std}).



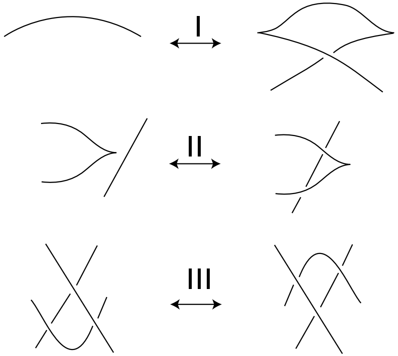
Front projection of a Legendrian graph.



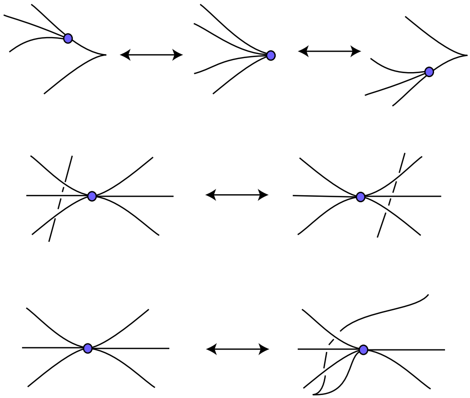
front projection near v



Two generic front projections of a Legendrian graph are related by:



Reidemeister moves



Moves given by mutual positions of edges and vertices

We extend tb and rot to Legendrian graphs.

We first define the invariants for piecewise smooth Legendrian knots (i.e. cycles in a Legendrian graph).

For a Legendrian graph Γ:

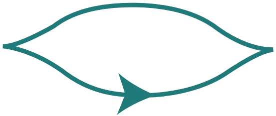
$\text{tb}(\Gamma) =$ ordered set of the tbs of the cycles of Γ.

$\text{rot}(\Gamma) =$ ordered set of the rots of the oriented cycles of Γ.

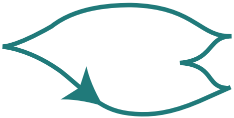
Certain types of Legendrian knots and links are determined by the classical invariants tb and rot in (S^3, ξ_{std}):

- the unknot (Eliashberg, Fraser)
- torus knots and, figure-eight knot (Etnyre, Honda)
- links consisting of the unknot and a cable of that unknot (Ding, Geiges)

Such knots are called Legendrian simple.



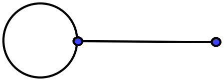
tb $= -1$
rot $= 0$



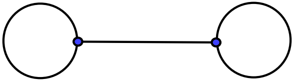
tb $= -2$
rot $= -1$

Are there graphs that are Legendrian simple? (i.e. Will the invariants tb and rot determine the Legendrian type of a graph?)

A Legendrian graph is called topologically trivial if it is ambient isotopic to a planar graph.



the lollipop graph



the handcuff graph

Work with Elena Pavelescu:

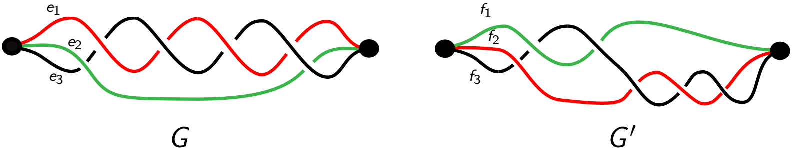
- A pair (tb, rot) determines exactly two Legendrian classes for the lollipop graph.
- A pair (tb, rot) determines exactly four Legendrian classes for the handcuff graph.

(a) (b)

(c) (d)

Work with Elena Pavelescu:

The two θ-graphs have the same invariants but they are not Legendrian isotopic.



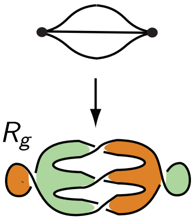
Key: The signed cyclic order of the edges around a vertex is unchanged under Legendrian isotopy.

Let g be a Legendrian graph. Roughly, a ribbon for g is a compact oriented surface R_g containing g in its interior such that:

1. the contact structure is tangent to R_g along g,
2. transverse to $R \setminus g$, and
3. ∂R_g is a transverse knot or link.

The contact framing $\overline{R_g}$ is the underlying unoriented surface of the Legendrian ribbon.

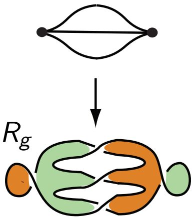
The Legendrian ribbon R_g



Features of R_g of a Legendrian graph

- invariant under Legendrian isotopy

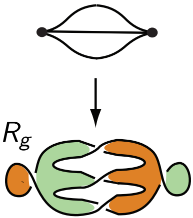
The Legendrian ribbon R_g



Features of R_g of a Legendrian graph

- invariant under Legendrian isotopy
- contains $\text{tb}(g)$
- contains the signed cyclic order of the edges at the vertices

The Legendrian ribbon R_g



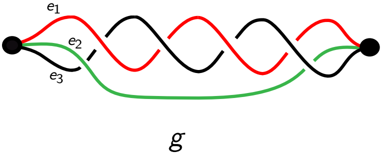
Features of R_g of a Legendrian graph

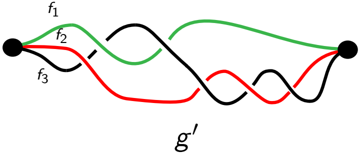
- invariant under Legendrian isotopy
- contains $\mathrm{tb}(g)$
- contains the signed cyclic order of the edges at the vertices

The contact framing $\overline{R_g}$ contains $\mathrm{tb}(g)$ and the cyclic order of the edges at the vertices.

Theorem (Lambert-Cole, O'D)

Let G be a trivalent planar graph. The pair $(R_g, rot(g))$ is a complete set of invariants for topologically trivial Legendrian embeddings $g : G \to S^3$.

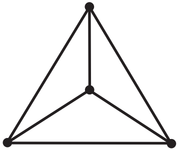


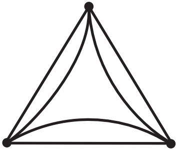


Theorem (Lambert-Cole, O'D)

Let G be a trivalent planar graph. For topologically trivial Legendrian embeddings of G:

1. *if G contains K_4 or Δ_2 as a minor, then the pair $(\overline{R}_g, rot)$ is a complete set of invariants*
2. *if G is 3-connected, the pair (tb, rot) is a complete set of invariants.*





Thank you!

Convex Surface Theory

Let G_1, G_2 be Legendrian graphs lying on convex spheres S_1, S_2 in (S^3, ξ_{std}). Suppose that there is a diffeomorphism $i : S_1 \to S_2$ that sends G_1 diffeomorphically to G_2 and the dividing set Γ_{S_1} diffeomorphically to the dividing set Γ_{S_2}. Then G_1 and G_2 are Legendrian isotopic in S^3.