Combinatorics of intrinsic linking

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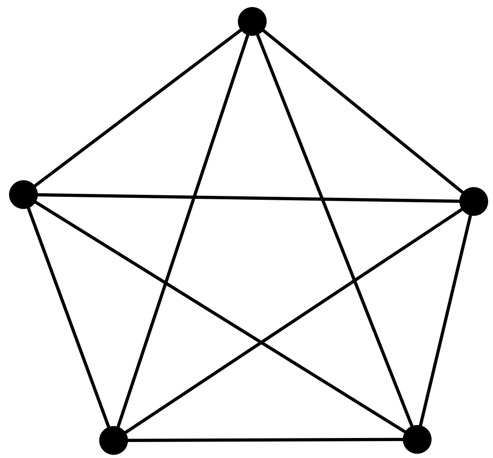
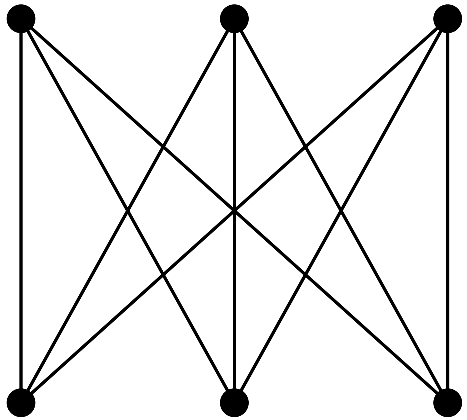
Theorem (Kuratowski 1929; Pontriagin '28, O.Frink-P.A.Smith '30)
A graph embeds in the plane iff it contains no subgraph homeomorphic to K_5 *or* $K_{3,3}$*.*



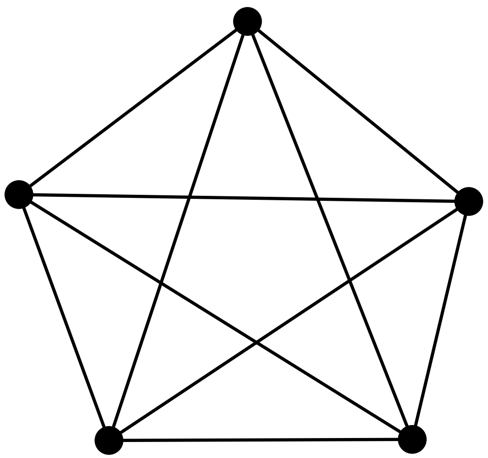
K_5



$K_{3,3}$

Remark (Wagner, 1937). An equivalent statement is:

Theorem (Kuratowski 1929; Pontriagin '28, O.Frink-P.A.Smith '30)
A graph embeds in the plane iff it ~~contains no subgraph homeomorphic~~ has no minor isomorphic to K_5 or $K_{3,3}$.



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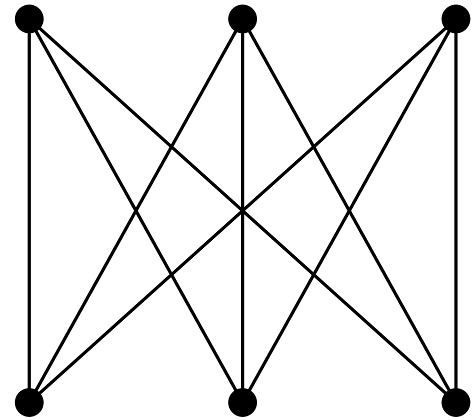
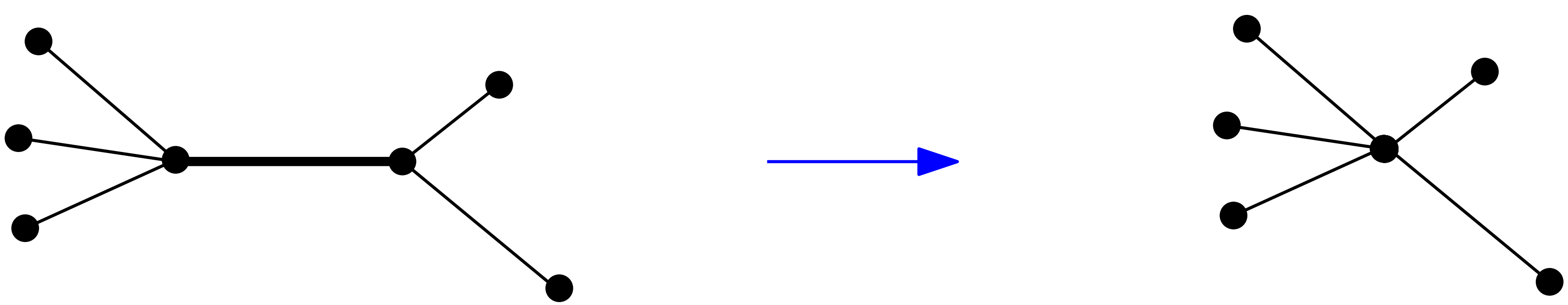


$K_{3,3}$

A *minor* of a graph G is a graph obtained from a subgraph of G by a sequence of edge contractions.



An embedding of a graph into $\mathbb{R}^3$ is *knotless* if (the image of) every cycle is unknotted, and *panelled* if every cycle bounds a disk whose interior is disjoint from the graph.

An embedding of a polyhedron into $\mathbb{R}^m$ is *linkless* if for every two disjoint closed subpolyhedra of the image, one is contained in an m-ball disjoint from the other one.

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In the sense of topology: a space triangulated by a simplicial complex.

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(a) If a graph embeds in $\mathbb{R}^3$ so that no pair of its disjoint cycles is linked with an odd linking number, then it admits a panelled embedding in $\mathbb{R}^3$.

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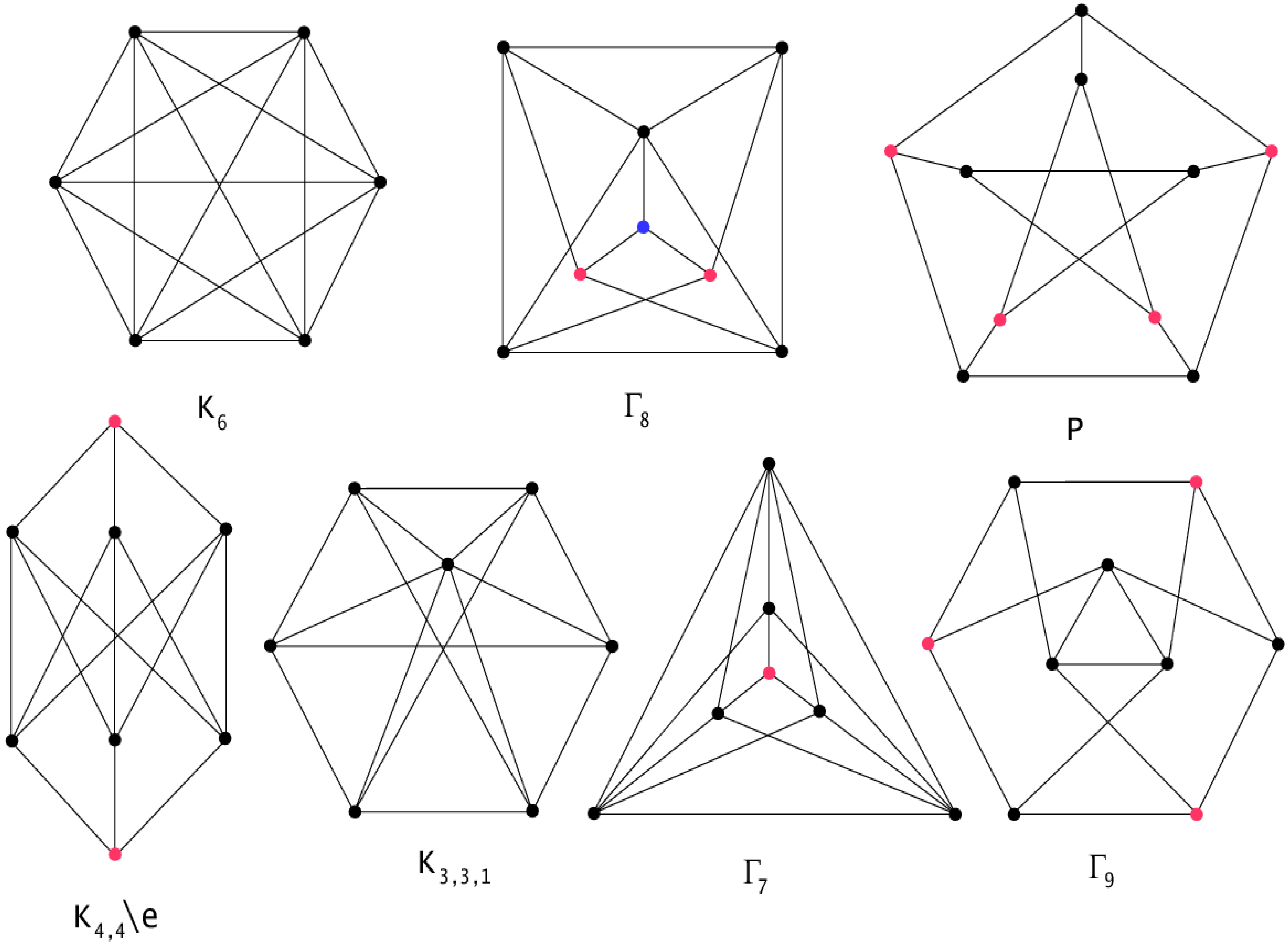
(c) A graph admits a linkless embedding in $\mathbb{R}^3$ if and only if it has no minor among the seven graphs known as the **Petersen family***.*


K_6
Γ_8
P
$K_{4,4}\backslash e$
$K_{3,3,1}$
Γ_7
Γ_9

A *cell complex* is a finite CW-complex whose attaching maps are PL embeddings. It is *atomic* if inclusions of cells are determined by inclusions of their vertex sets. (Cf. graphs without loops and multiple edges.)

Note: $\emptyset$ is NOT a cell.

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A finite cell complex B is *dichotomial* if it is atomic, and for each cell A of B, there exists another (the "opposite") cell $\bar{A}$ of B whose vertices are precisely all the vertices of B that are not in A.

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Theorem 1. *Every dichotomial cell complex is PL homeomorphic to a sphere of some dimension.*

There exist precisely two dichotomial 3*-spheres; their* 1*-skeleta are* K_5 *and* $K_{3,3}$*.*

There exist precisely six dichotomial 4*-spheres; their* 1*-skeleta are precisely the graphs of the Petersen family apart from* $K_{4,4} \setminus e$*.*

Example. K_5 is the 1-skeleton of $\partial\Delta^4$, and K_6 is the 1-skeleton of $\partial\Delta^5$.

Example. The dichotomial 3-sphere whose 1-skeleton is $K_{3,3}$

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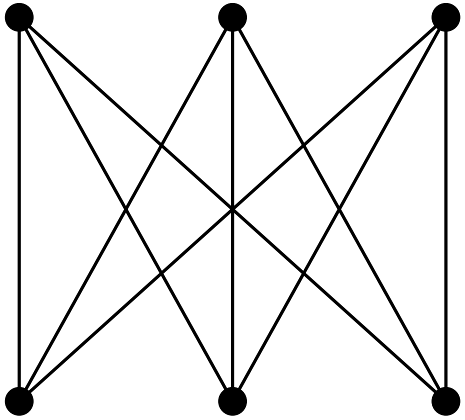
We start with the $K_{3,3}$ and attach to it 2-cells and 3-cells



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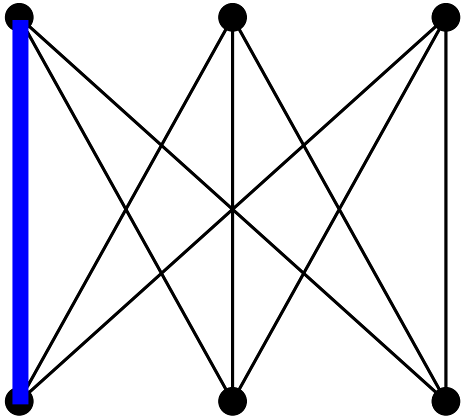

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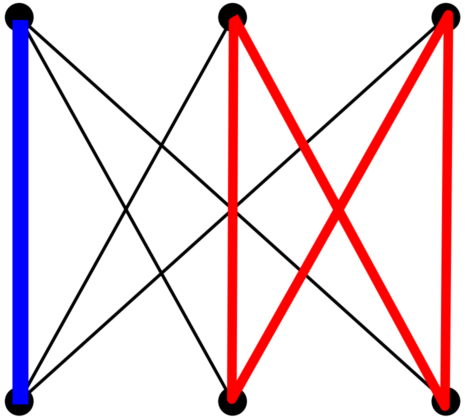

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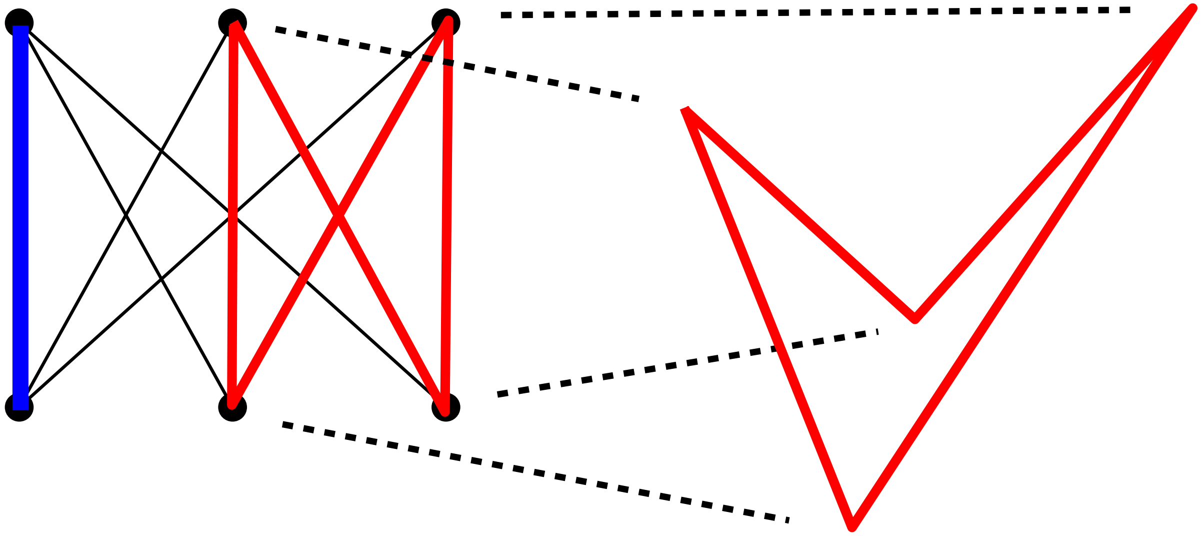

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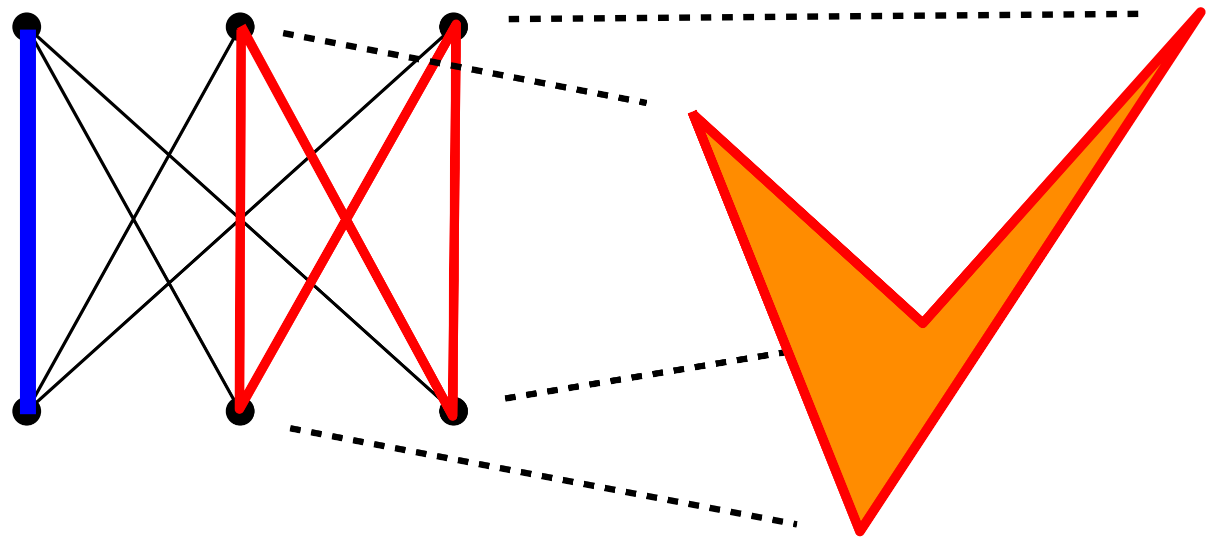

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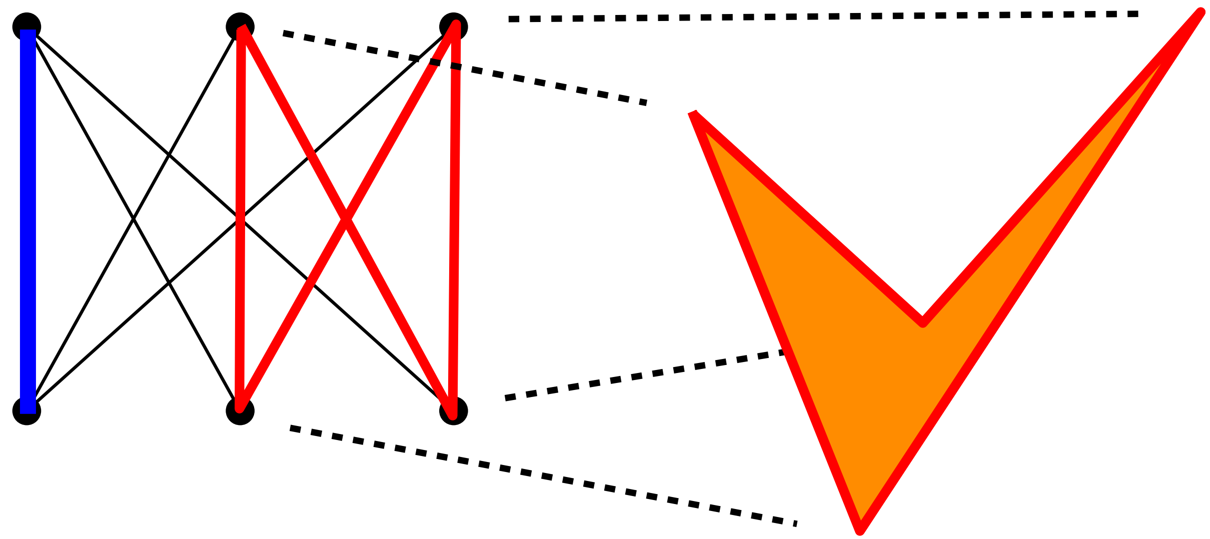

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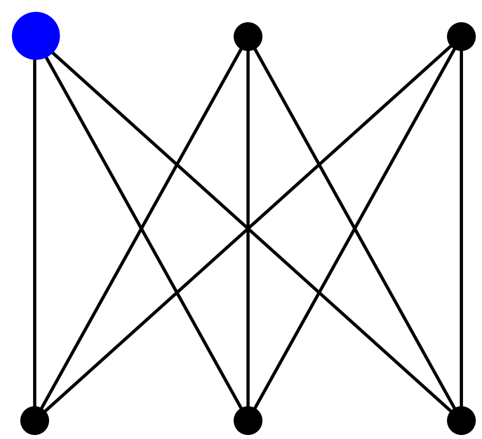

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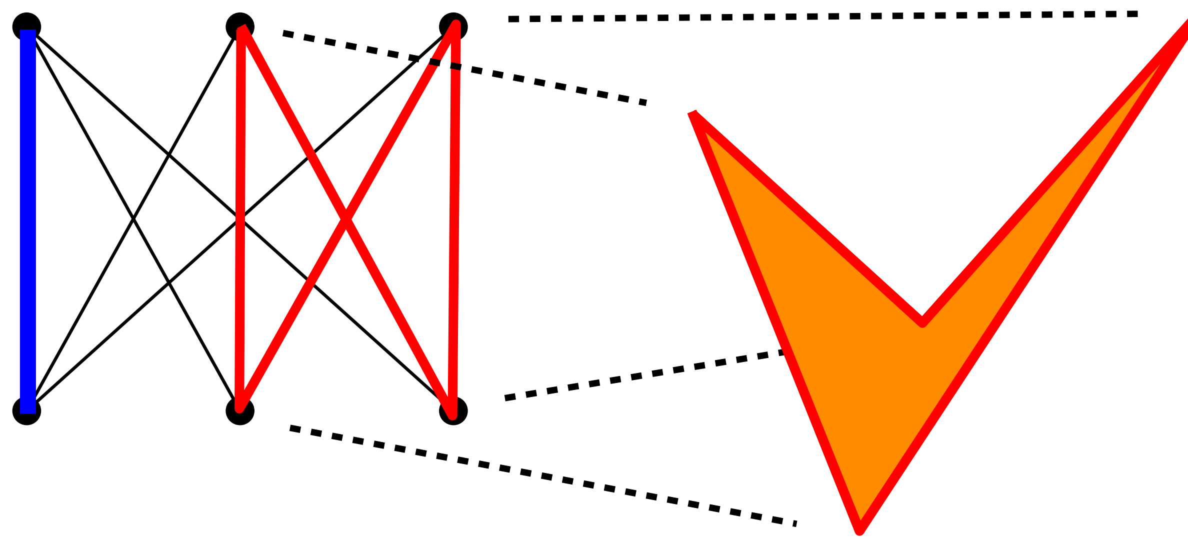

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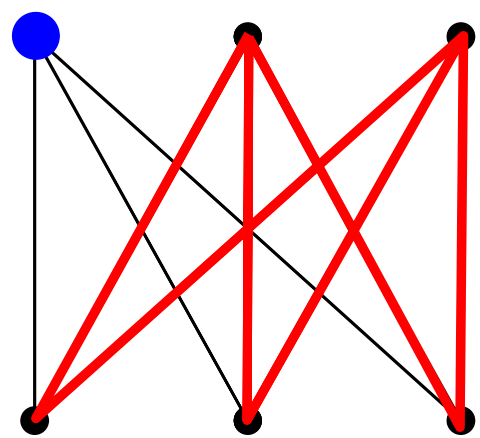

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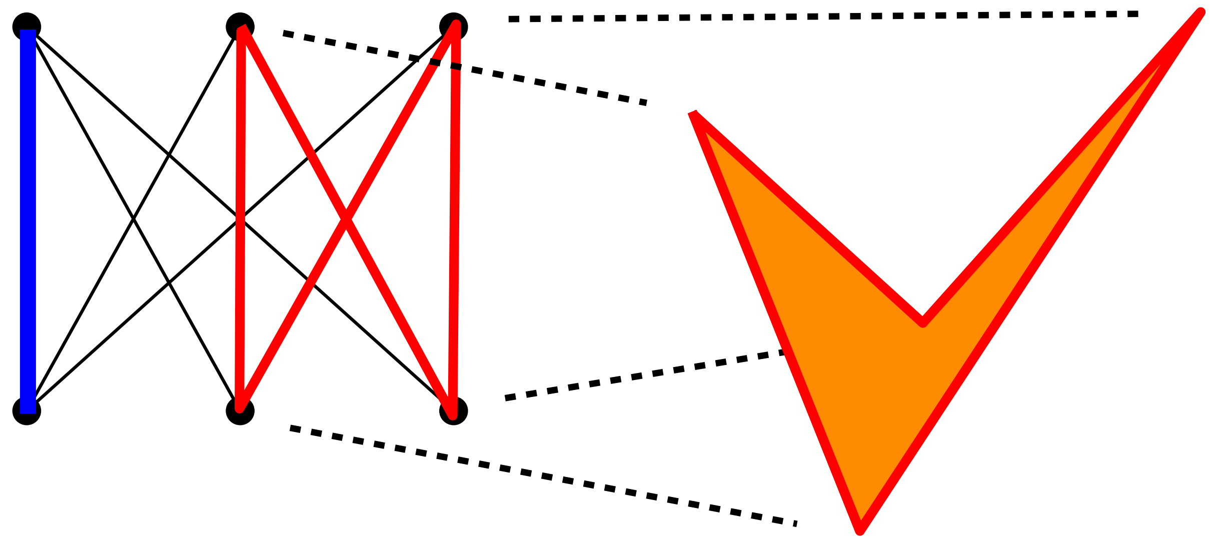

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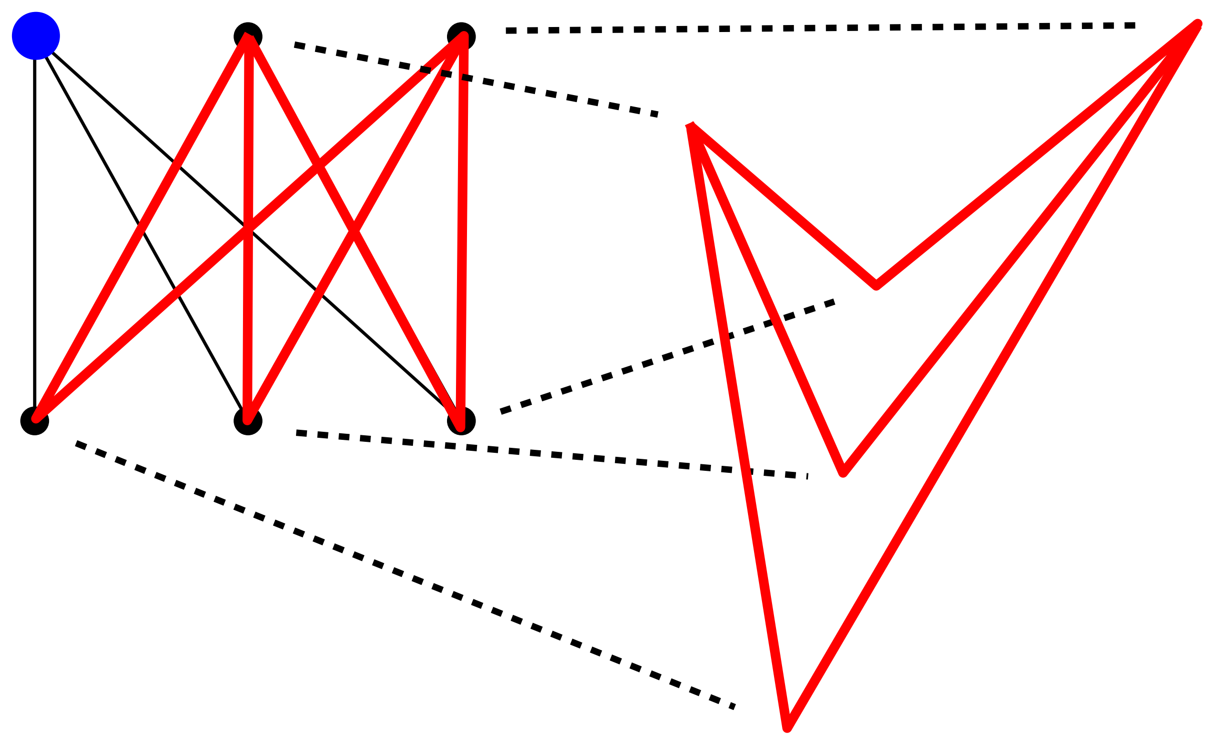

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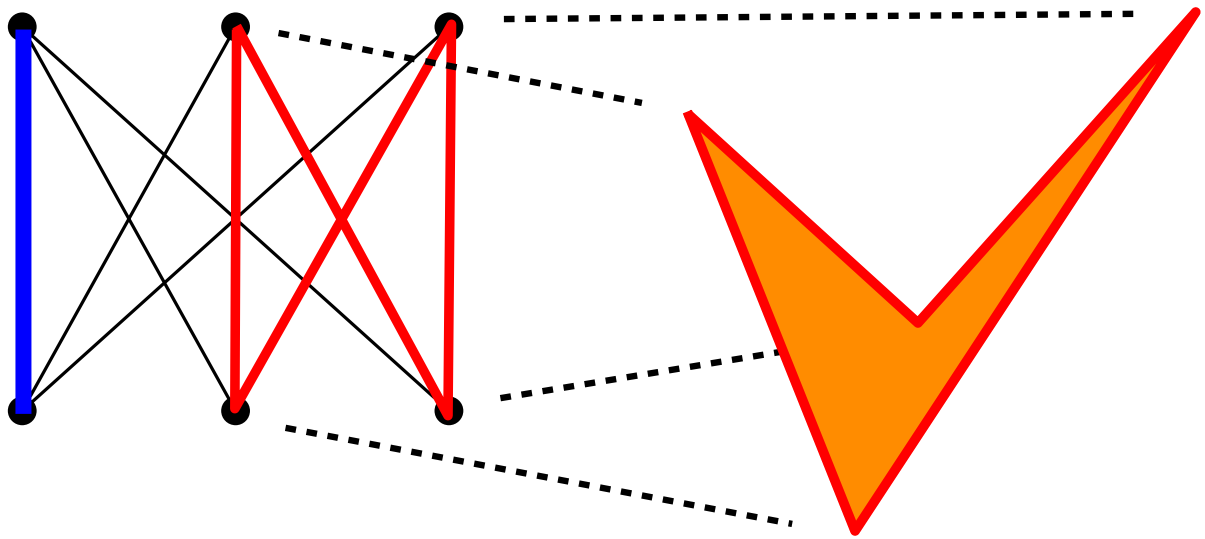

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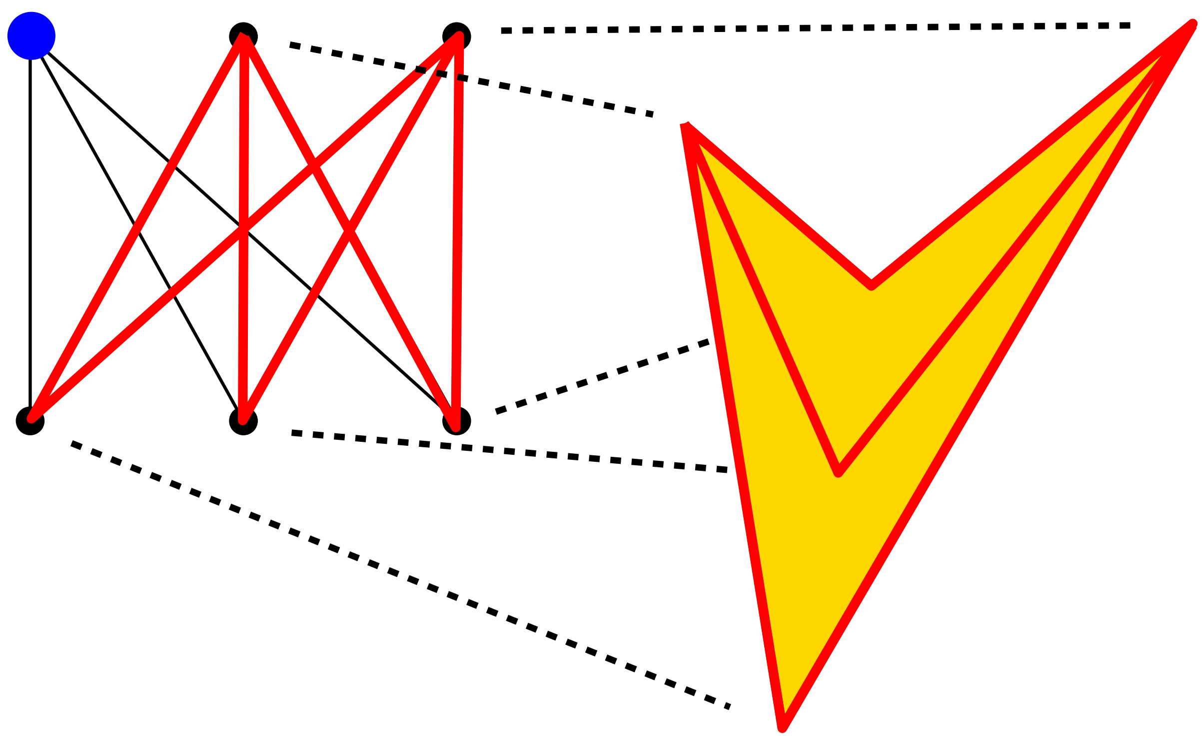

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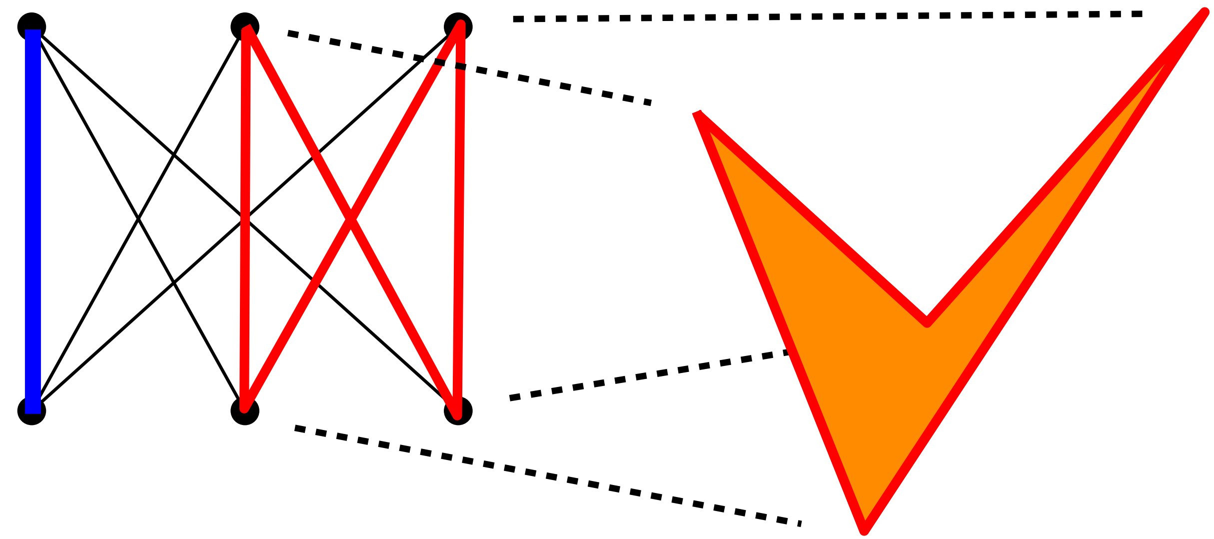

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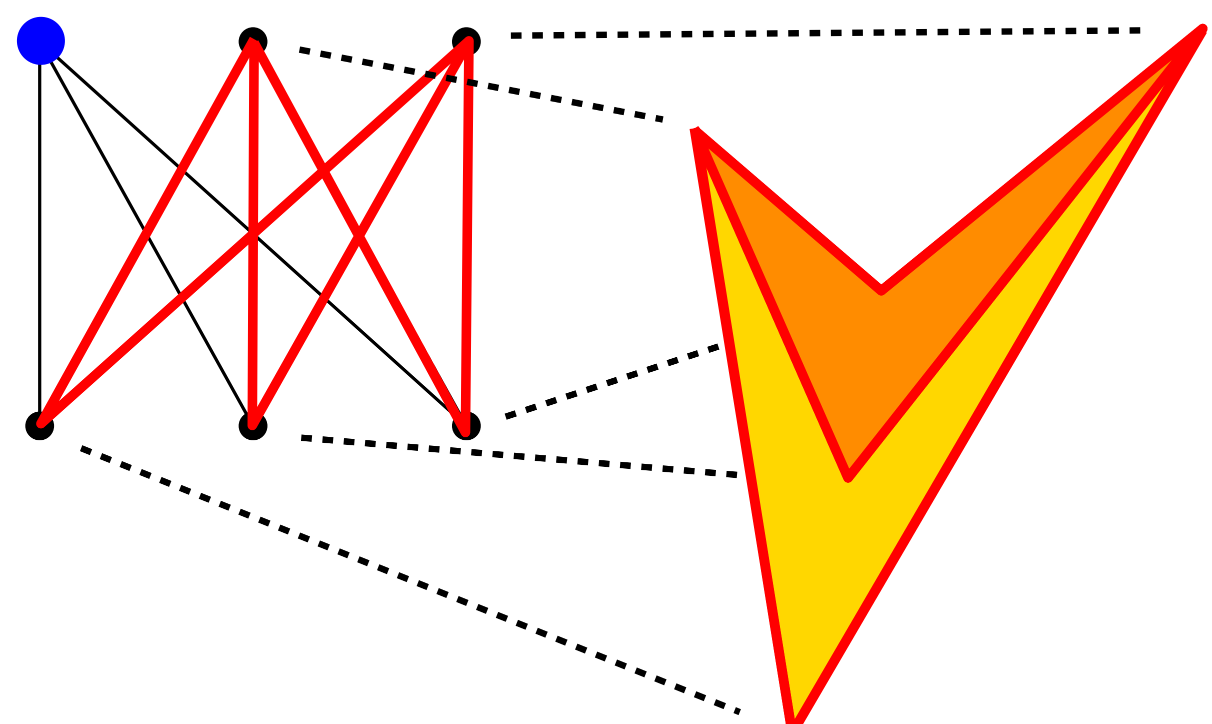

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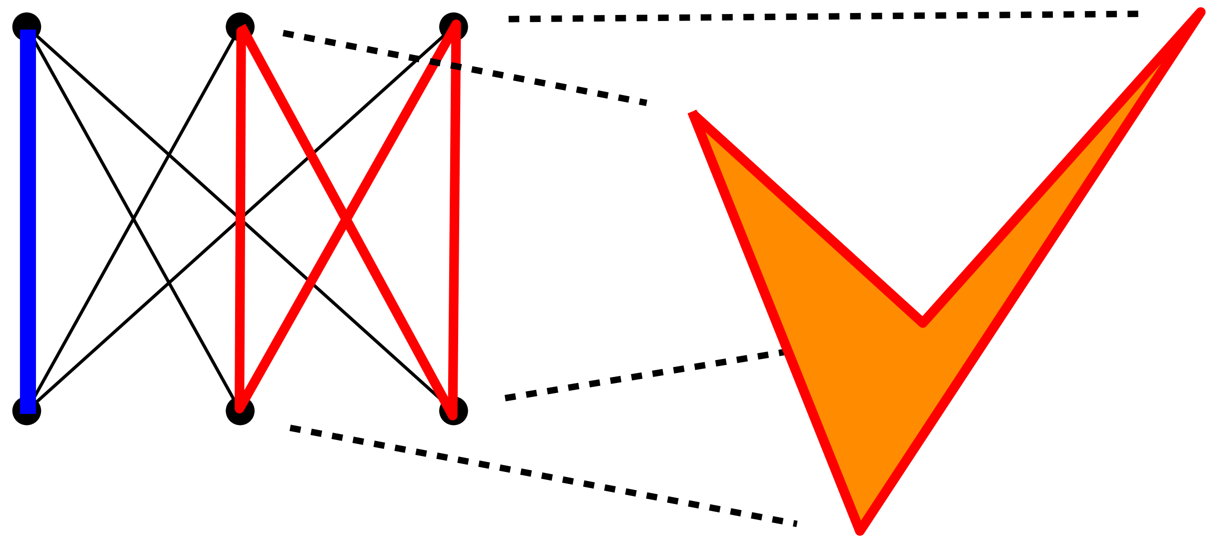

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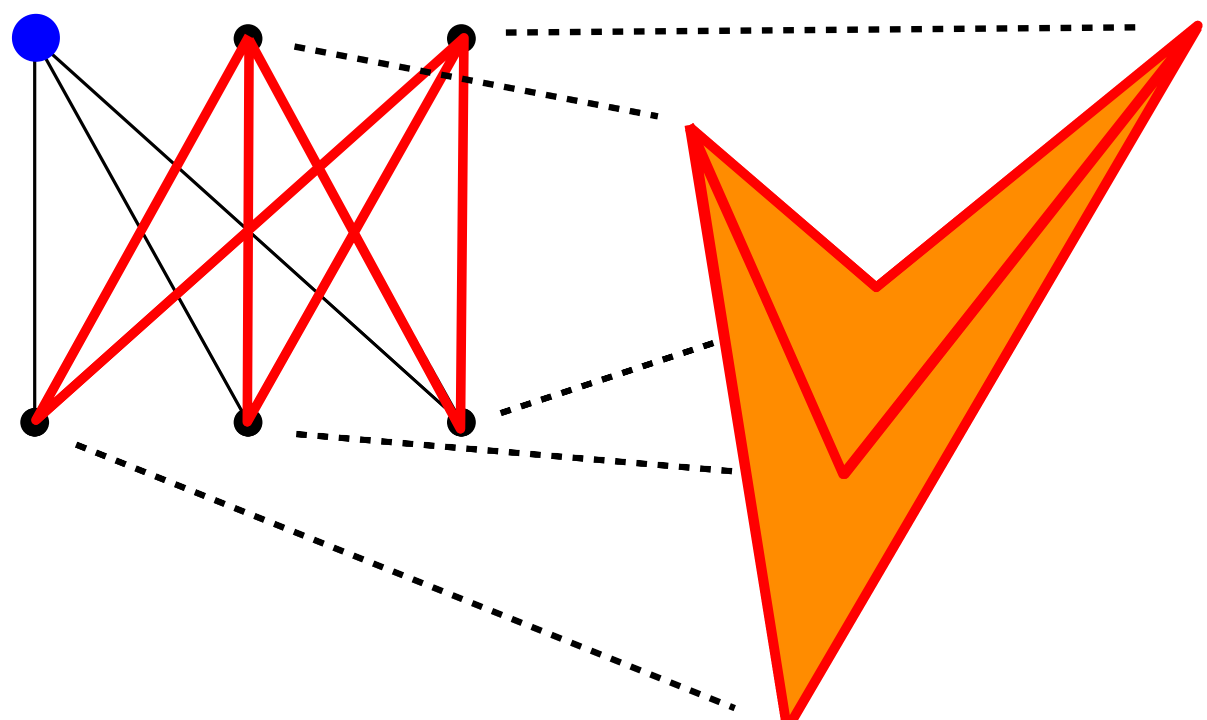

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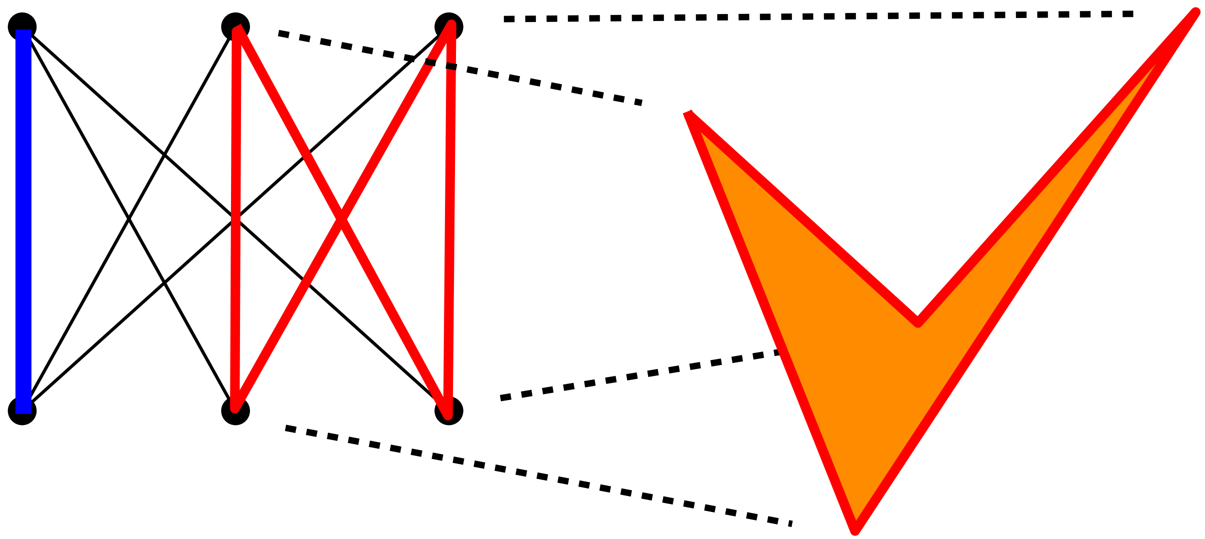

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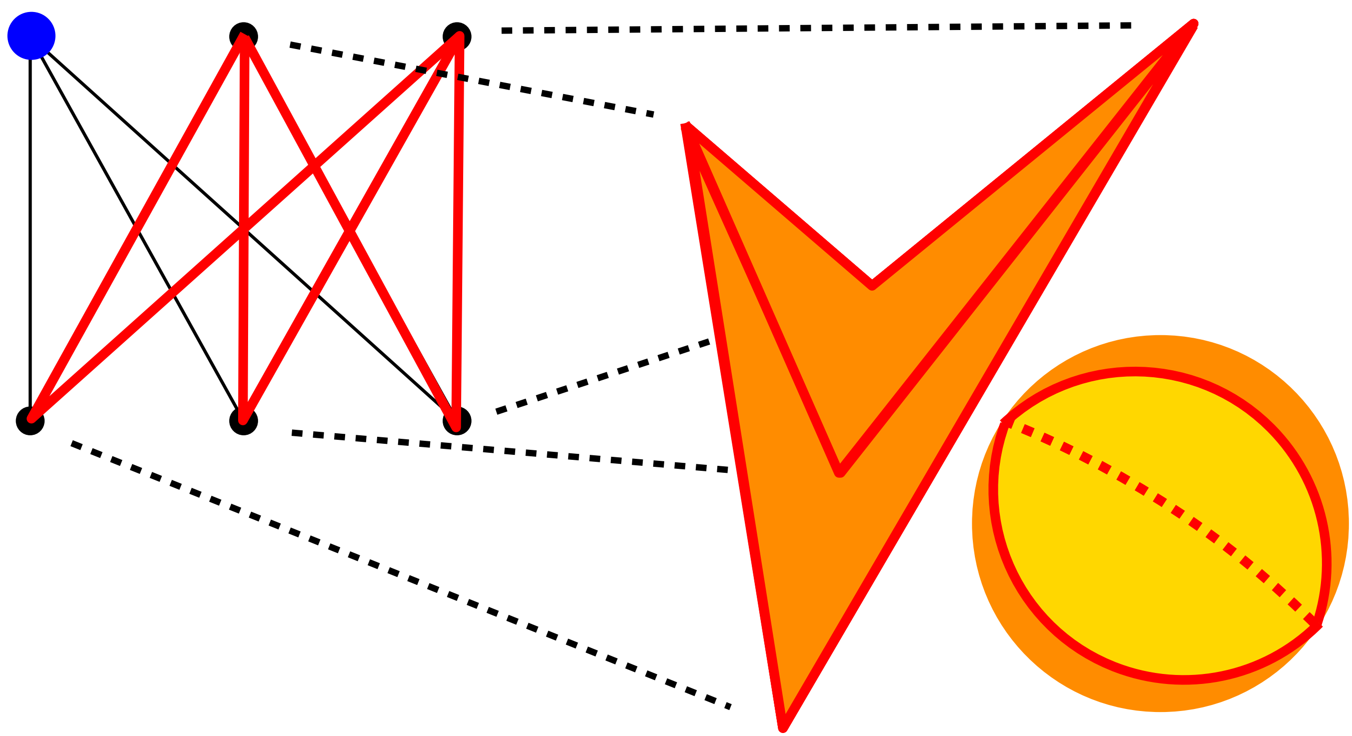

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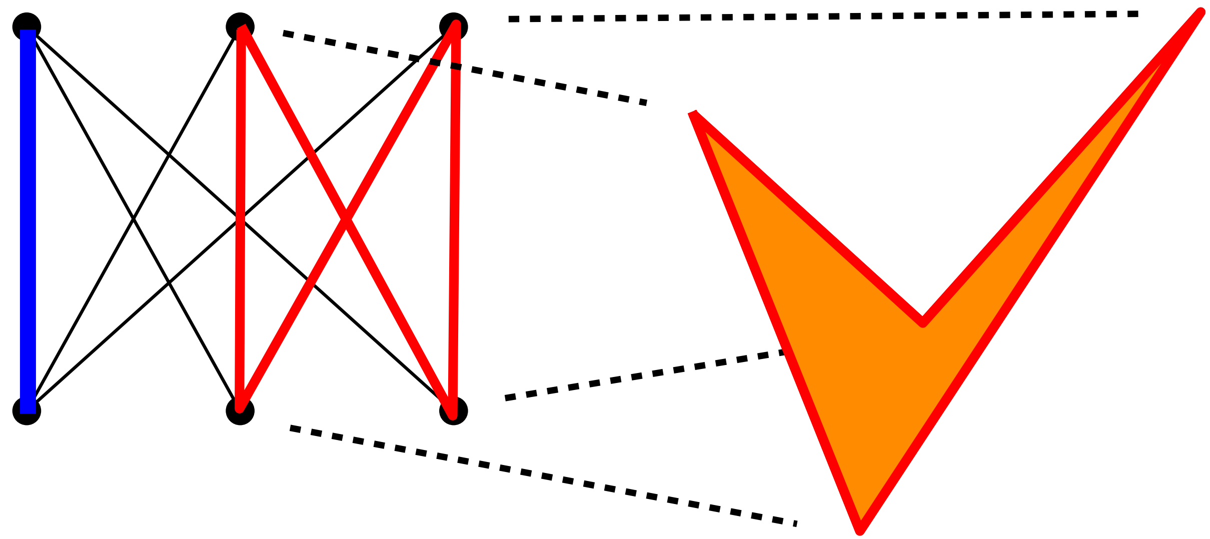

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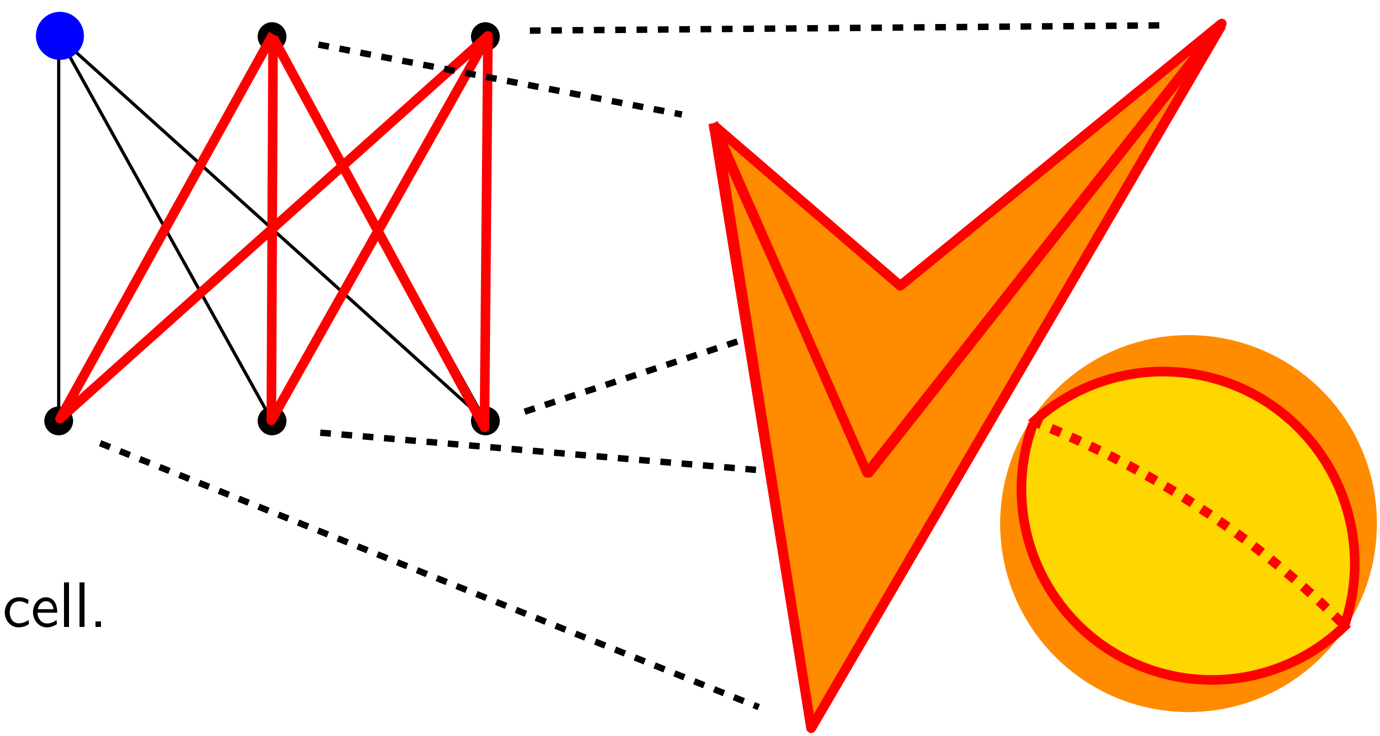

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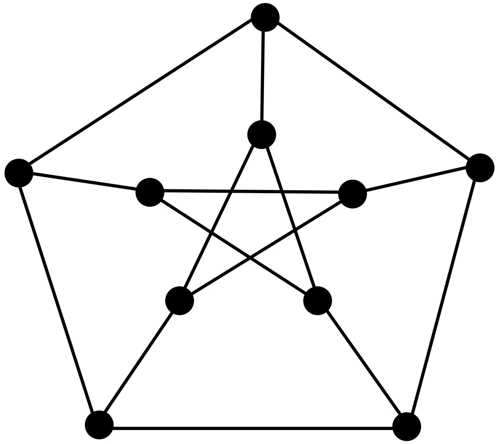

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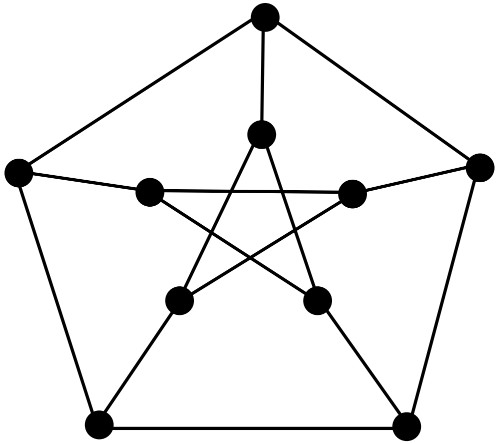
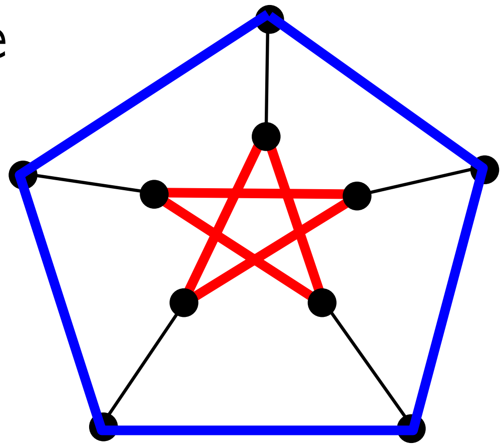
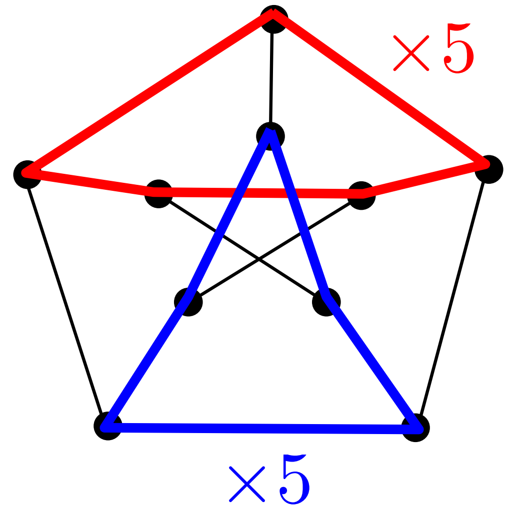


There are twelve 5-cycles.





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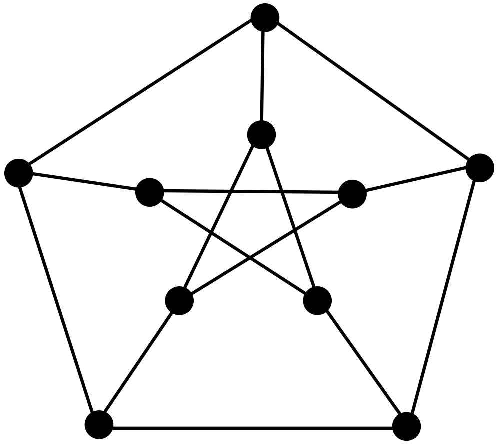

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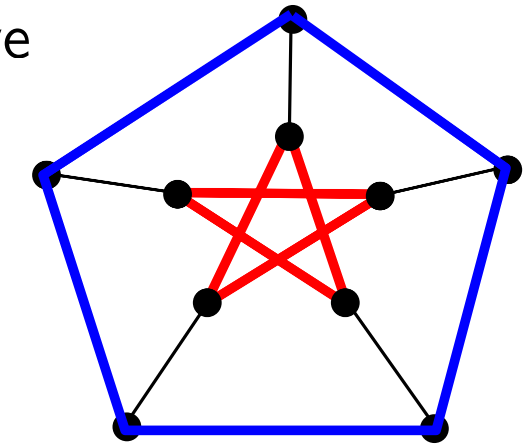
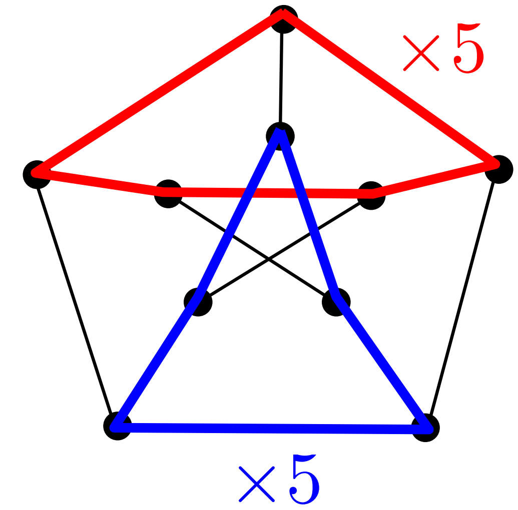
All of them are glued up by 2-cells.





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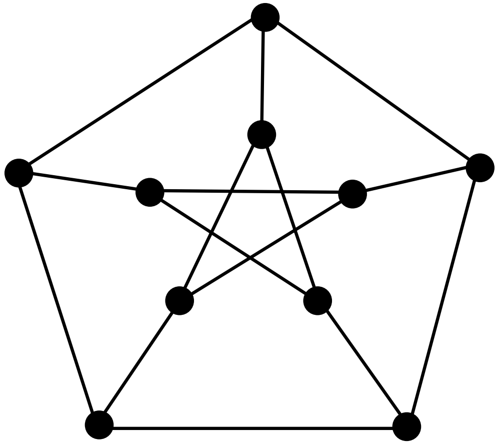

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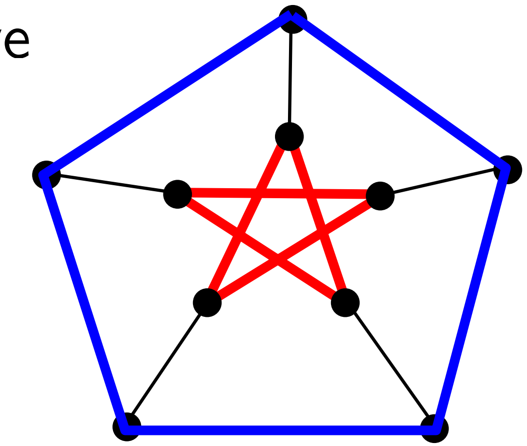

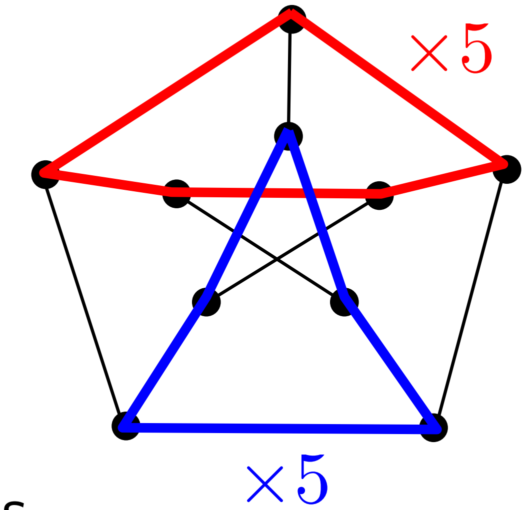

Now 2-cells are already split into pairs of opposite cells.

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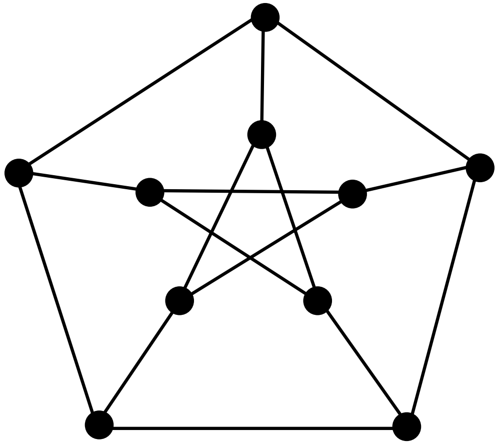

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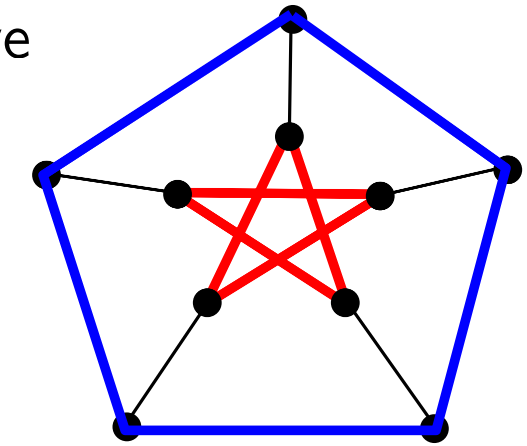

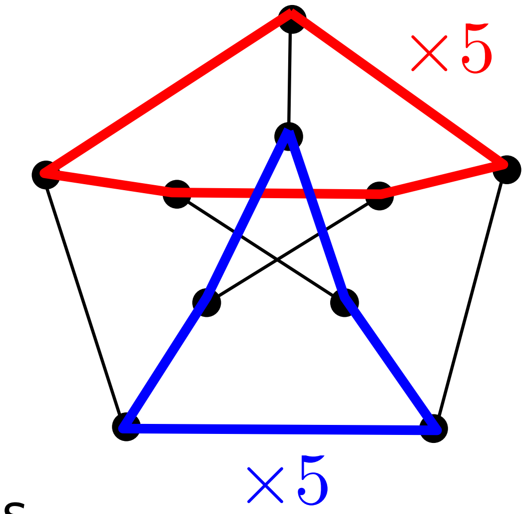

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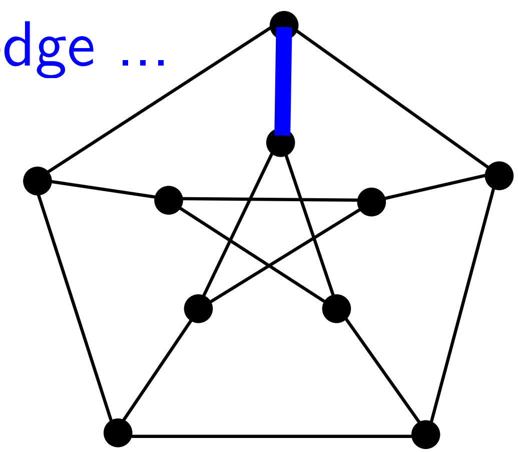

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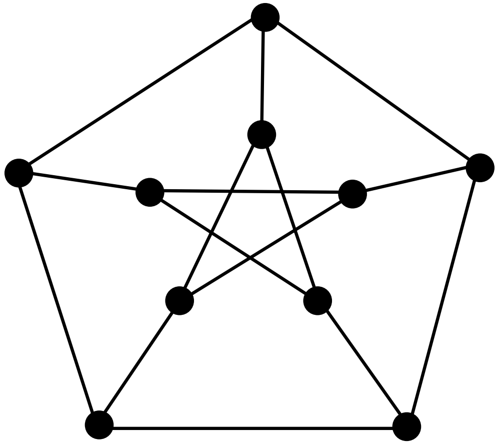

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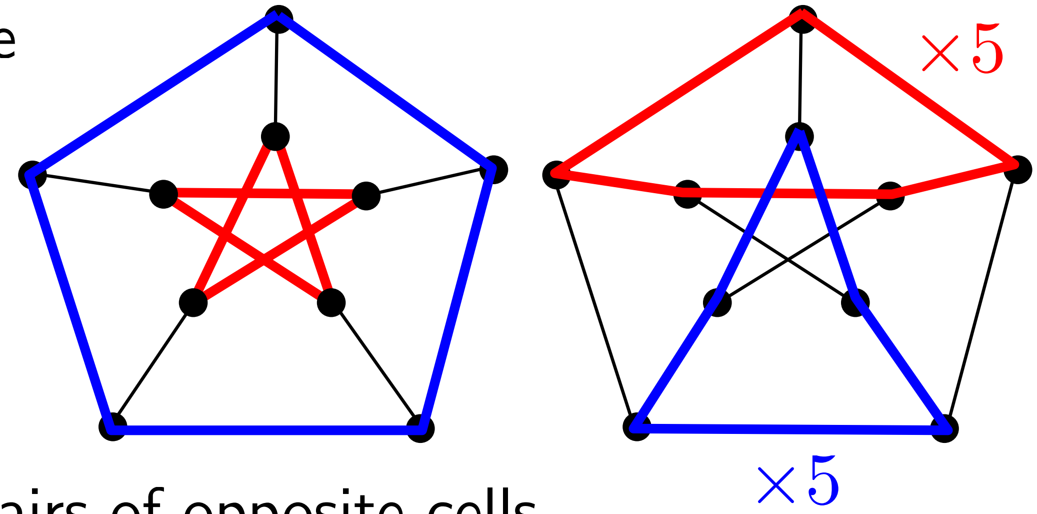

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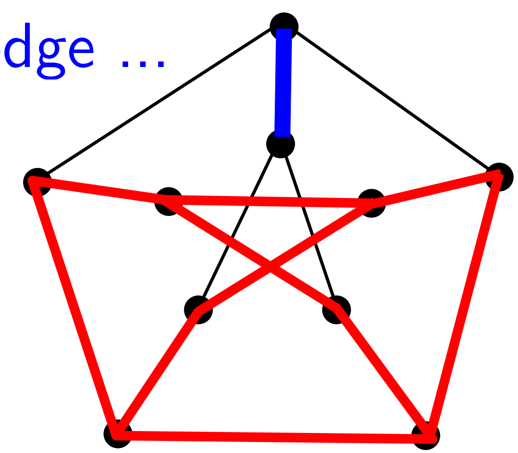

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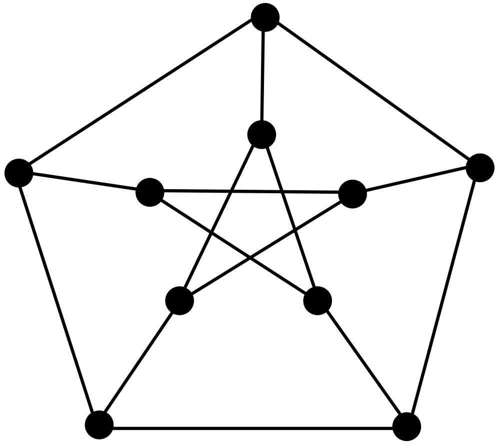

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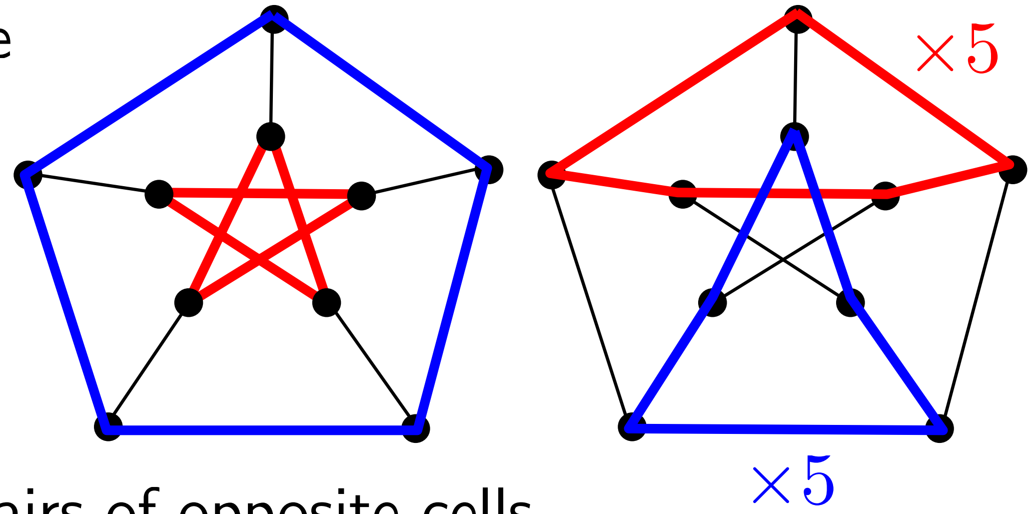

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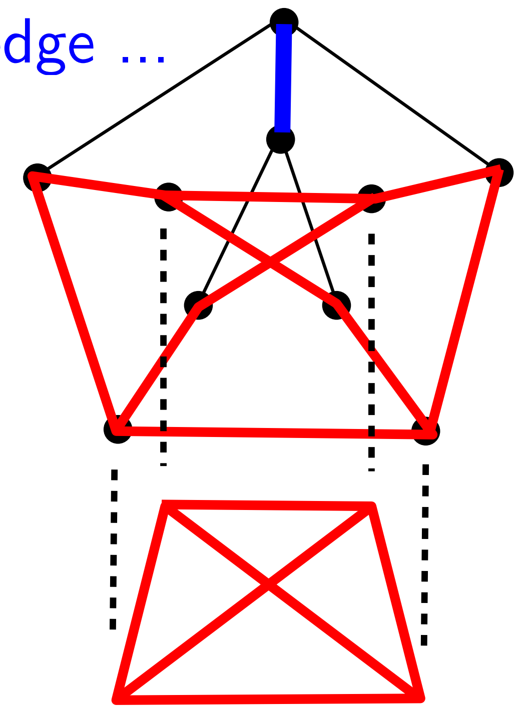
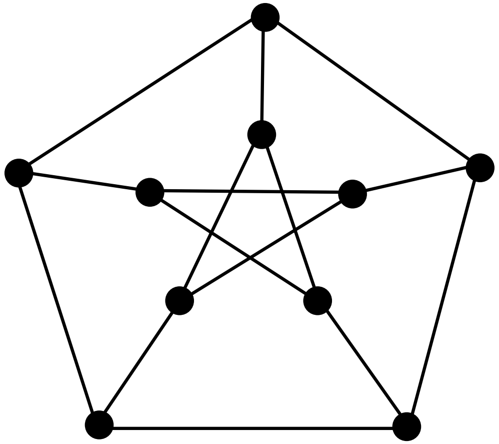
Opposite to each edge ...

is a subdivision of the 1-skeleton of the 3-simplex ...

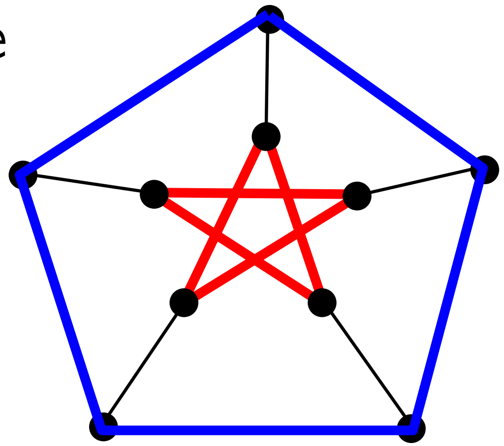


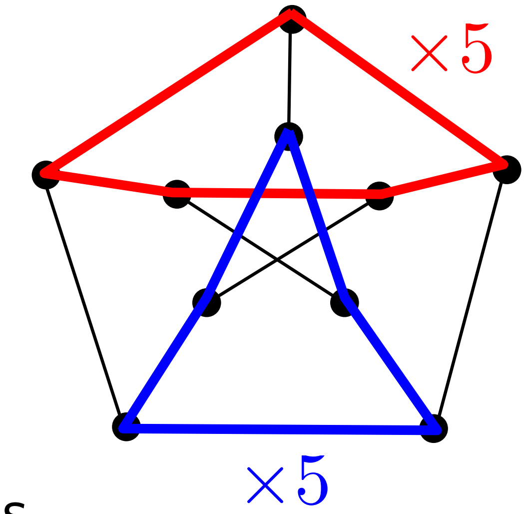
Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph



There are twelve 5-cycles.

All of them are glued up by 2-cells.





Now 2-cells are already split into pairs of opposite cells.

Opposite to each edge ...

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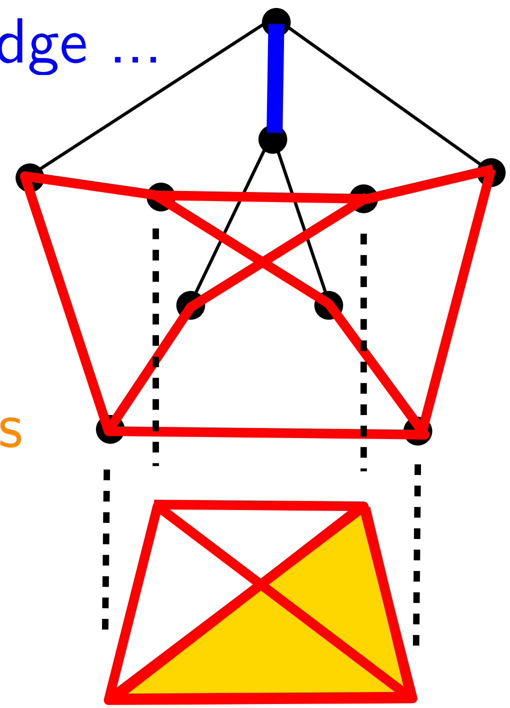
with missing 2-faces glued up by 2-cells.



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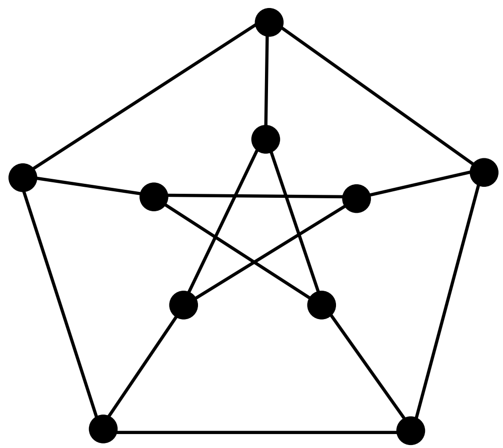

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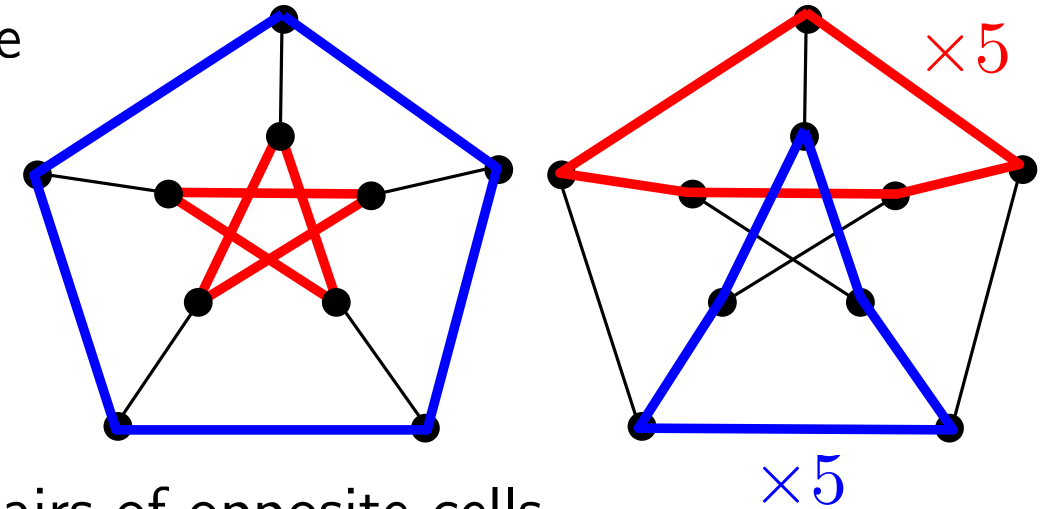

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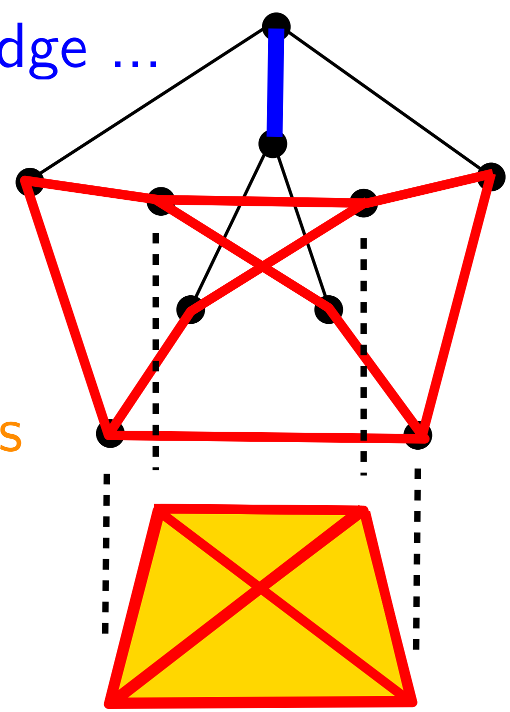

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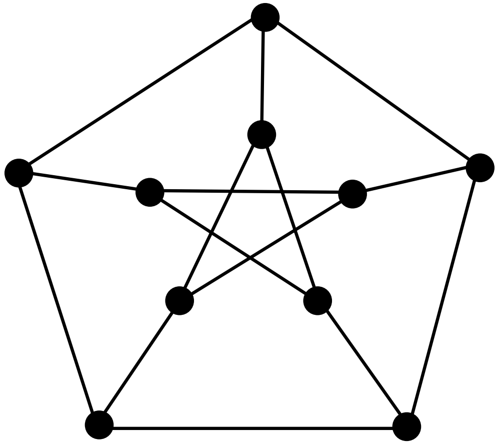

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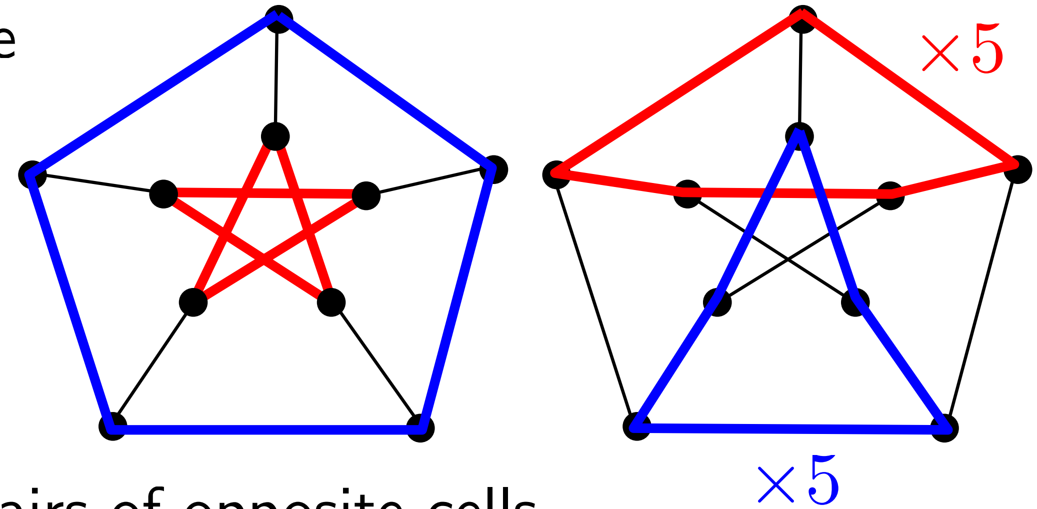

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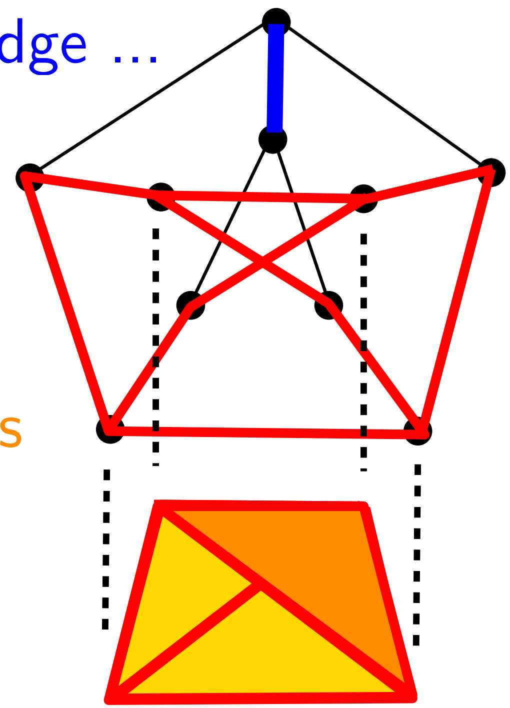

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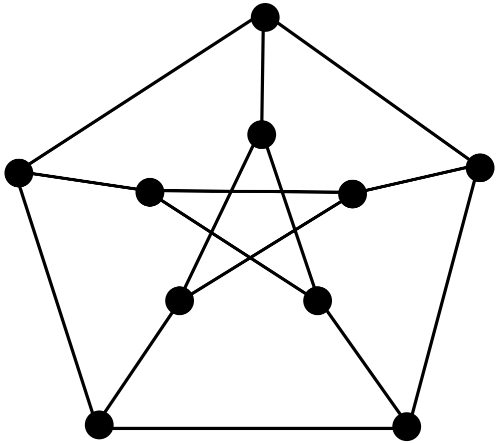

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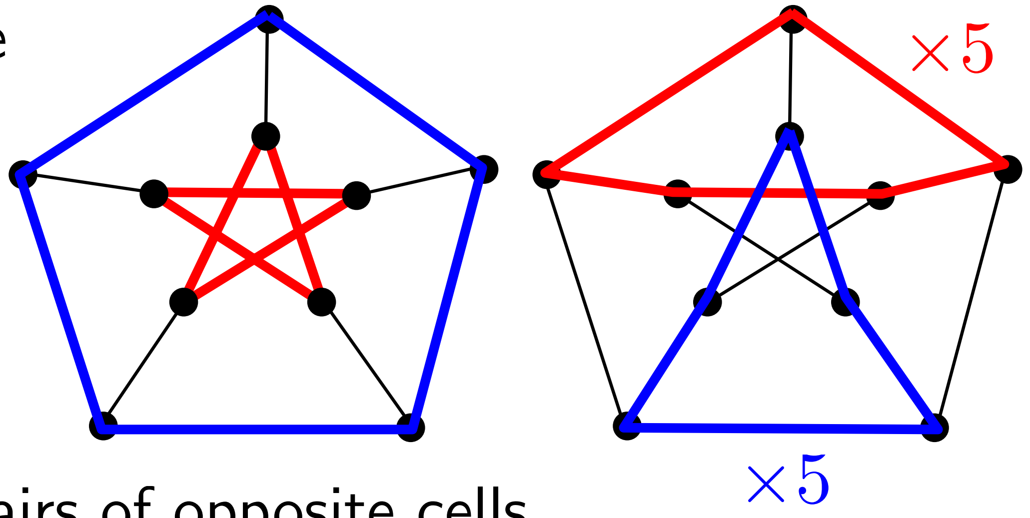

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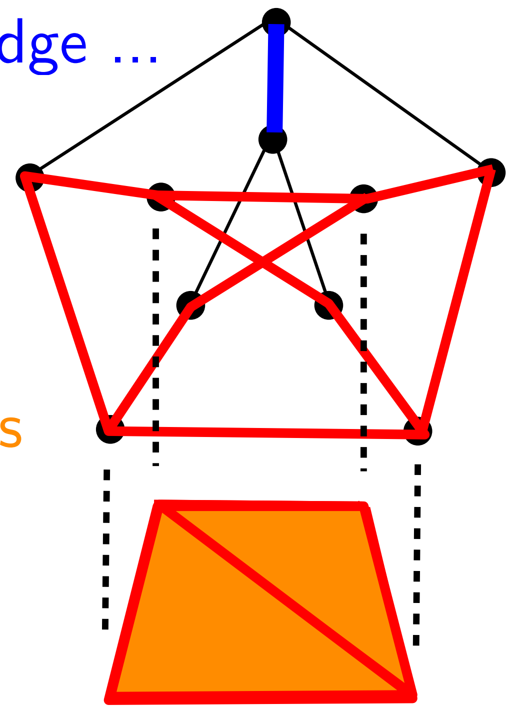

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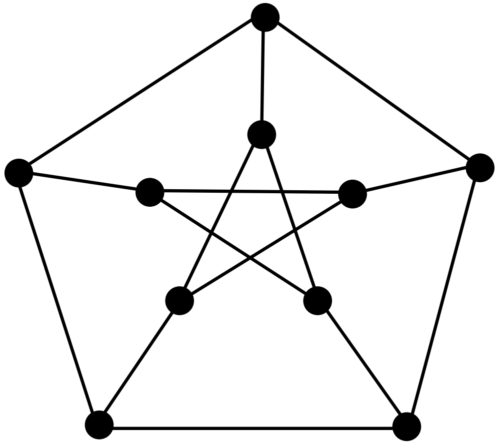

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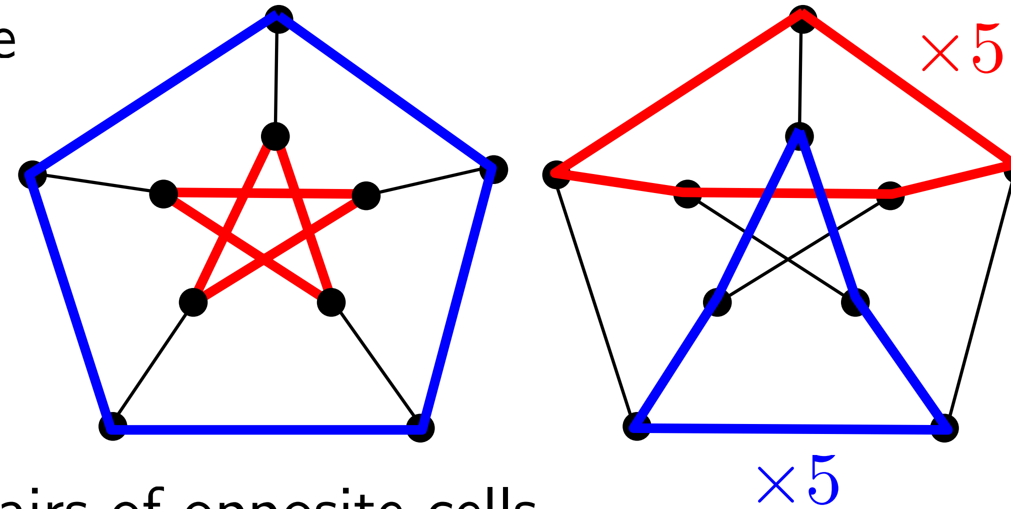

Now 2-cells are already split into pairs of opposite cells.

Opposite to each edge ...

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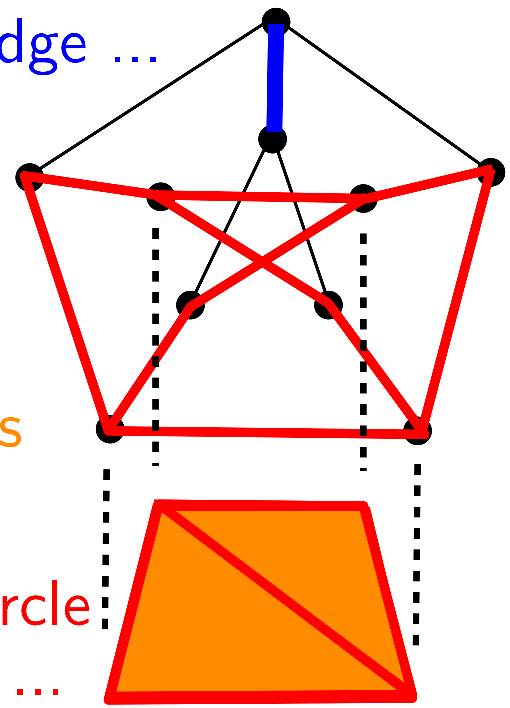
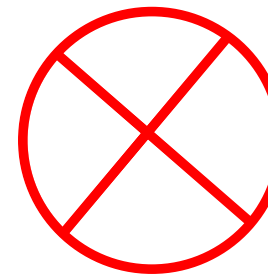
In other words, a circle with two diameters ...





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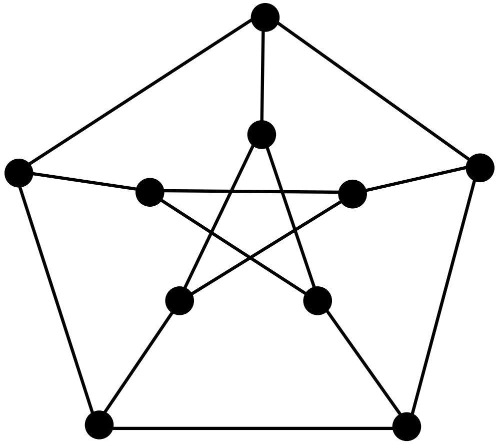

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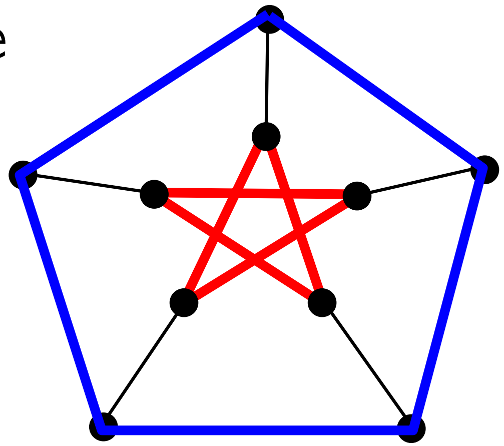

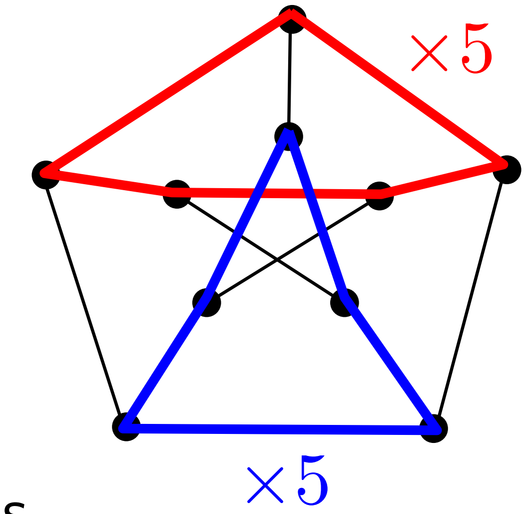

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Opposite to each edge ...

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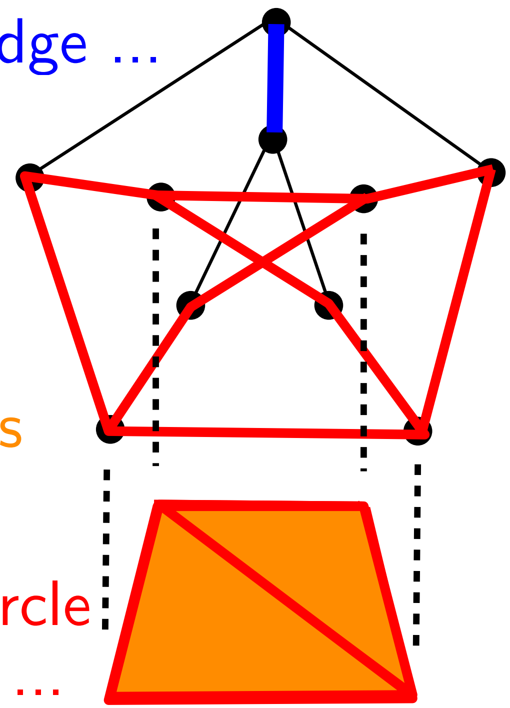
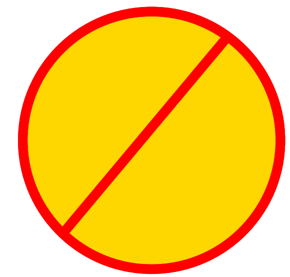
In other words, a circle with two diameters ...

which is glued up by two disks.





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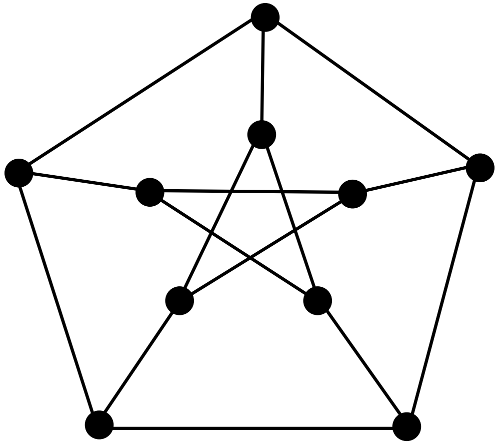

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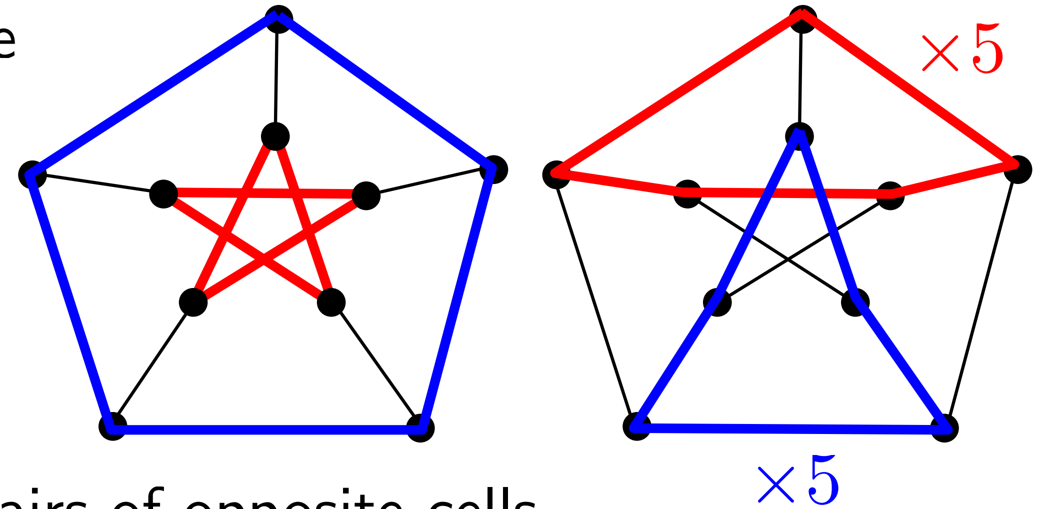

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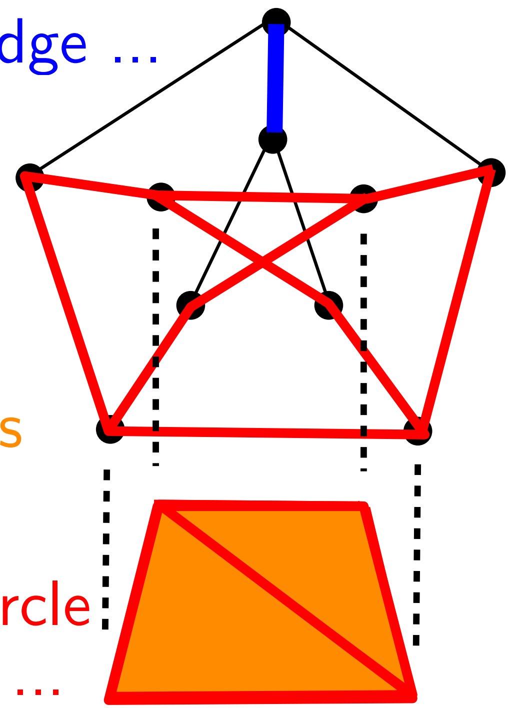

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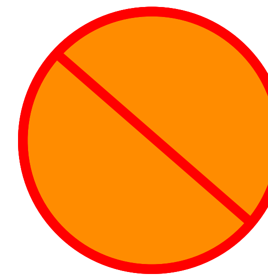
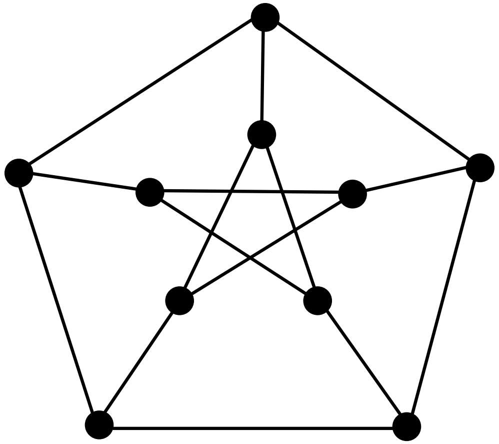


which is glued up by two disks.

The resulting 2-sphere is glued up by a 3-ball.

Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph



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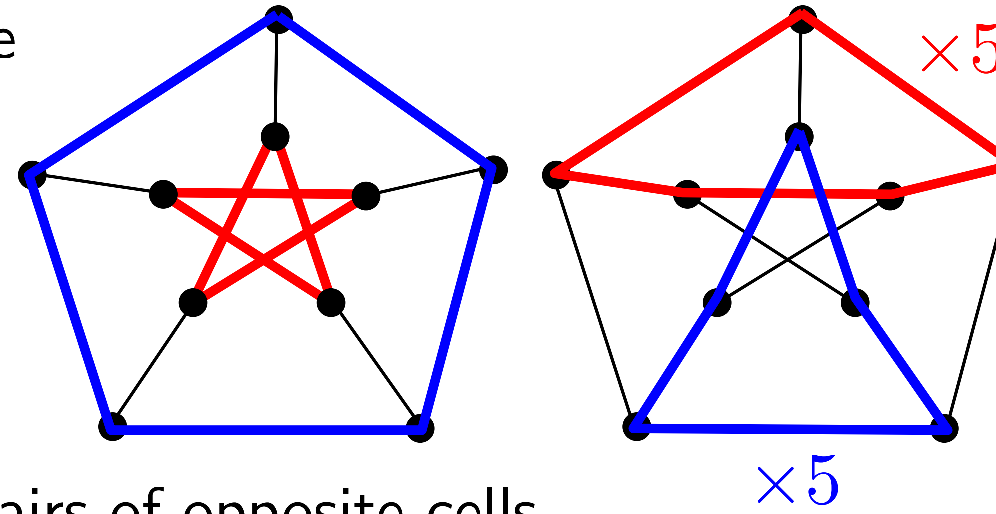

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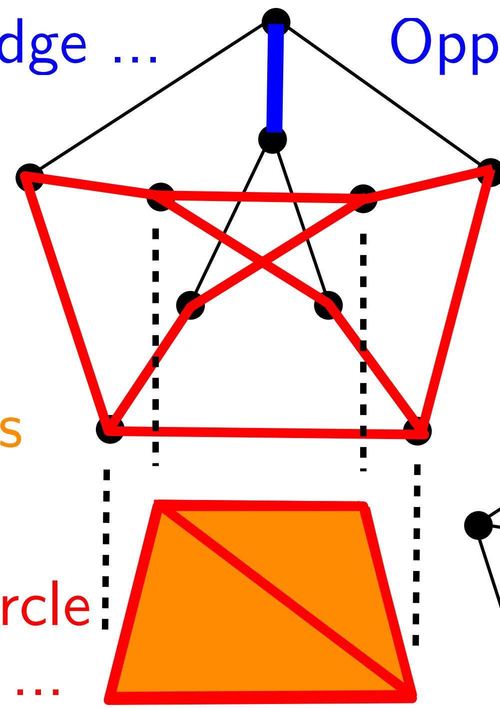
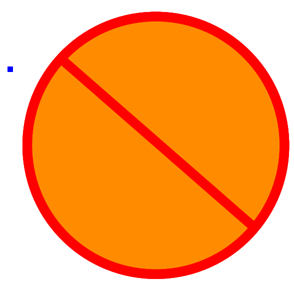
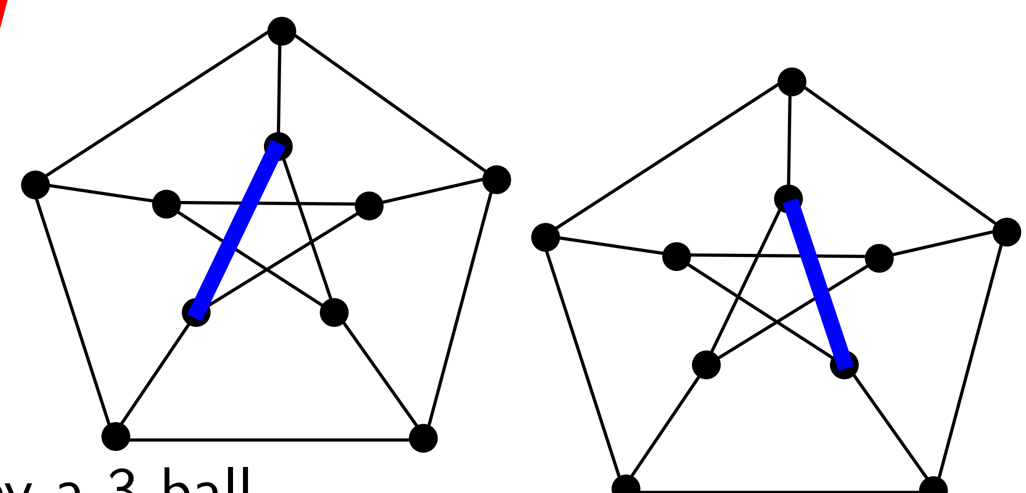


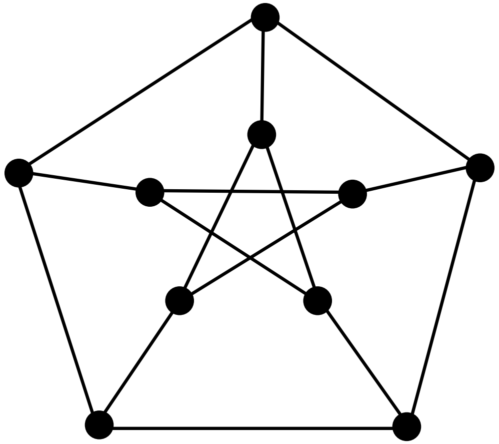
Opposite to adjacent edges ...





The resulting 2-sphere is glued up by a 3-ball.

Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph



There are twelve 5-cycles.

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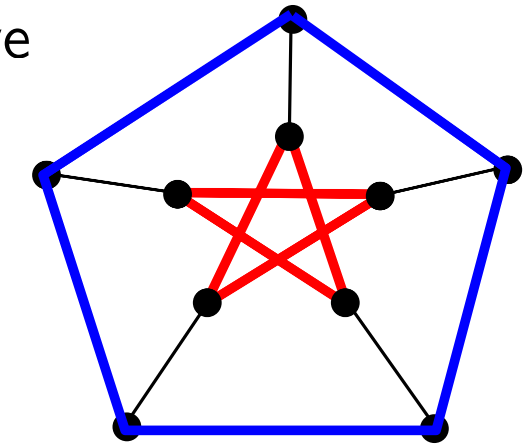

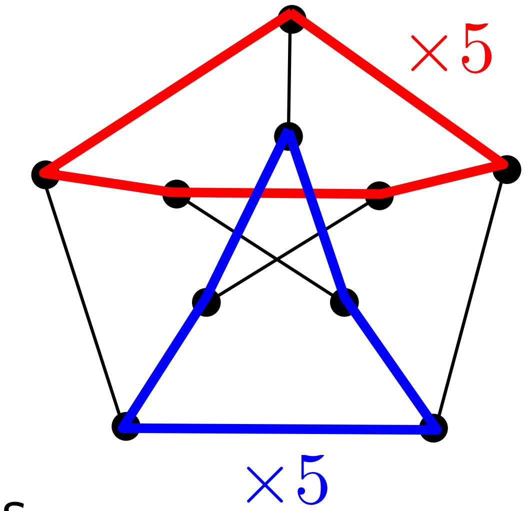

Now 2-cells are already split into pairs of opposite cells.

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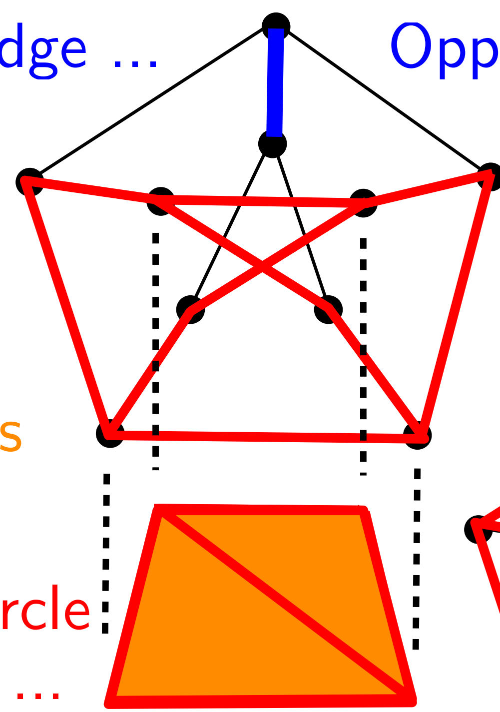

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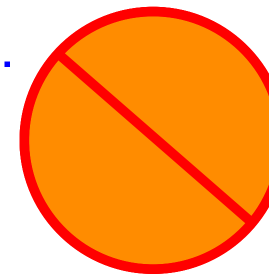
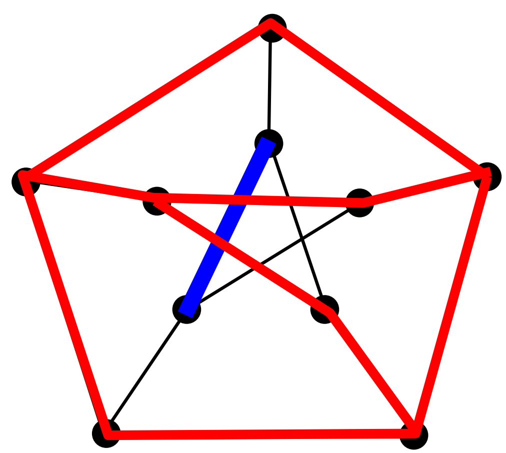


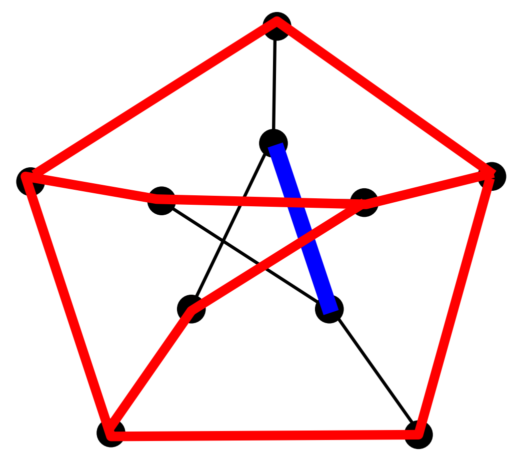
is a subdivision of the 1-skeleton of the 3-simplex ...

with missing 2-faces glued up by 2-cells.

In other words, a circle with two diameters ...

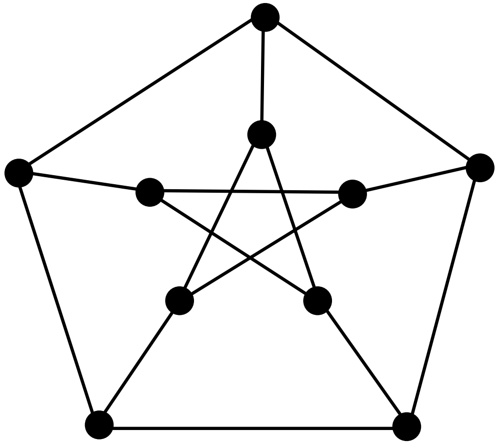
which is glued up by two disks.





The resulting 2-sphere is glued up by a 3-ball.

Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph



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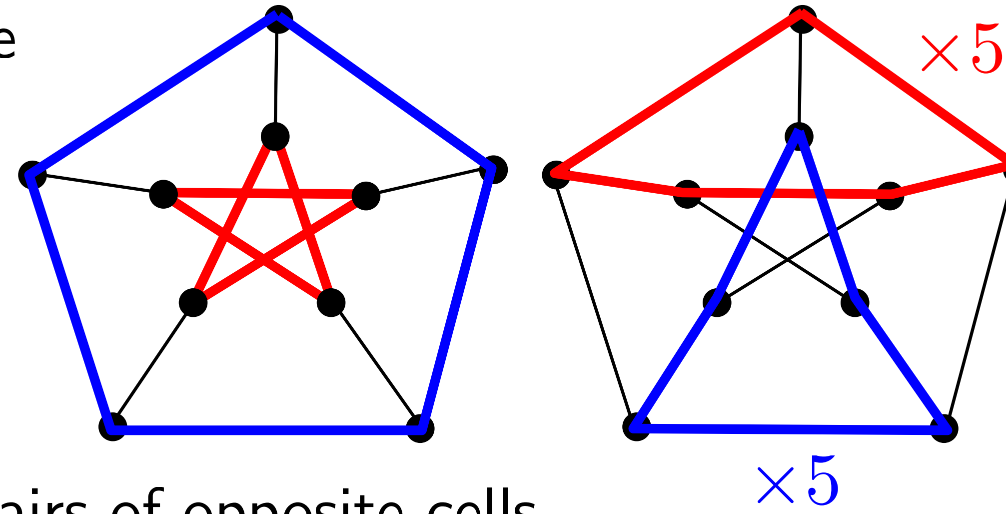

Now 2-cells are already split into pairs of opposite cells.

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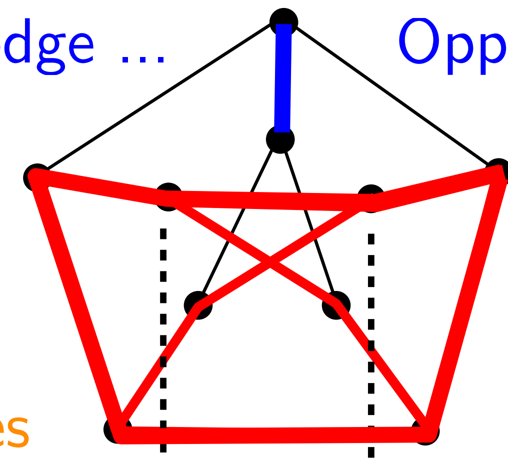

is a subdivision of the 1-skeleton of the 3-simplex ...

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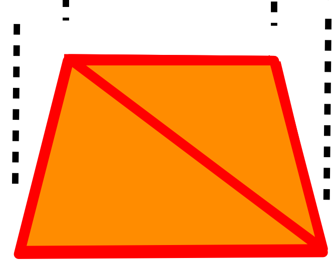
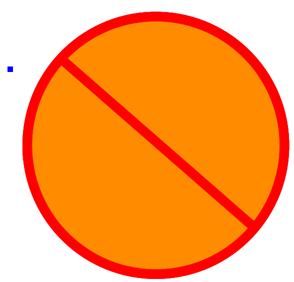


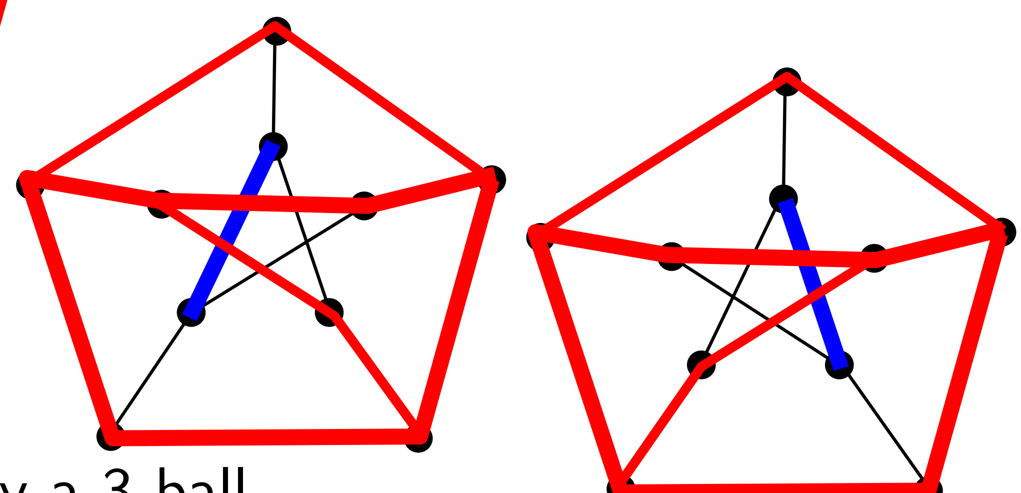
In other words, a circle with two diameters ...

which is glued up by two disks.

Opposite to adjacent edges ...

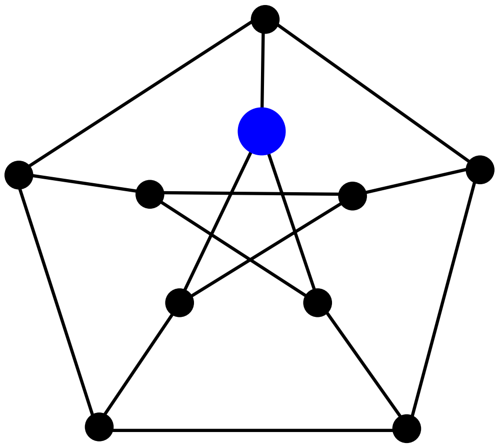


is the same circle with two diameters: the same one and a different one.



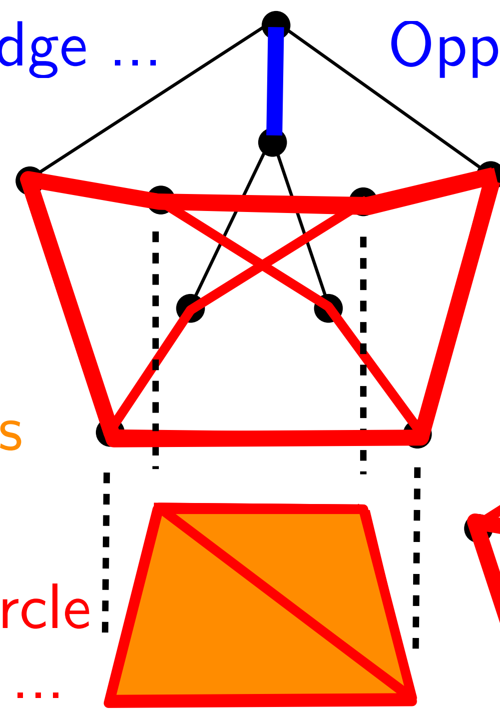
The resulting 2-sphere is glued up by a 3-ball.

Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph



Opposite to each vertex ...

Opposite to each edge ...



is a subdivision
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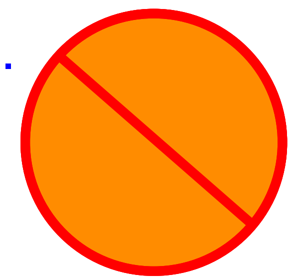
Opposite to adjacent edges ...



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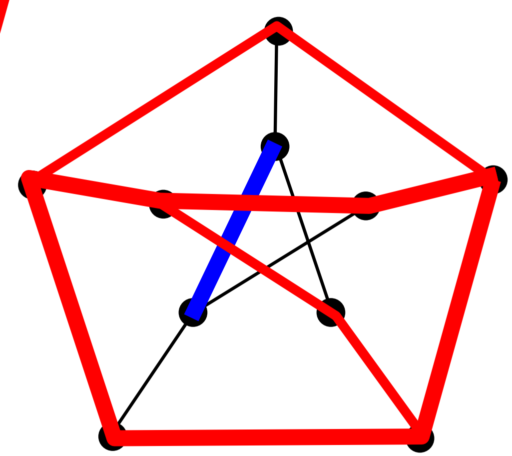
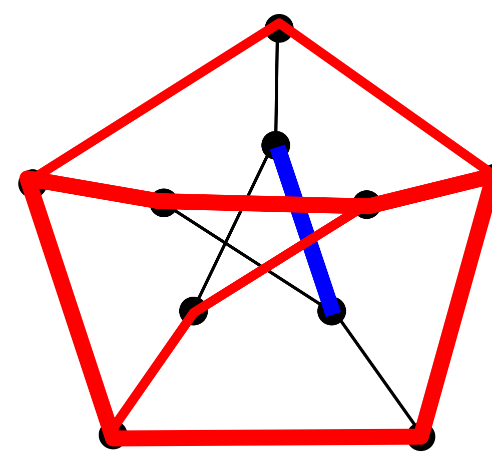
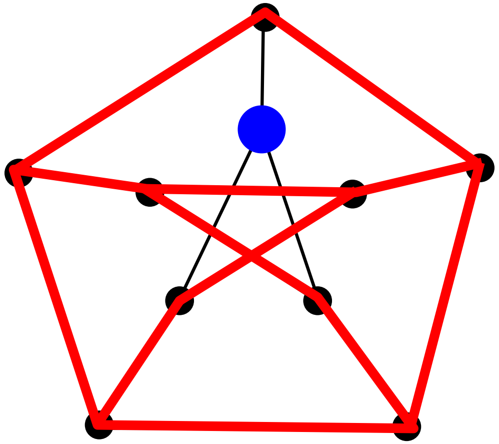




The resulting 2-sphere is glued up by a 3-ball.

Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph

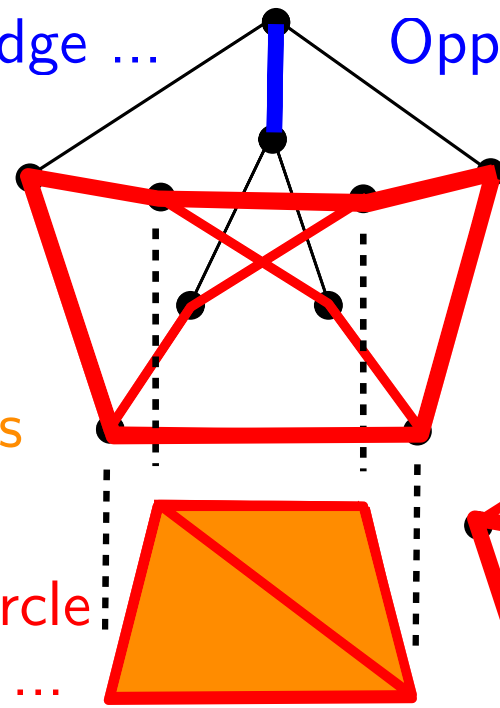


Opposite to each vertex ...

Opposite to each edge ...

is a subdivision
of the 1-skeleton
of the 3-simplex ...

with missing 2-faces
glued up by 2-cells.



In other words, a circle
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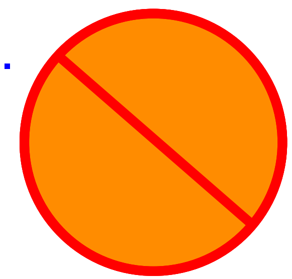
Opposite to adjacent edges ...



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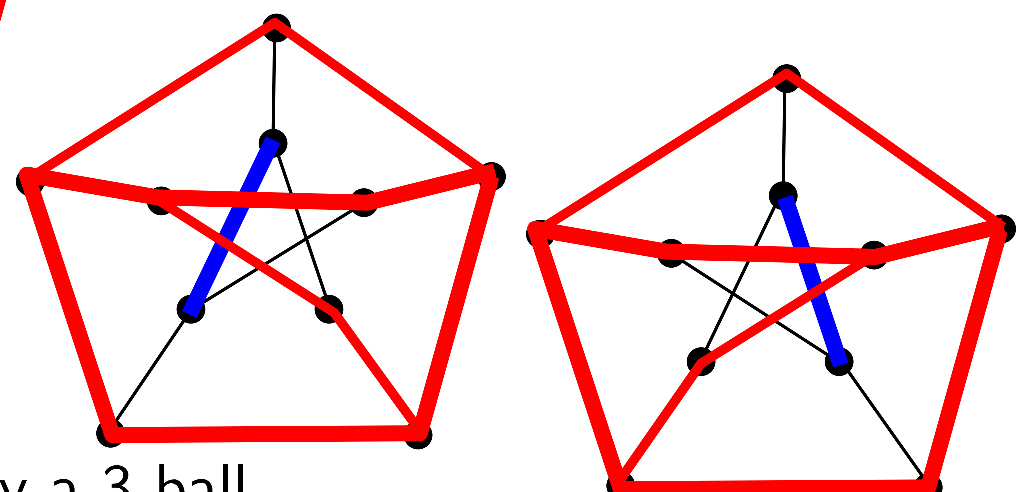

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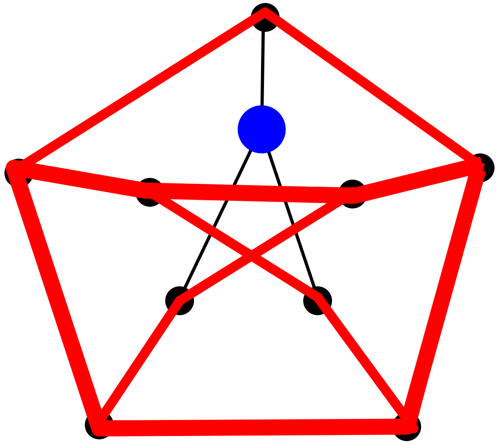
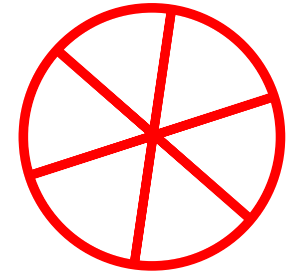


Opposite to each vertex ...

is a circle with three diameters ...



Opposite to each edge ...

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of the 3-simplex ...

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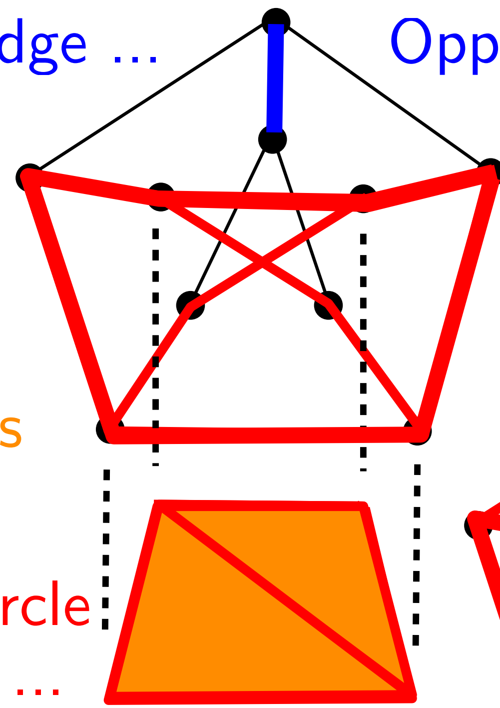
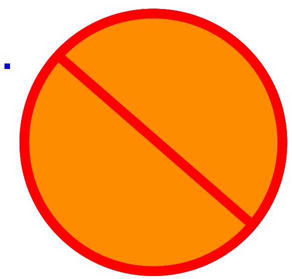
glued up by 2-cells.



In other words, a circle
with two diameters ...

which is glued up by two disks.

Opposite to adjacent edges ...



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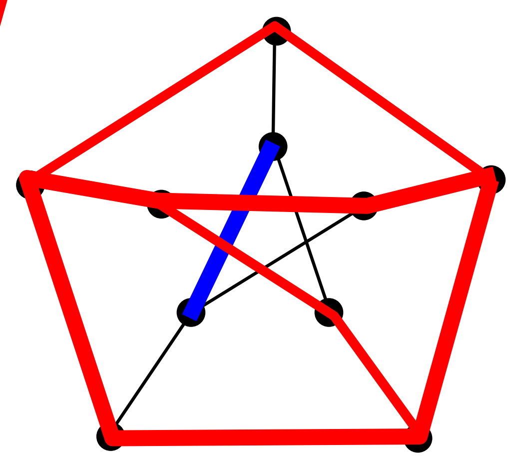
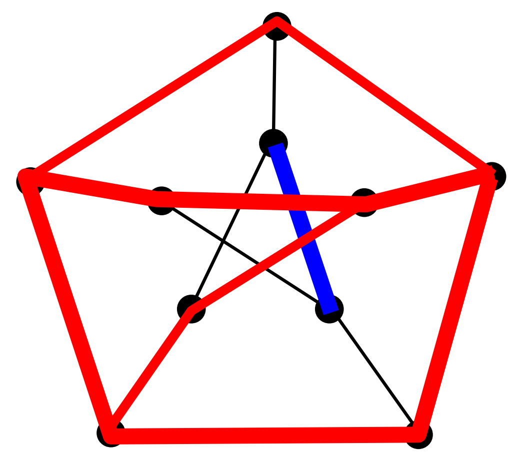
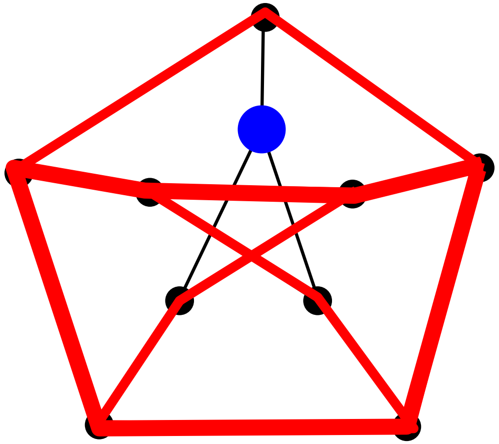




The resulting 2-sphere is glued up by a 3-ball.

Example. The dichotomial 4-sphere whose 1-skeleton is the Petersen graph



Opposite to each vertex ...

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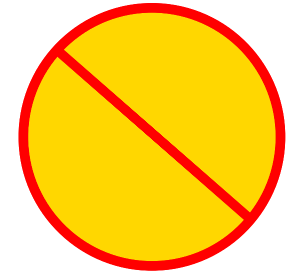

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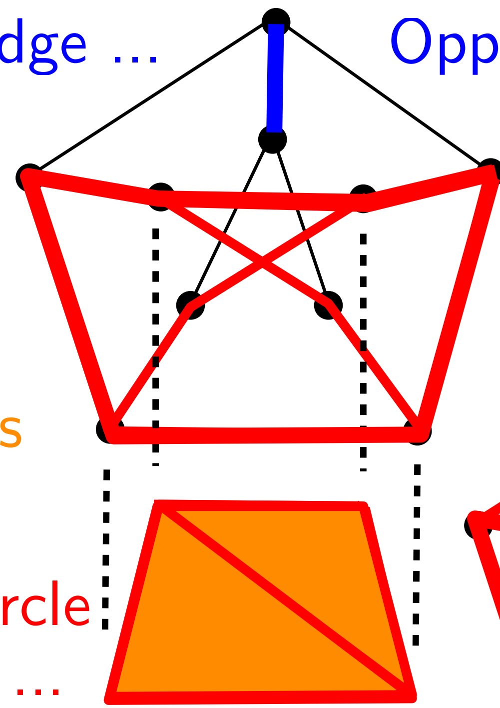

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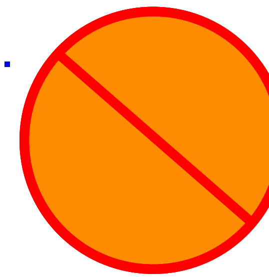

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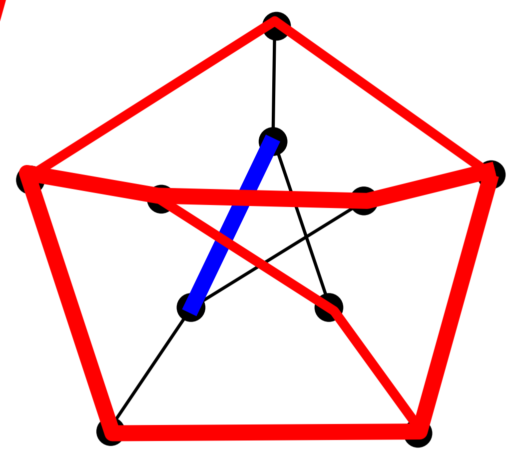

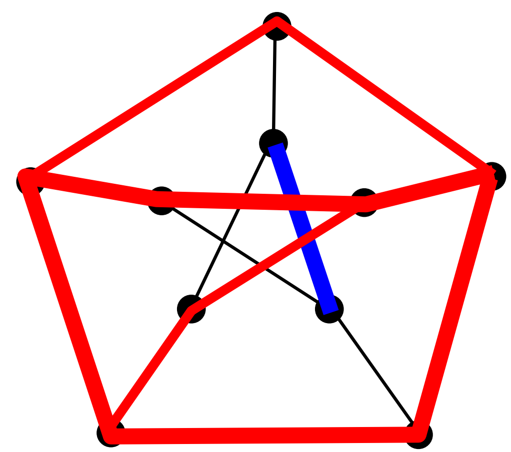

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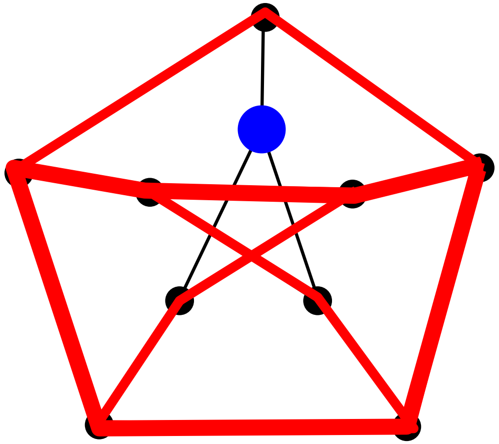

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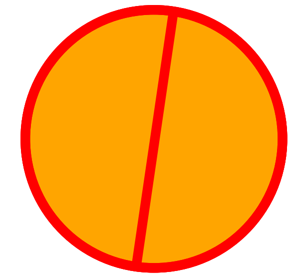

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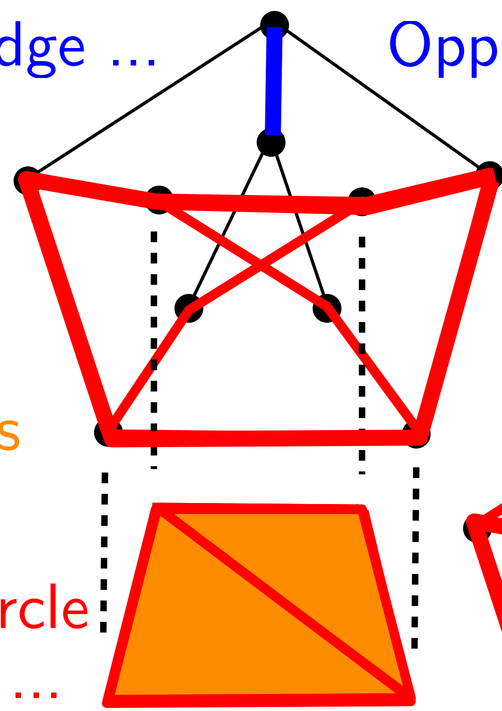

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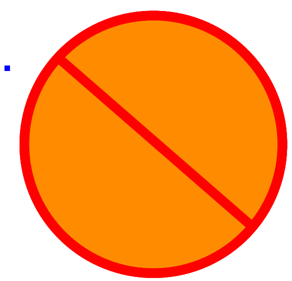

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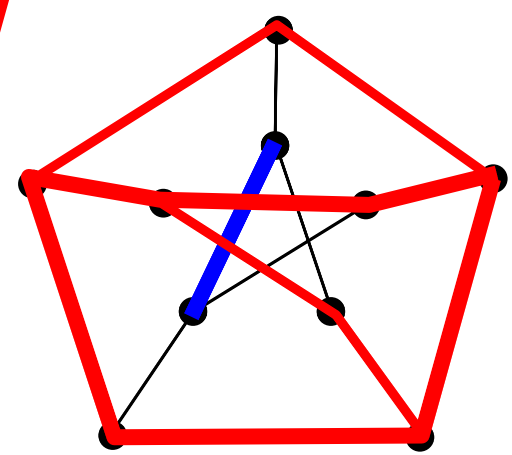

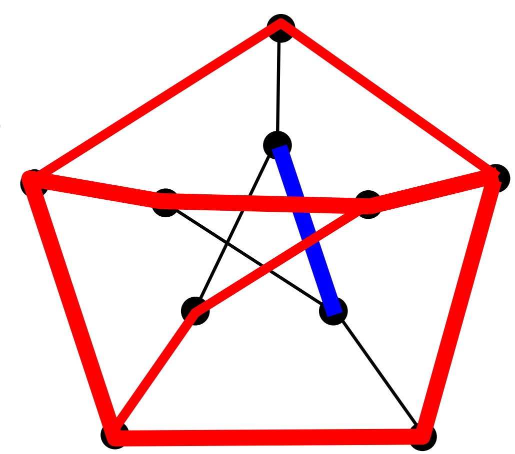

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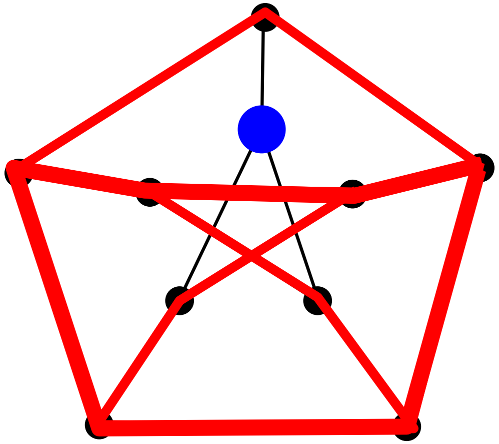

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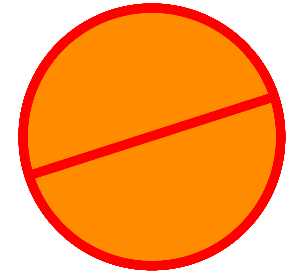

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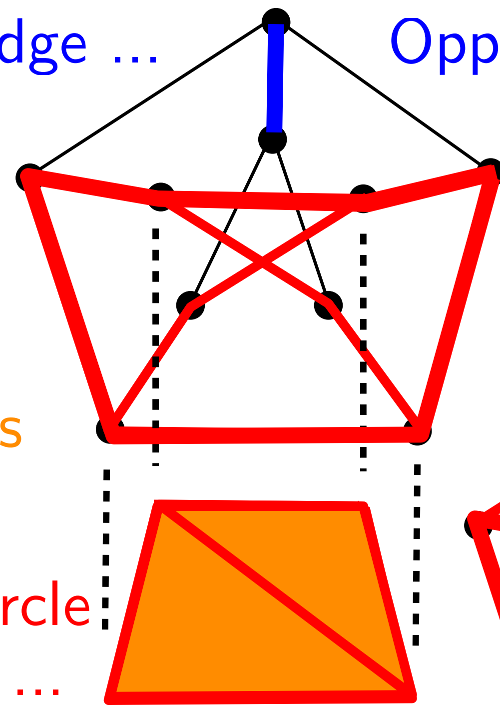

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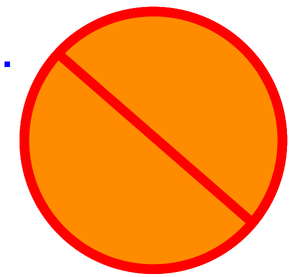

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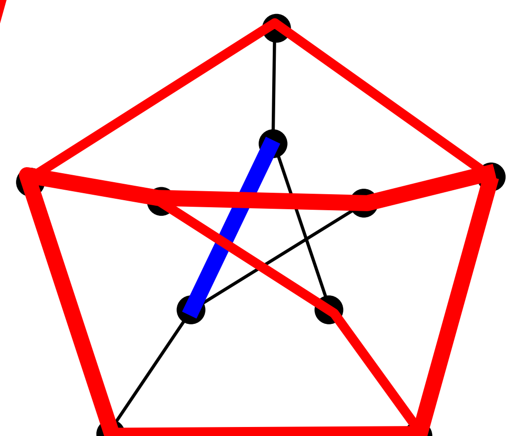

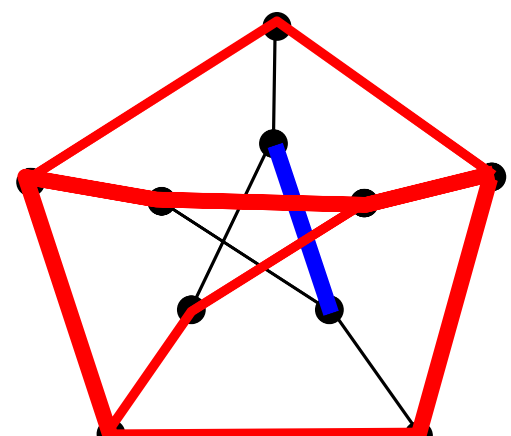

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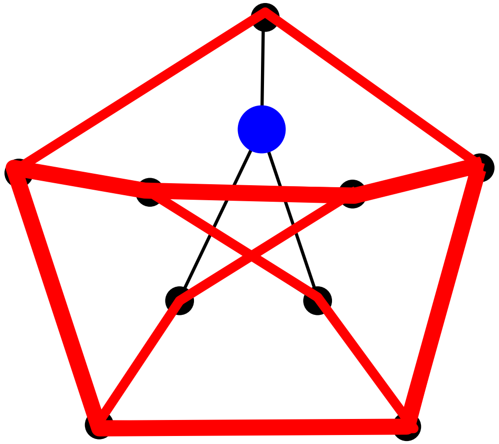

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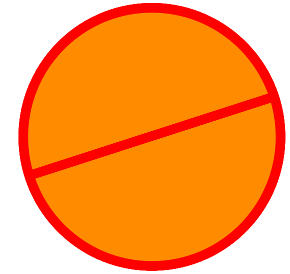

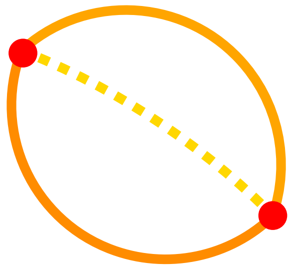

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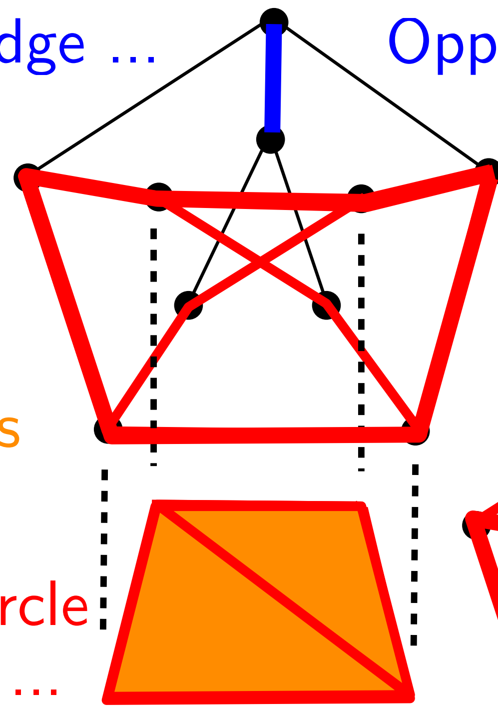

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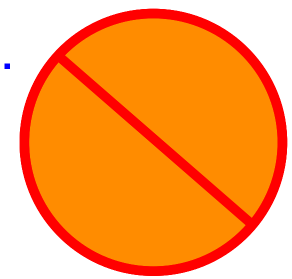

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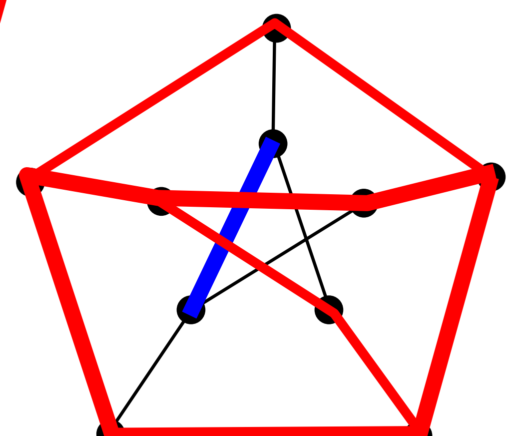

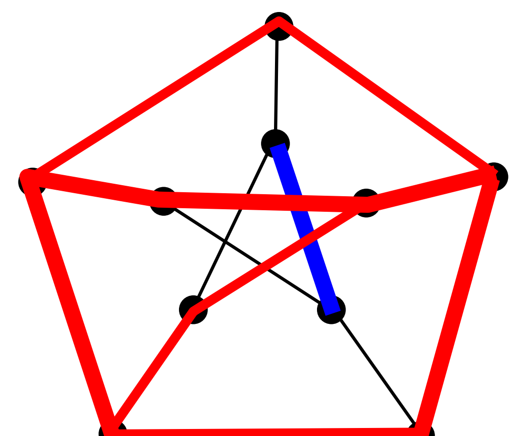

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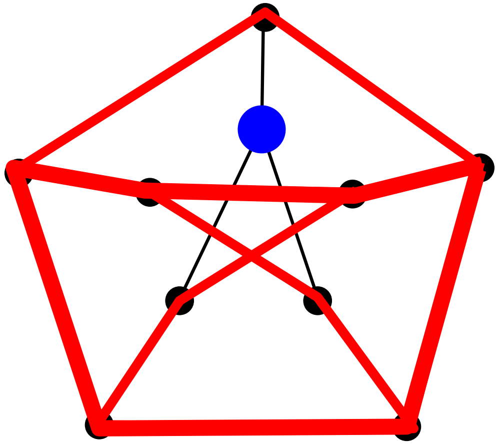

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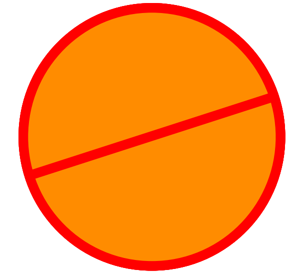

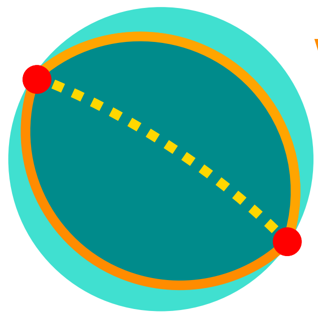

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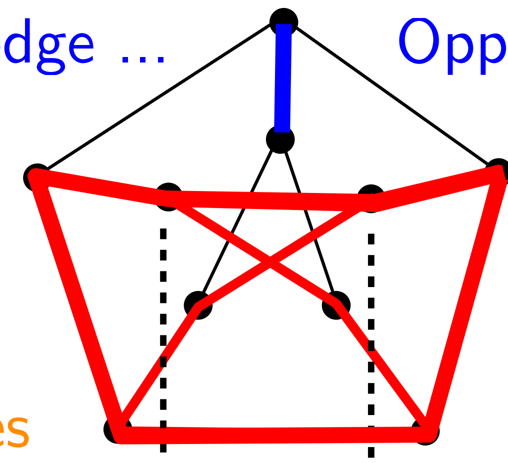

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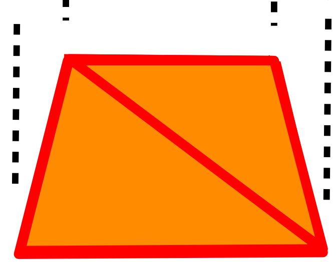

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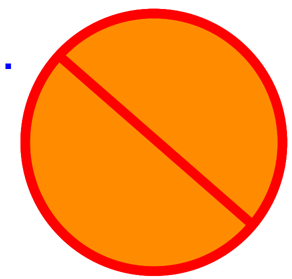

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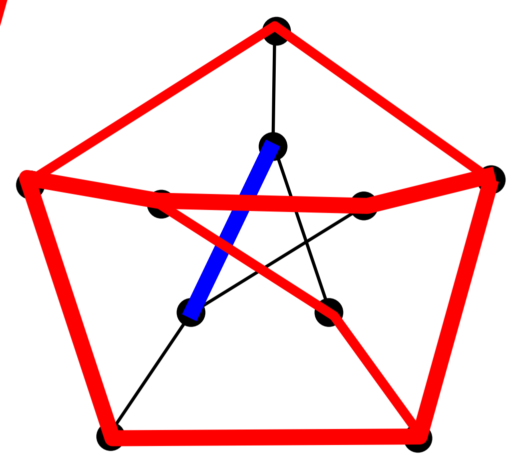

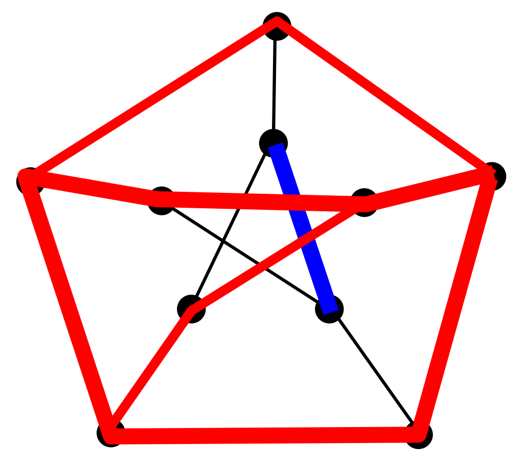

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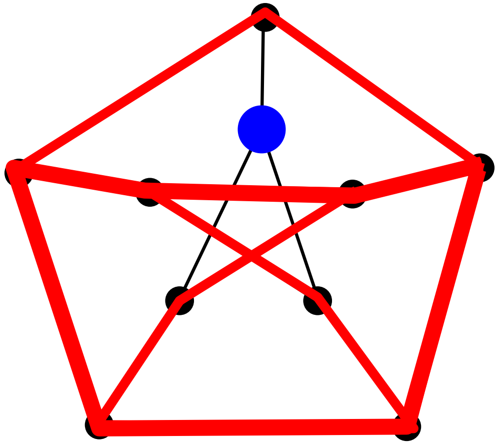

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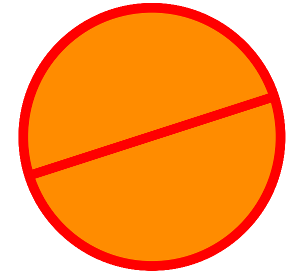

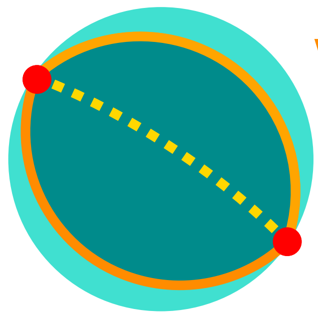

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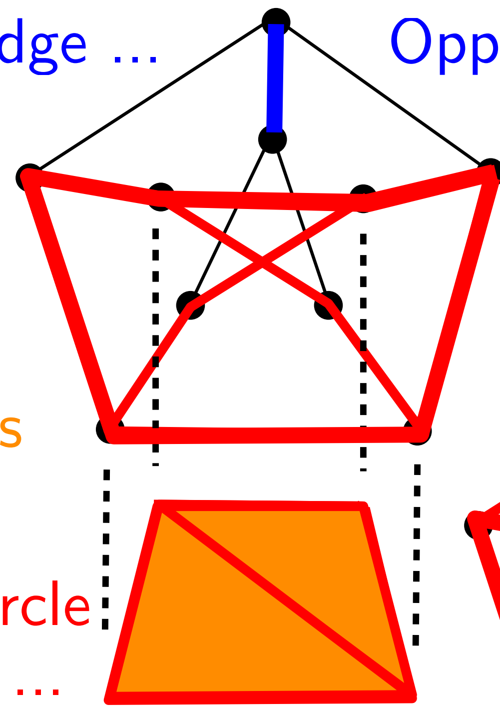

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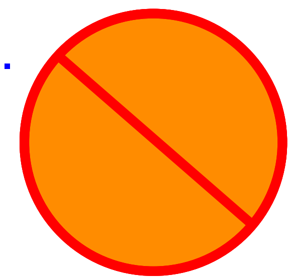

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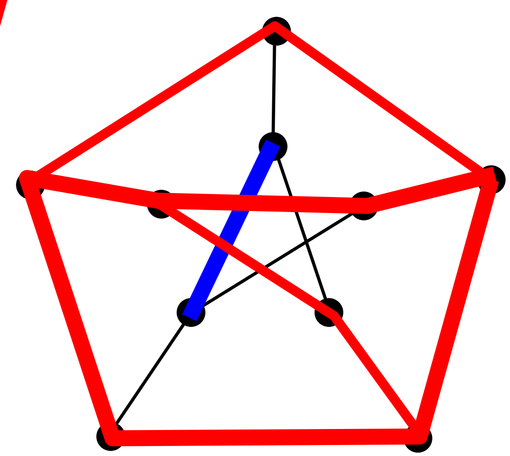

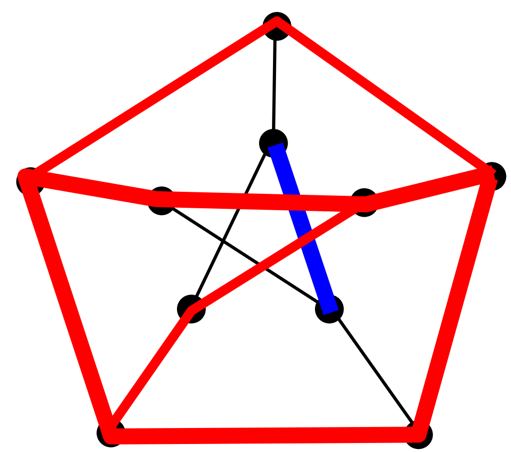

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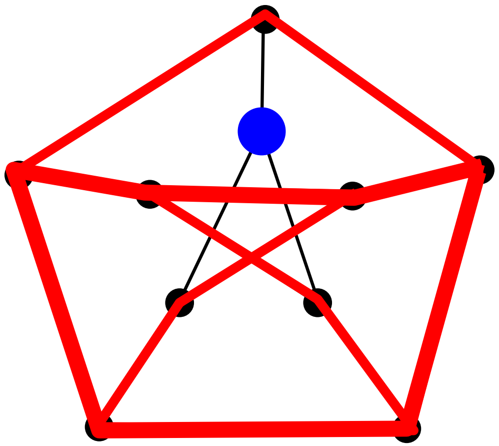

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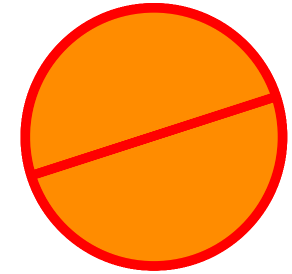

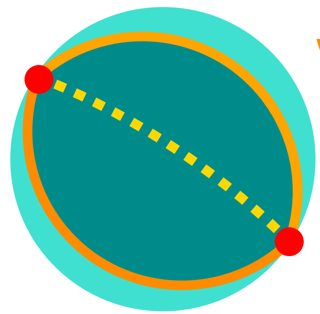

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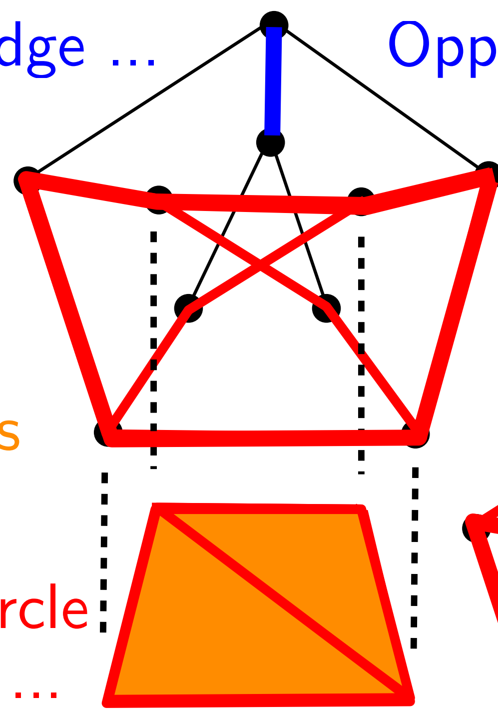

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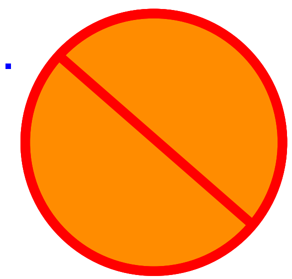

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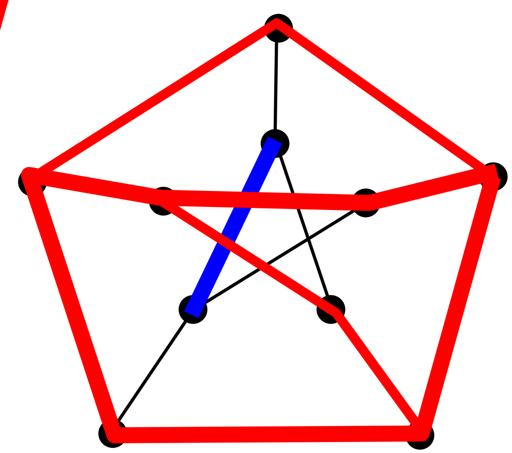

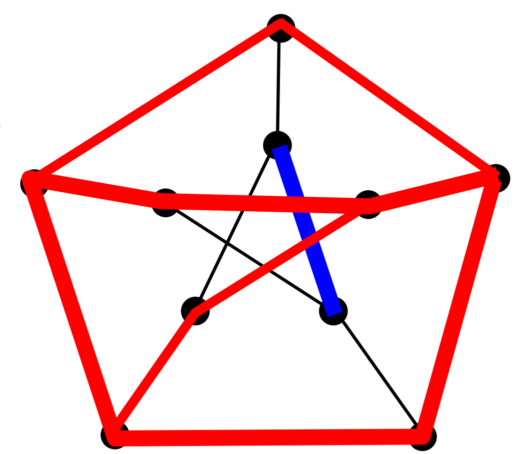

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Theorem (Flores, 1933) *Every m-obstructor does not embed in $\mathbb{R}^m$.*

Proof: Uses the Borsuk–Ulam theorem (1932).

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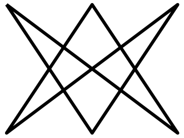

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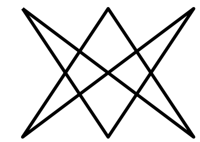 $\cong S^1$

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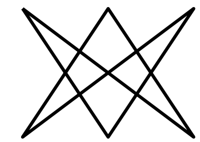
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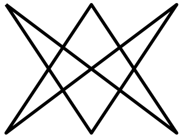 $\cong S^1$

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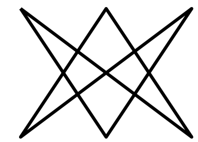 $\cong S^1$

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Theorem (Sarkaria, 1991) *The only $2n$-obstructors among simplicial complexes are n-dimensional joins of the type $F^{n_1} * \dots * F^{n_k}$.*

Theorem 2. (a) *The n-skeleton of every dichotomial $(2n+1)$-sphere is a $2n$-obstructor. Consequently it does not embed in $\mathbb{R}^{2n}$.*

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(i) K is linklessly embeddable in $\mathbb{R}^{2n+1}$;
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Example (Lovasz-Schrijver, Taniyama). The n-skeleton of Δ^{2n+3} does not linklessly embed in $\mathbb{R}^{2n+1}$.

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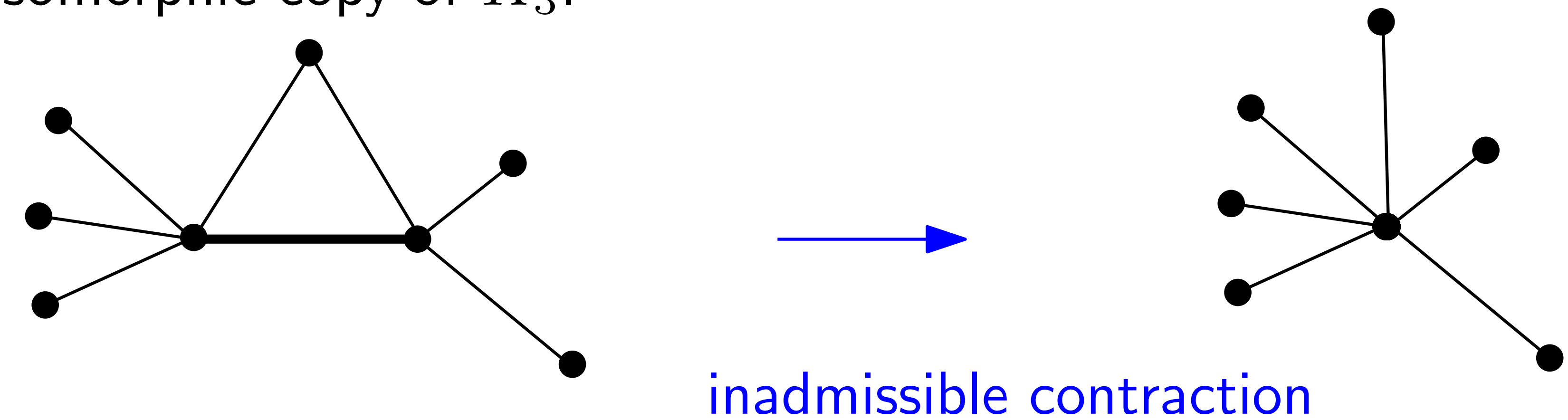
Van der Holst (2006): *(b)*, $n = 1$ and much of *(a)*, $n = 2$

Robertson–Seymor–Thomas (1993) *(b)*, $\Rightarrow$, $n = 1$

An edge contraction is *admissible* if the contracted edge is not contained in an isomorphic copy of K_3.



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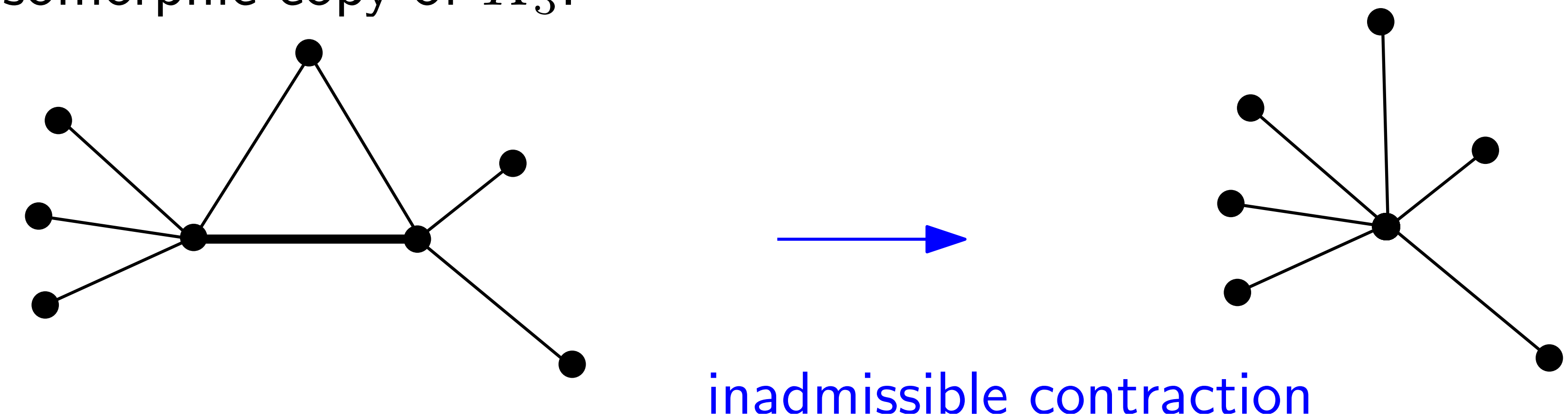

Lemma. *H is a minor of $G \iff H$ is obtained from a subgraph of G by a sequence of* **admissible** *edge contractions.*

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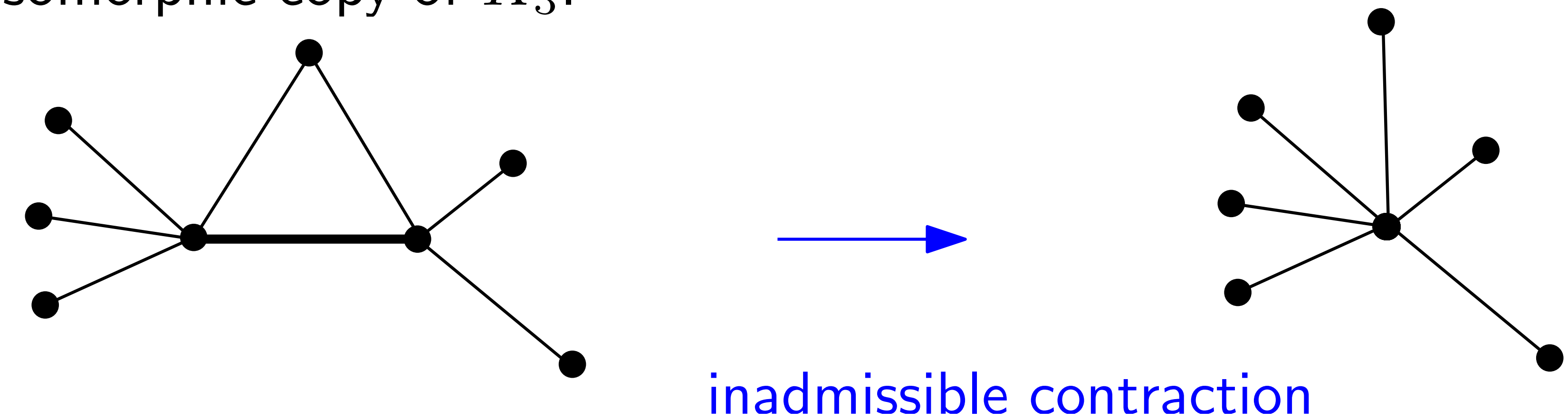

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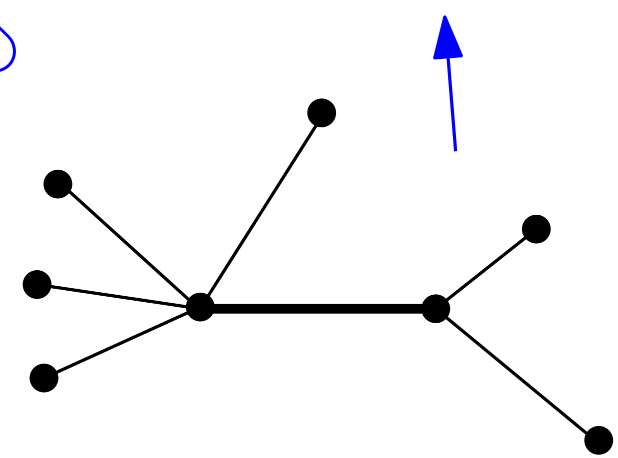

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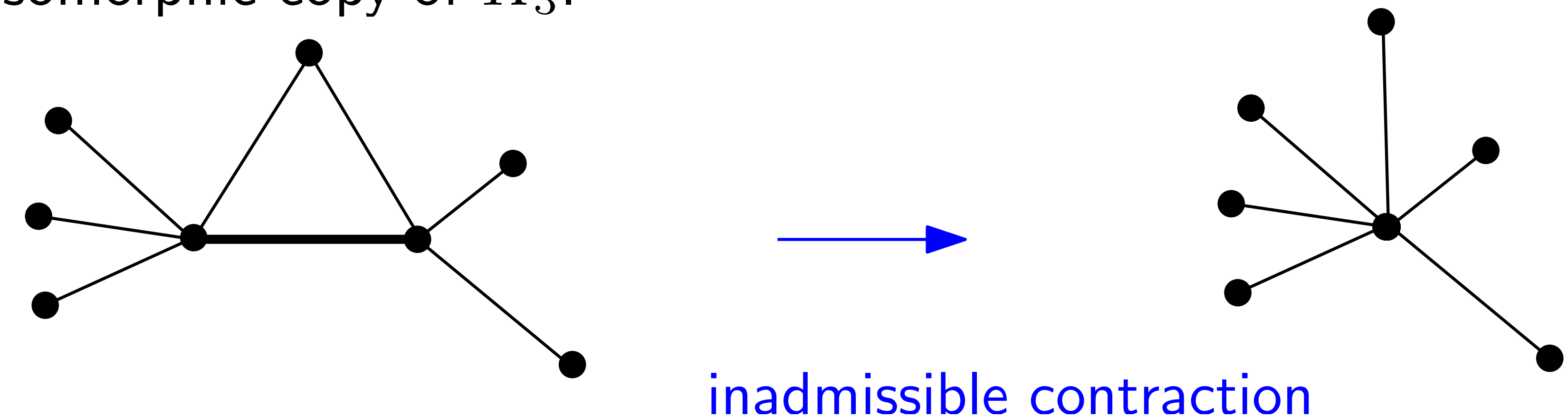

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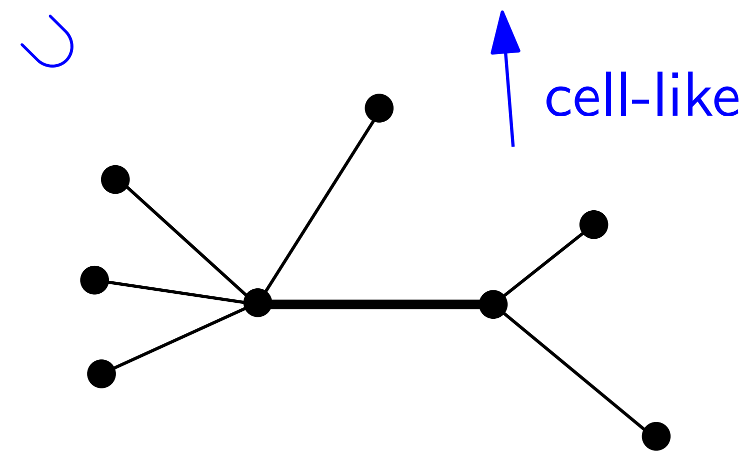

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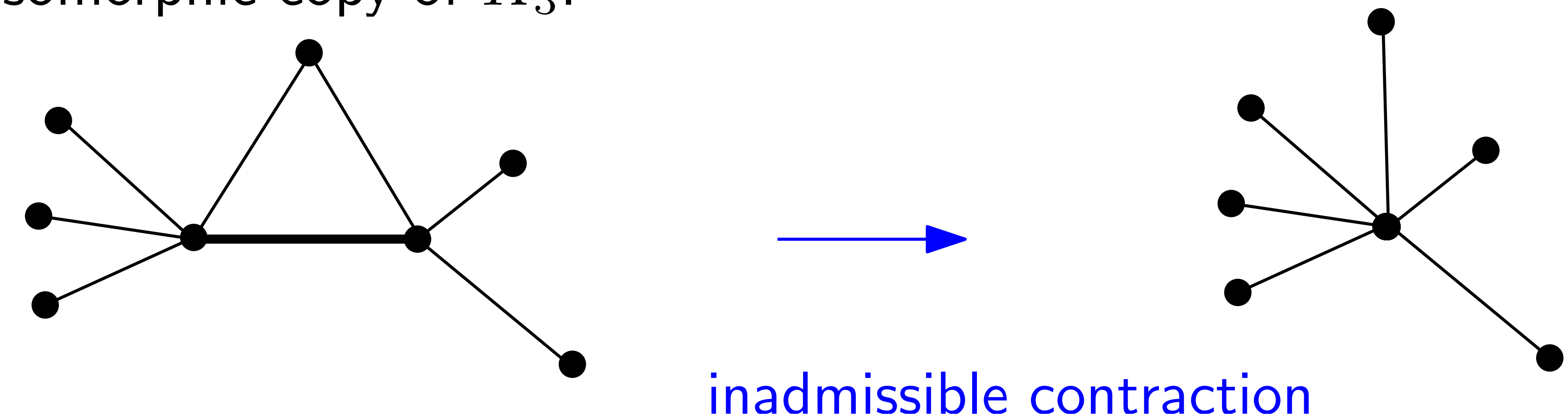

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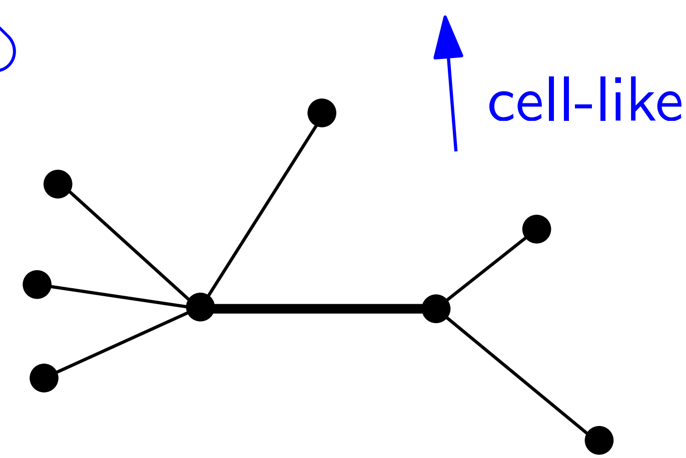

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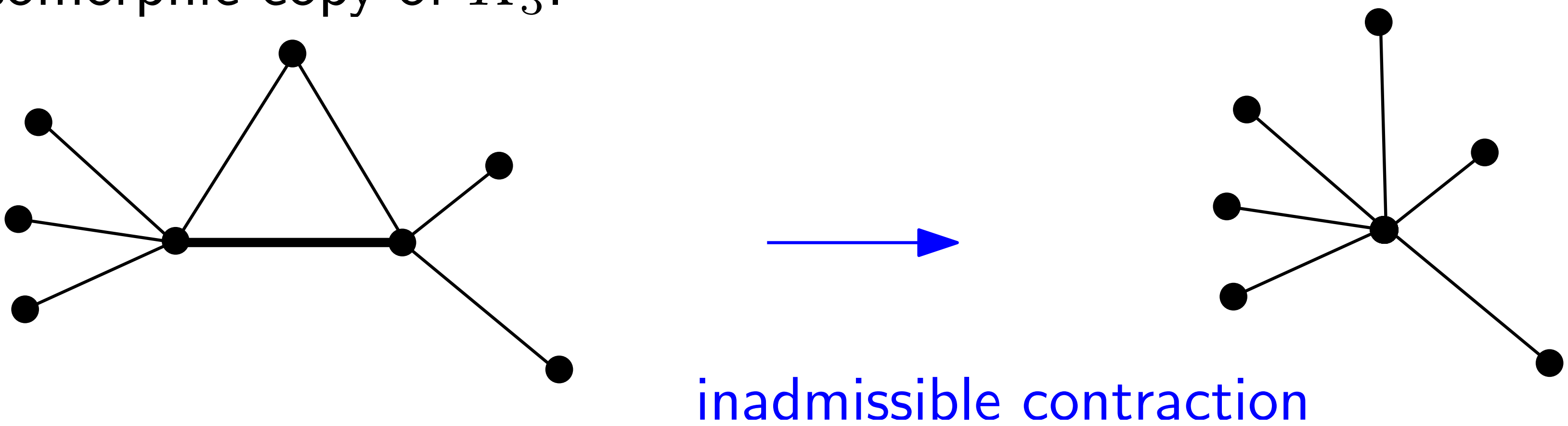

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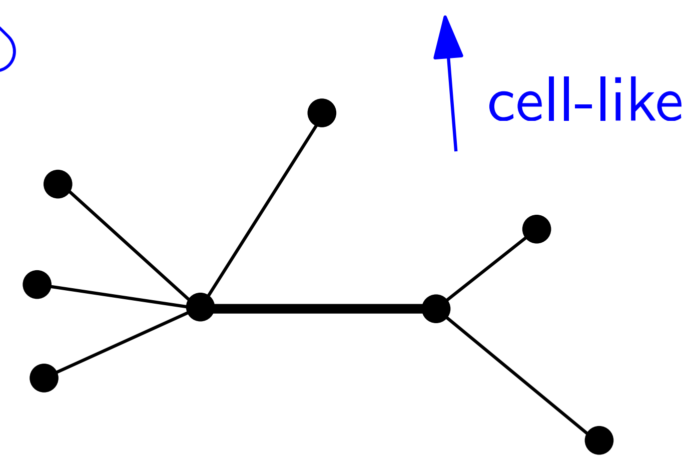

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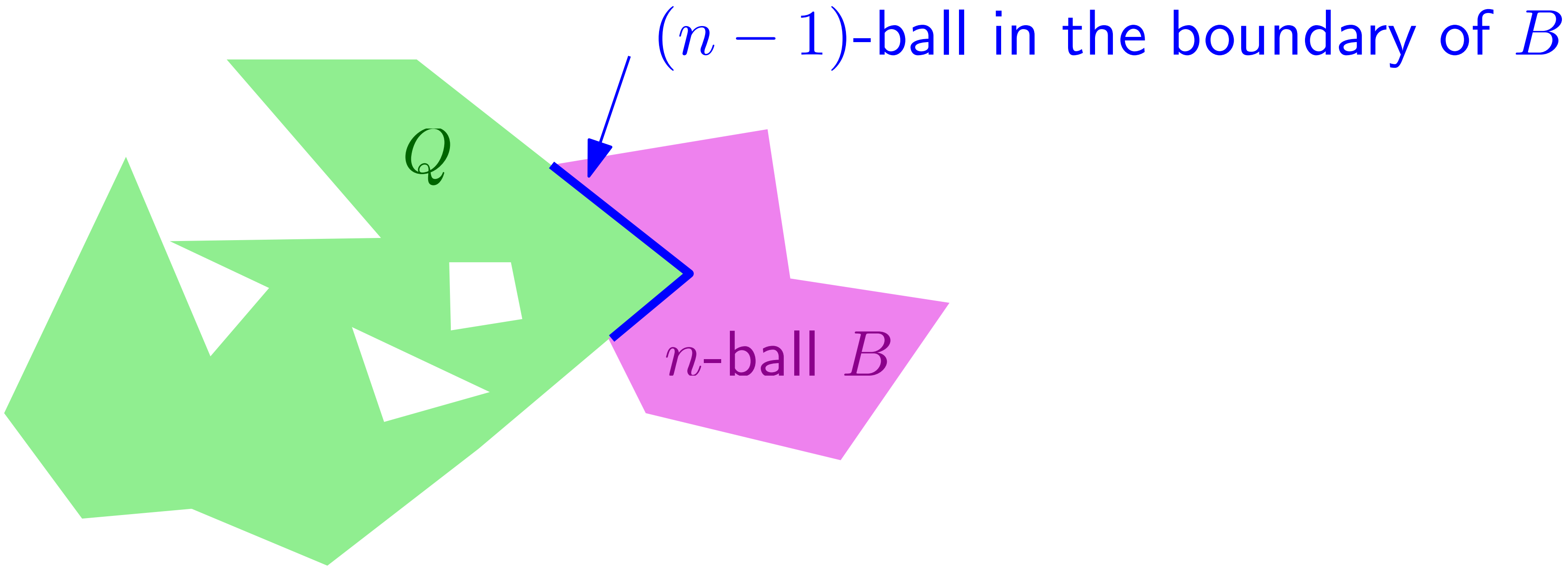
It is the definition of a minor via admissible contractions that will generalize to higher dimensions: admissible contraction $\rightsquigarrow$ zipping.

PART 3

DIGRESSION: COMBINATORICS OF COLLAPSING

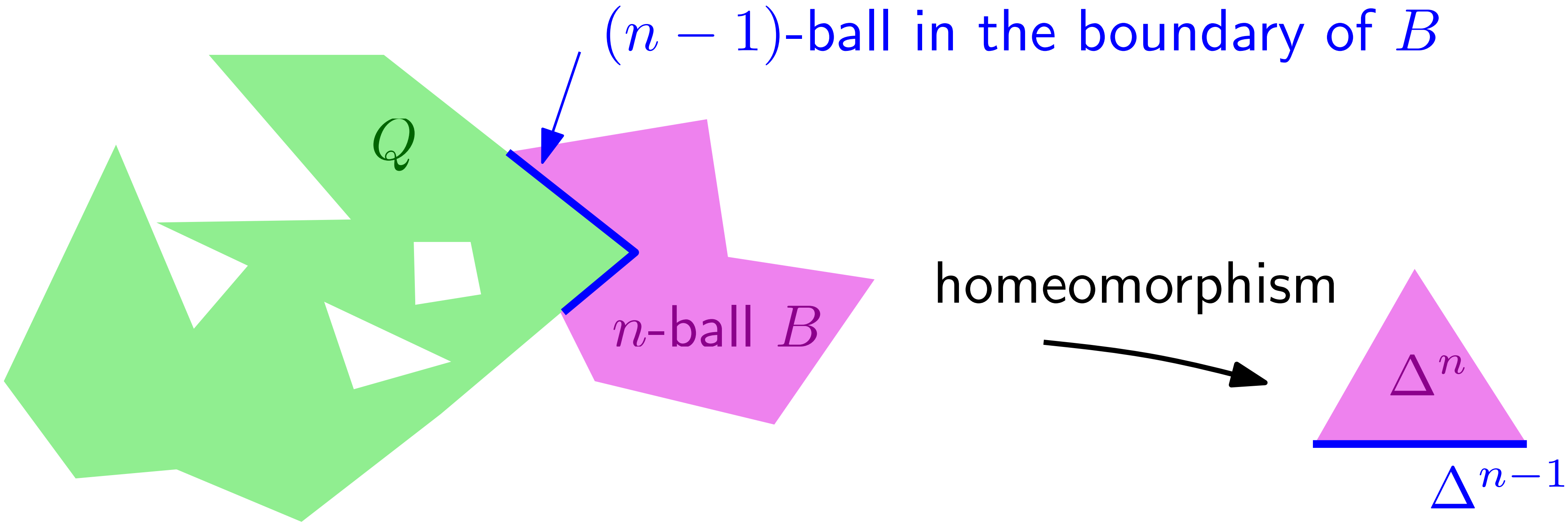
Collapsing

Elementary collapse:



Collapsing

A polyhedron P *elementarily collapses* onto a subpolyhedron Q if $P = Q \cup B$, where the pair $(B, B \cap Q)$ is homeomorphic to (Δ^n, Δ^{n-1}) for some n. Here Δ^n is the n-simplex and Δ^{n-1} is its facet.



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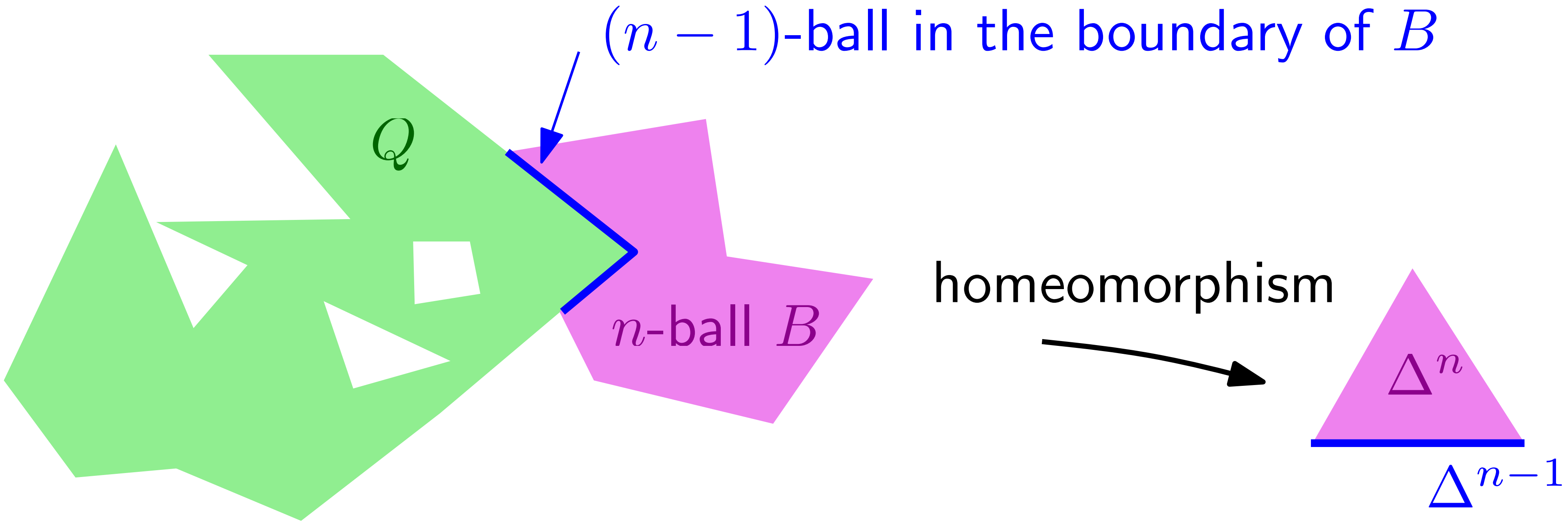
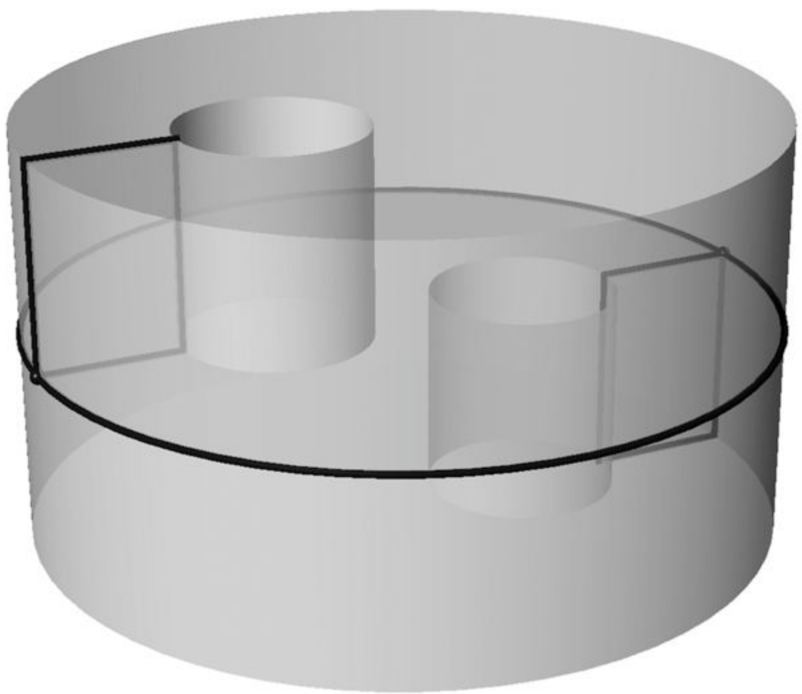


A *collapse* is a finite chain of elementary collapses.

A polyhedron is *collapsible* if it collapses onto a point.

Bing's house with two rooms

- B is not collapsible, because "there's nowhere to start from"



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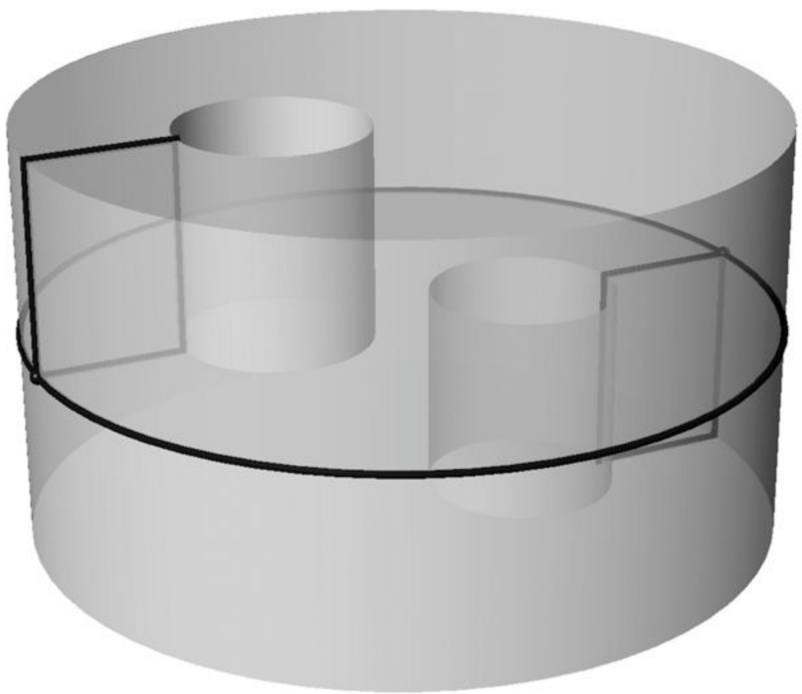

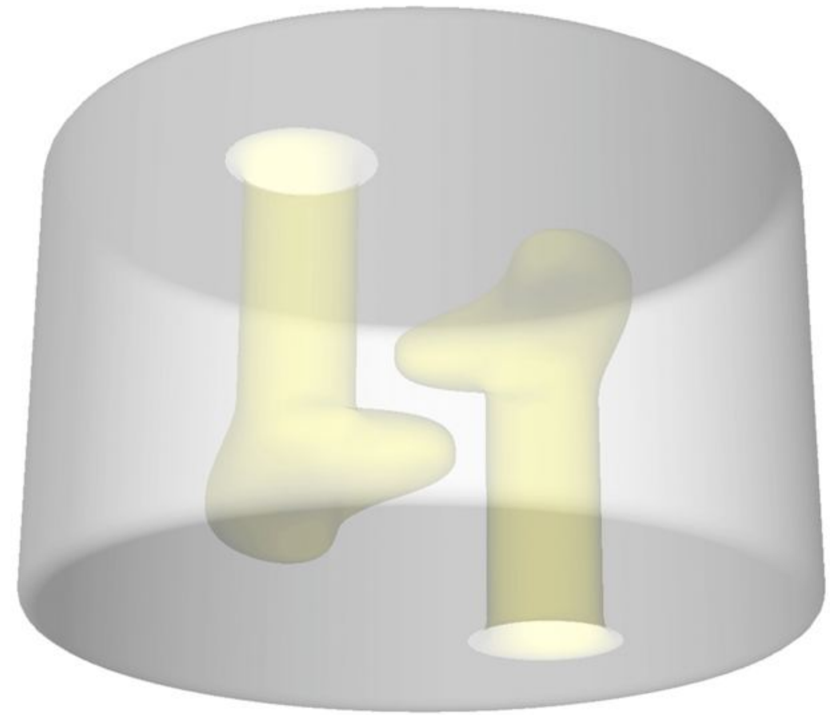

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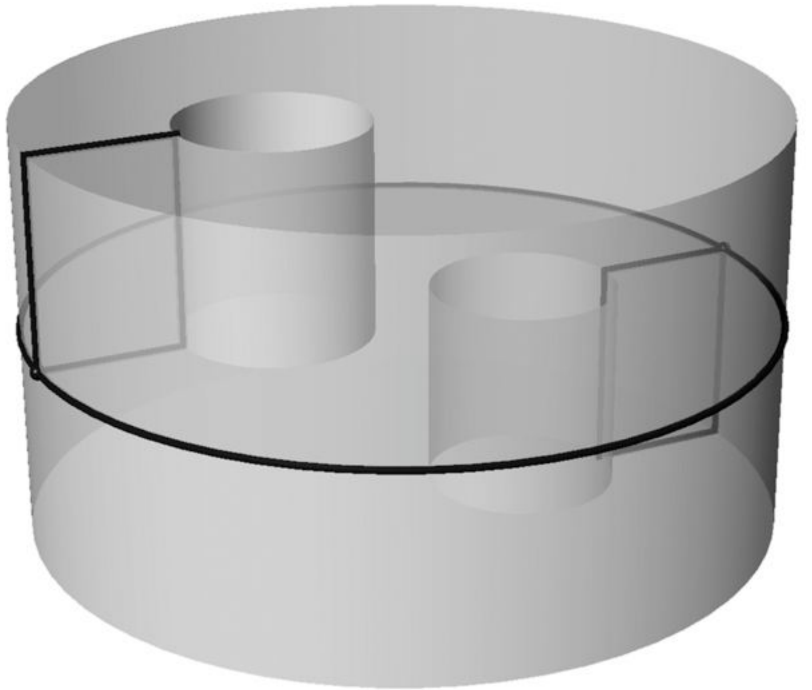

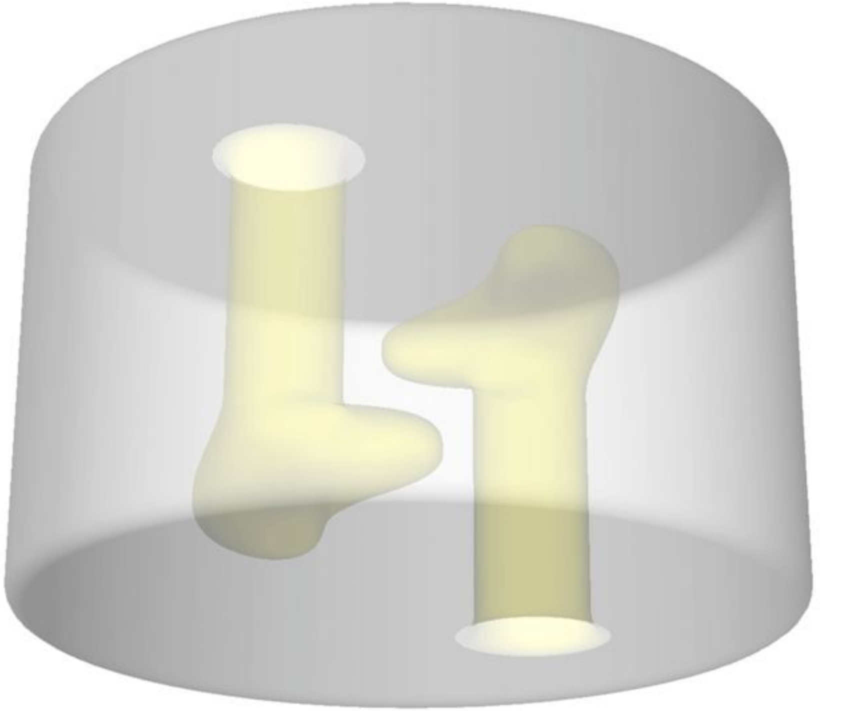

If P collapses onto Q, clearly there is a deformation retraction $P \to Q$. In particular, P and Q are homotopy equivalent.

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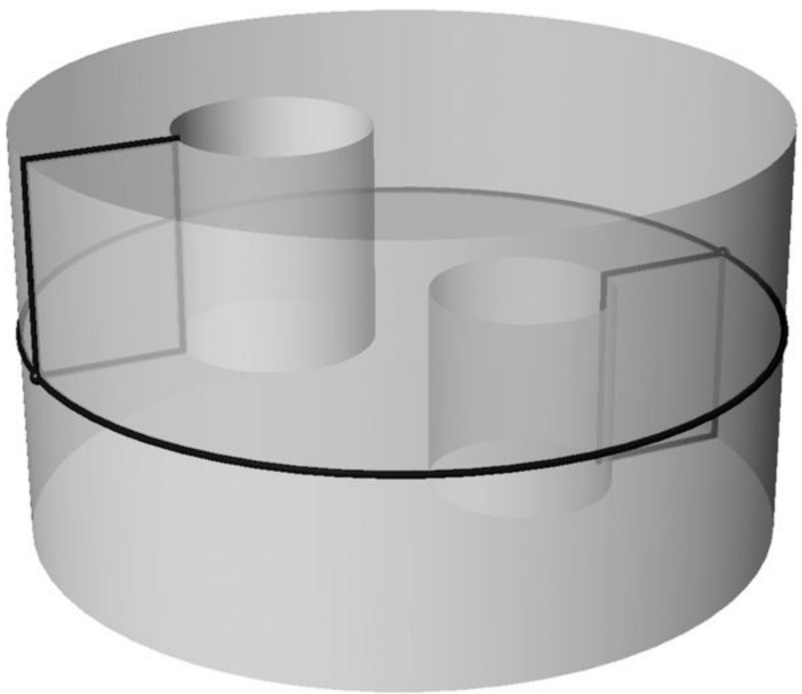

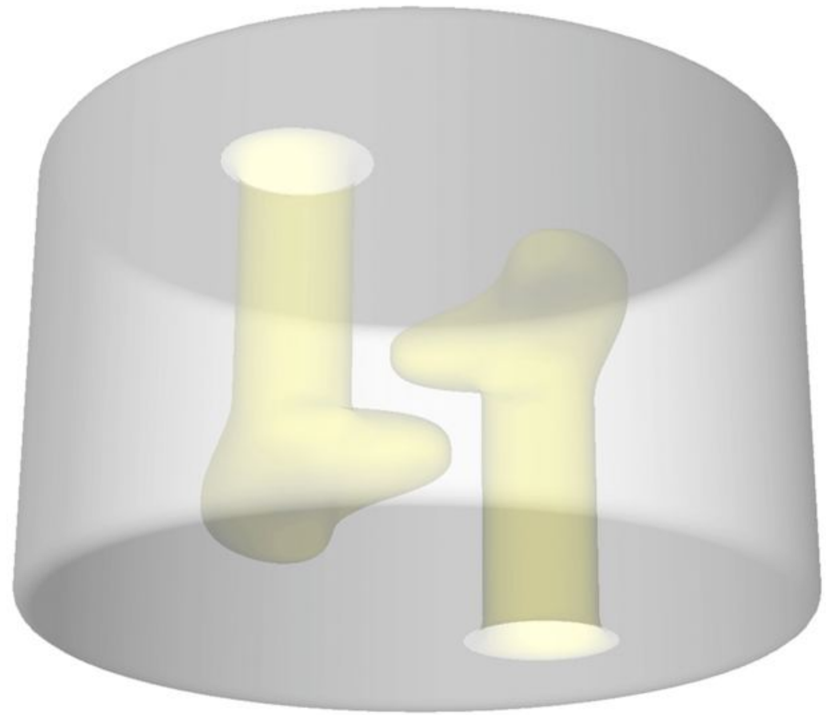

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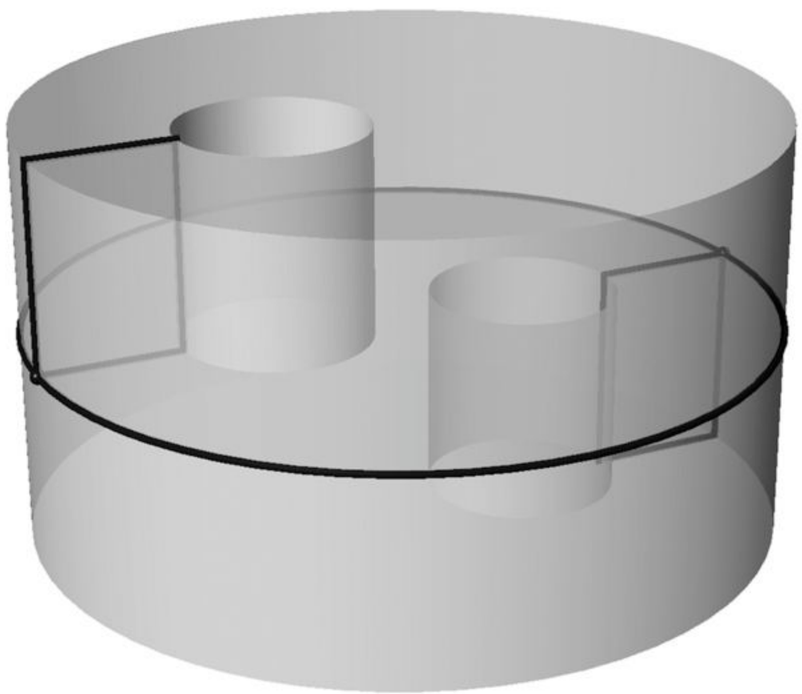

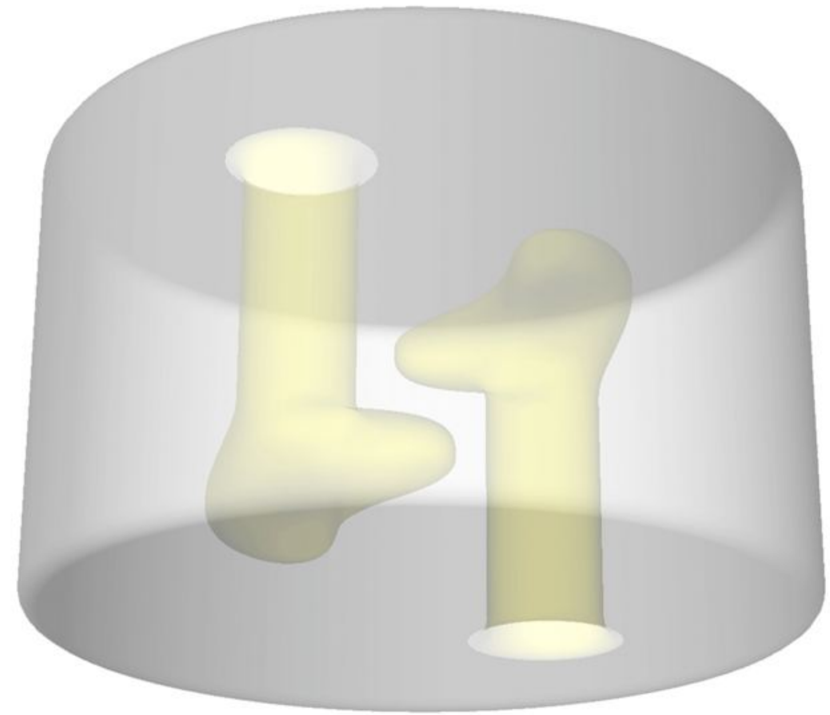

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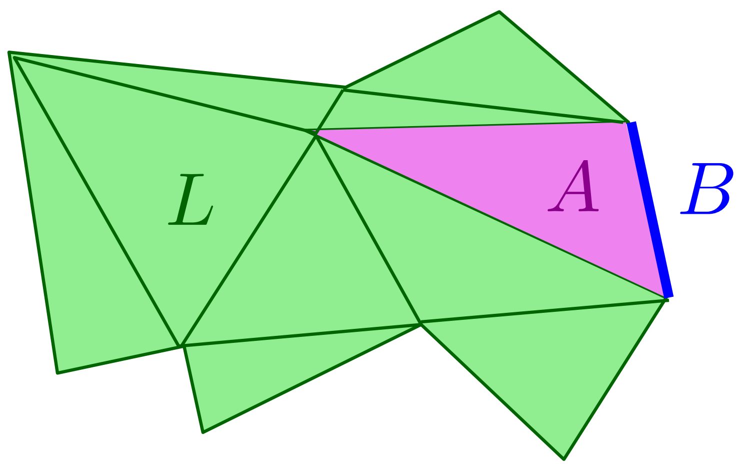
B is homotopy equivalent to the 3-ball, i.e. contractible

Combinatorial characterizations of collapsibility

A simplicial complex K *elementarily simplicially collapses* onto a subcomplex L if $K = L \cup A$, where A is a simplex of K and $L \cap A = \partial A \setminus B$, where B is a facet of A.

A *simplicial collapse* is a finite chain of elementary simplicial collapses.

Theorem (Whitehead, 1939) A polyhedron is collapsible if and only if it can be triangulated by a simplicial complex that is simplicially collapsible.



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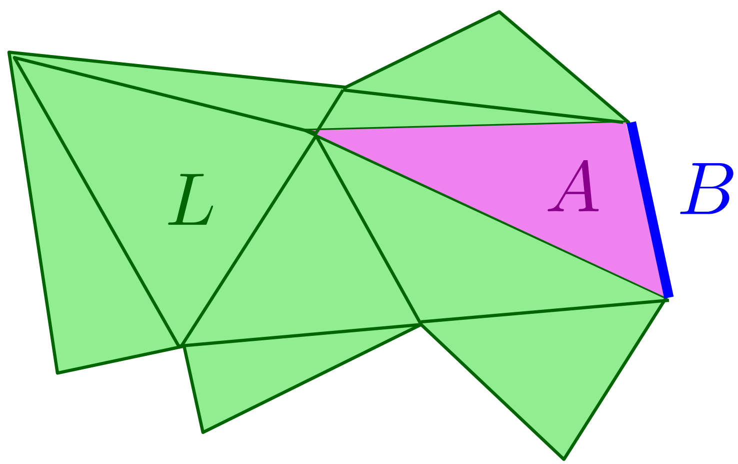

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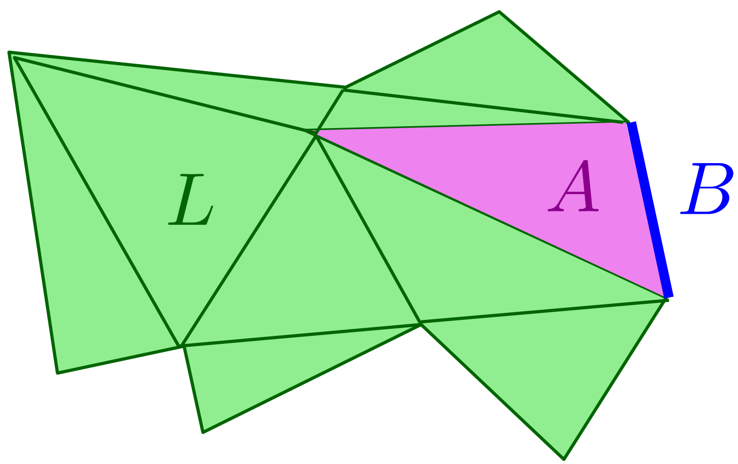

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2) In terms of contructible posets

A simplicial complex, and more generally a cell complex can be reconstructed from, and so will be identified with, the poset of its *nonempty* faces.

Constructible posets

We call a poset P *constructible* if either P has a greatest element or $P = Q \cup R$, where Q and R are order ideals (that is, if $p \leq q$ where $q \in Q$ then $p \in Q$; and similarly for R), each of Q, R and $Q \cap R$ is constructible, and every maximal element of $Q \cap R$ is covered by a maximal element of Q and by a maximal element of R.

For face posets of simplicial complexes (including the empty face!) this is Hochster's 1972 definition.

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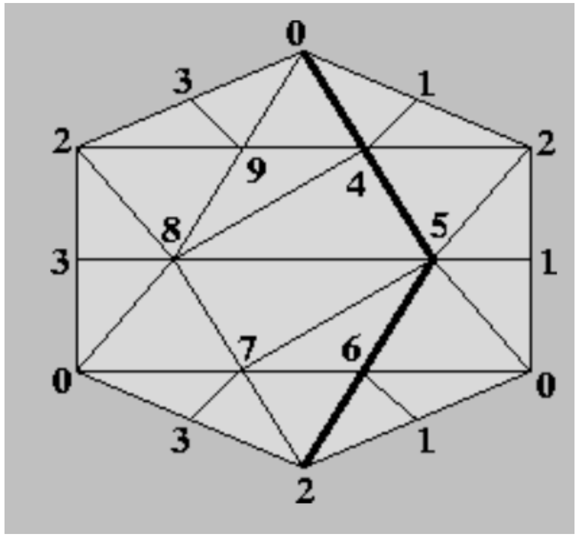

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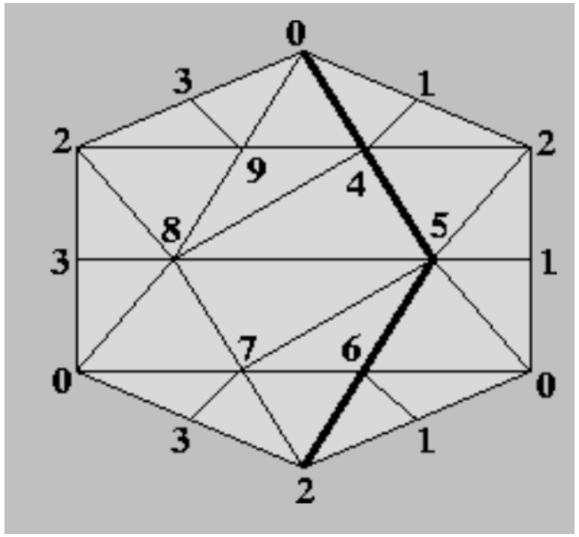

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Theorem 6. *The following are equivalent for a polyhedron X:*

(i) X is collapsible;

(ii) X is triangulated by a simplicial complex whose dual poset is constructible;

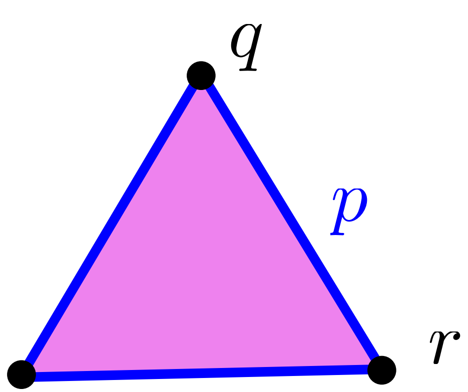
(iii) X is cellulated by a cell complex whose dual poset is constructible.

Zipping

Let P be a poset. Let $p \in P$ cover incomparable elements $q, r \in P$, where

- every $s < p$ with $s \neq q, r$ satisfies $s < q$ and $s < r$, and
- if $s > q$ and $s > r$ then $s \geq p$.

Then P *elementarily zips* onto the quotient of P by $(\{p, q, r\}, \leq)$.



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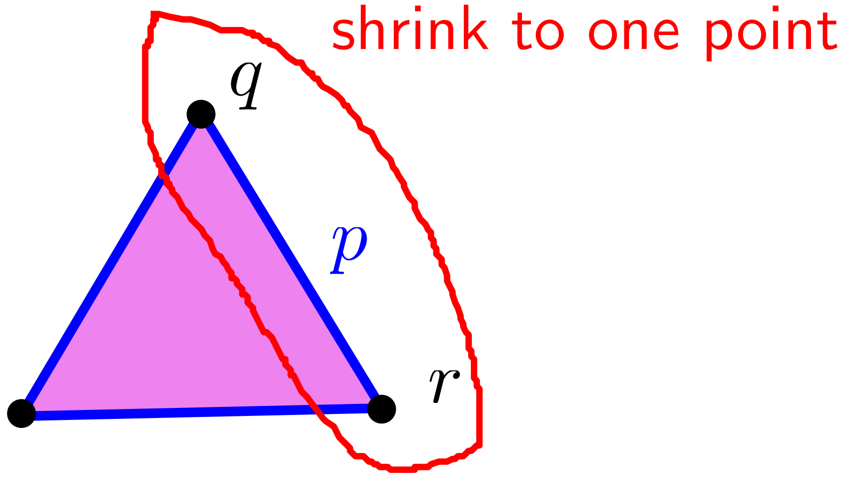

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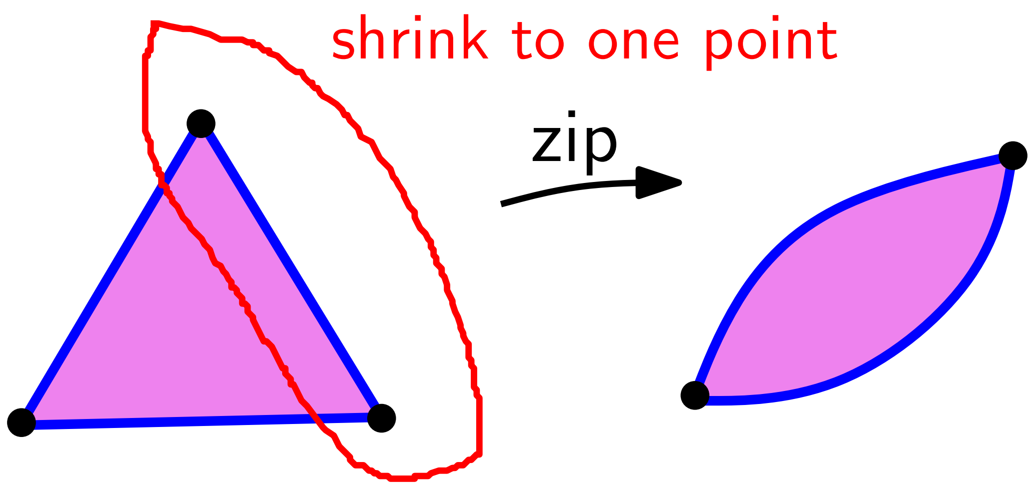

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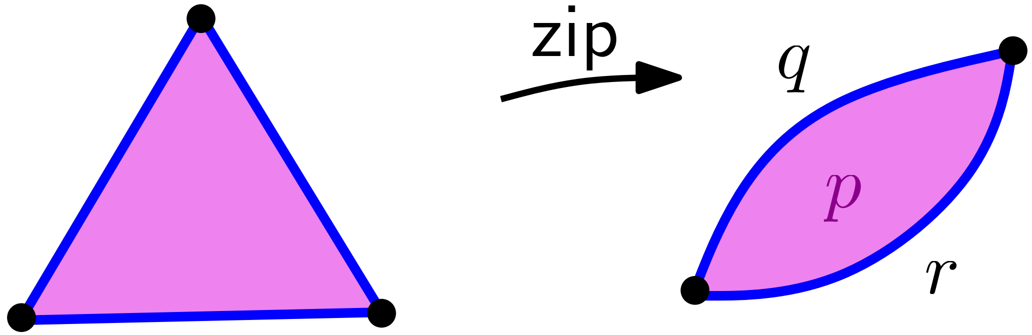

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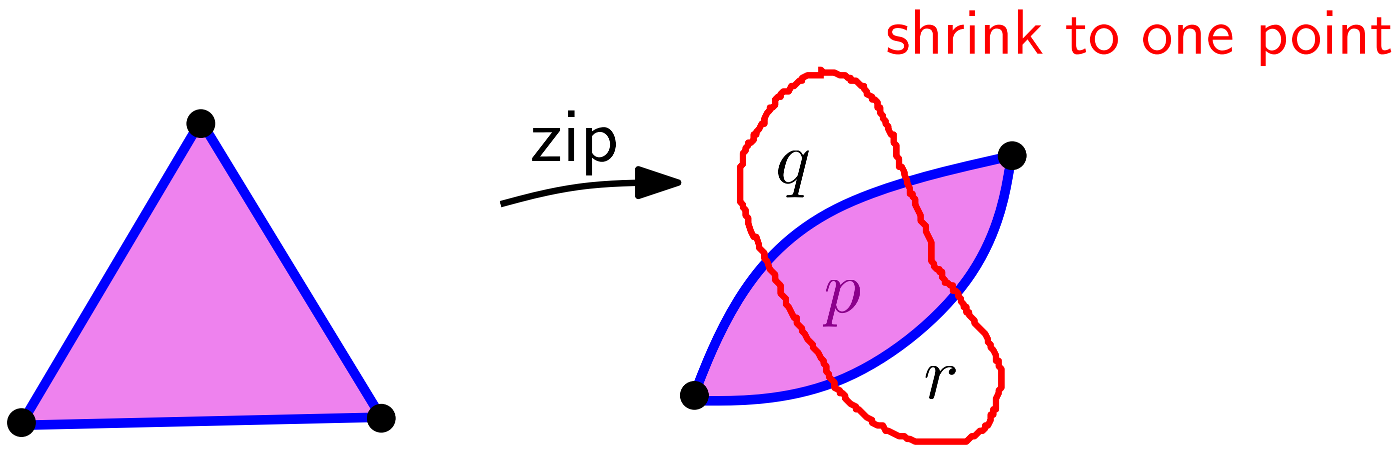

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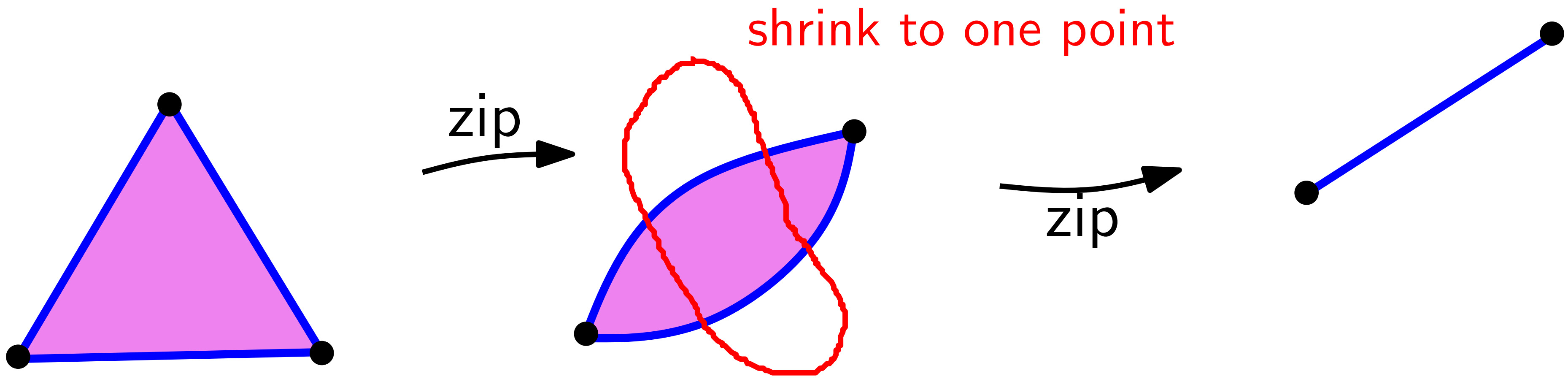

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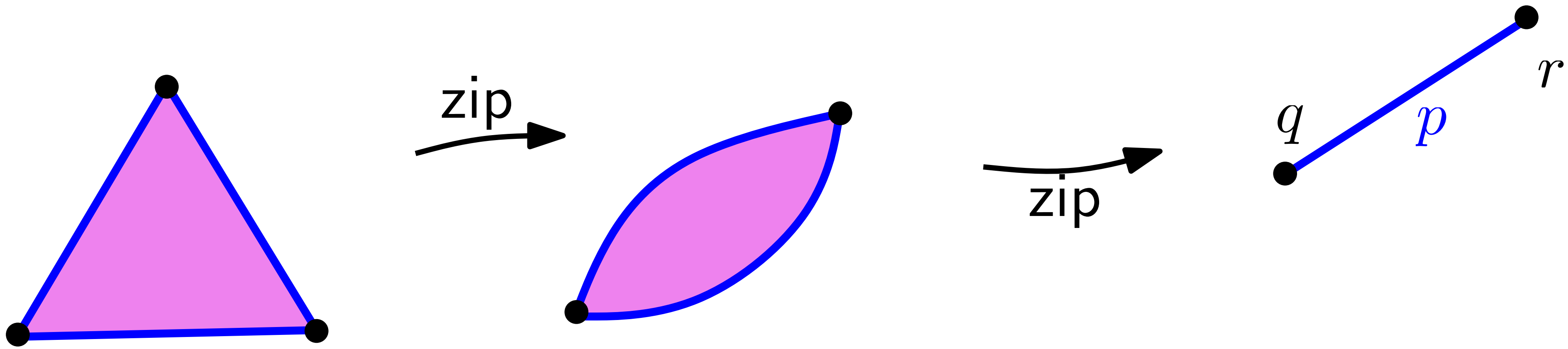

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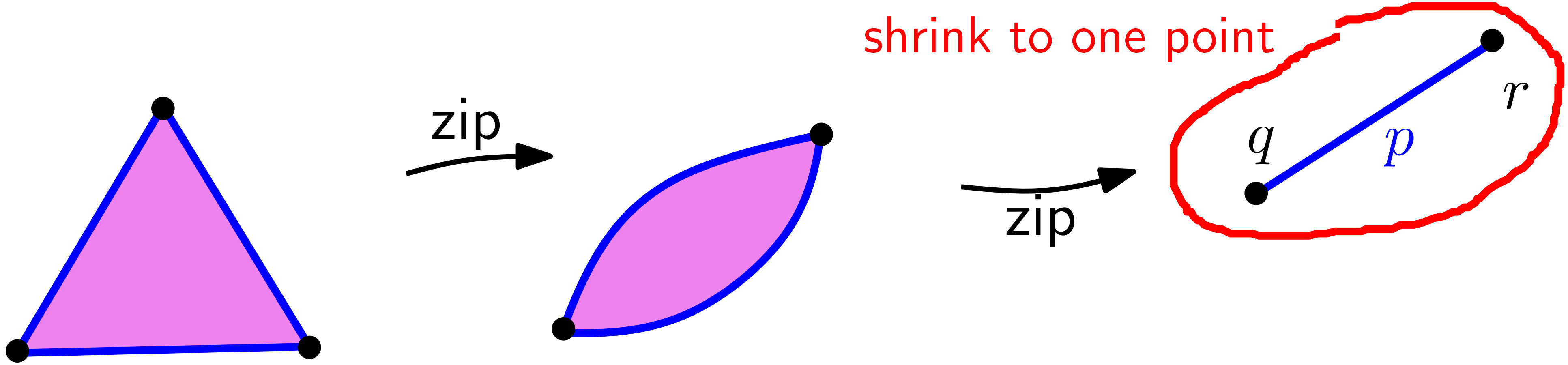

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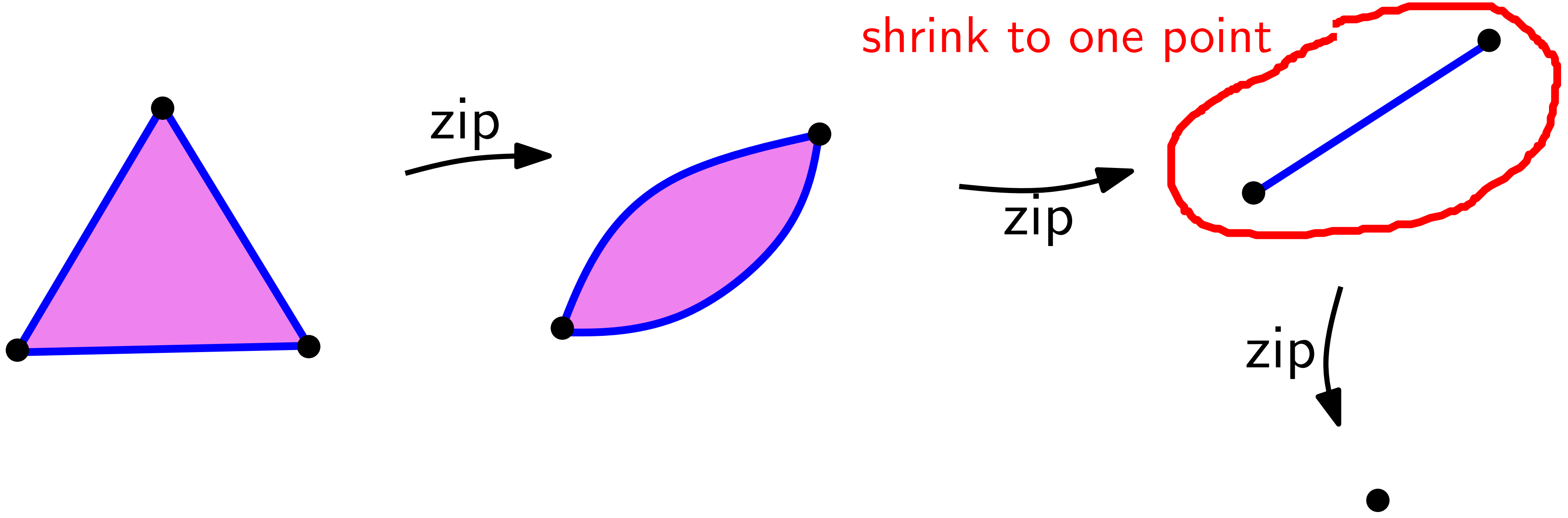

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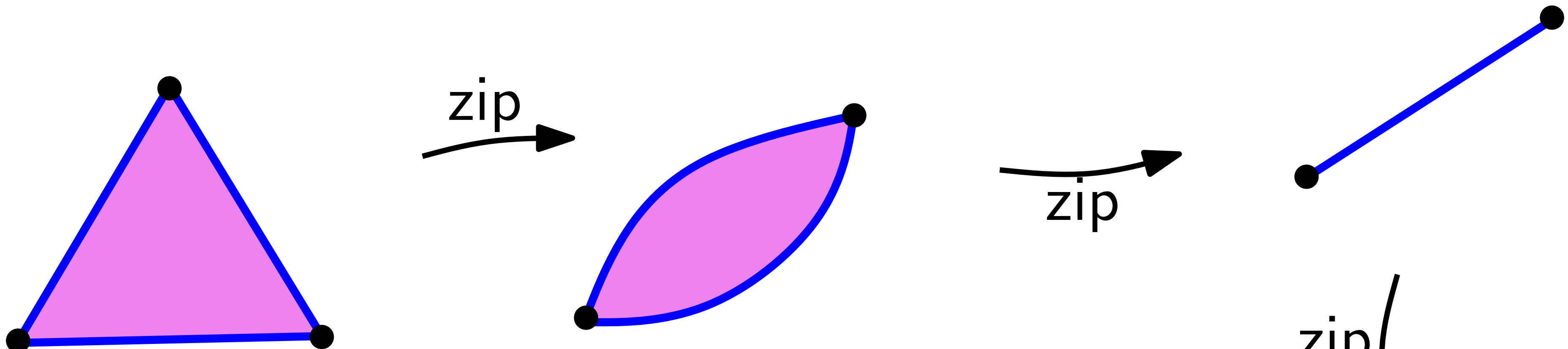

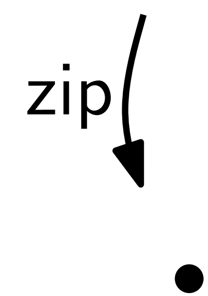

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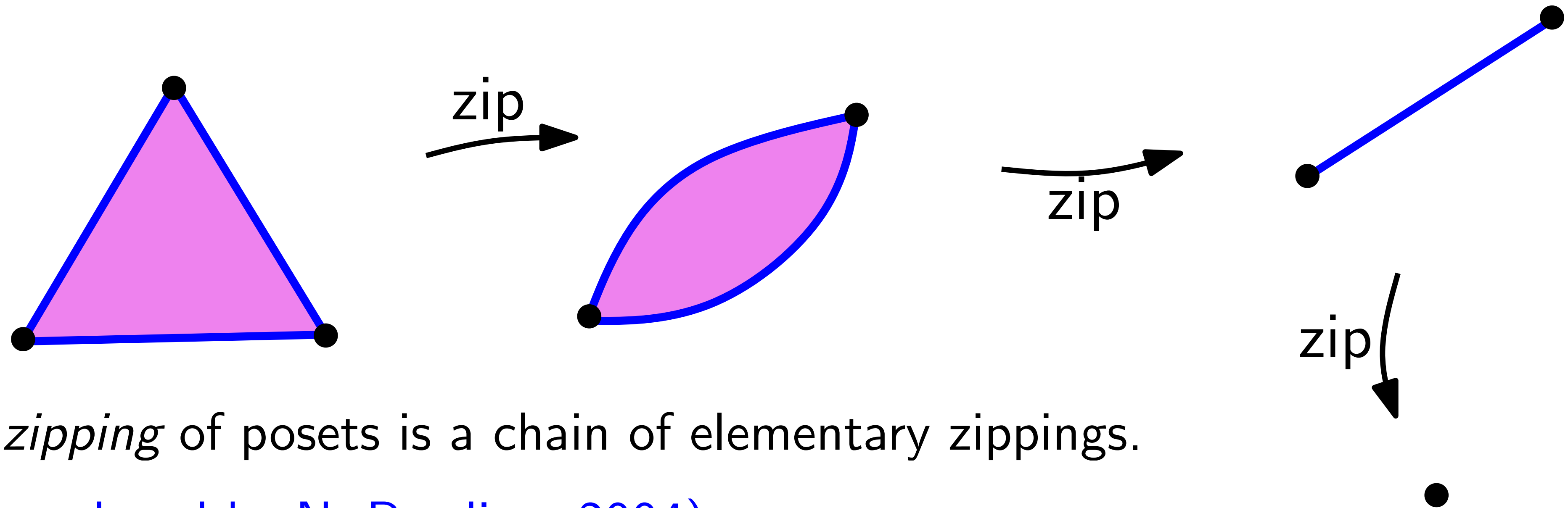

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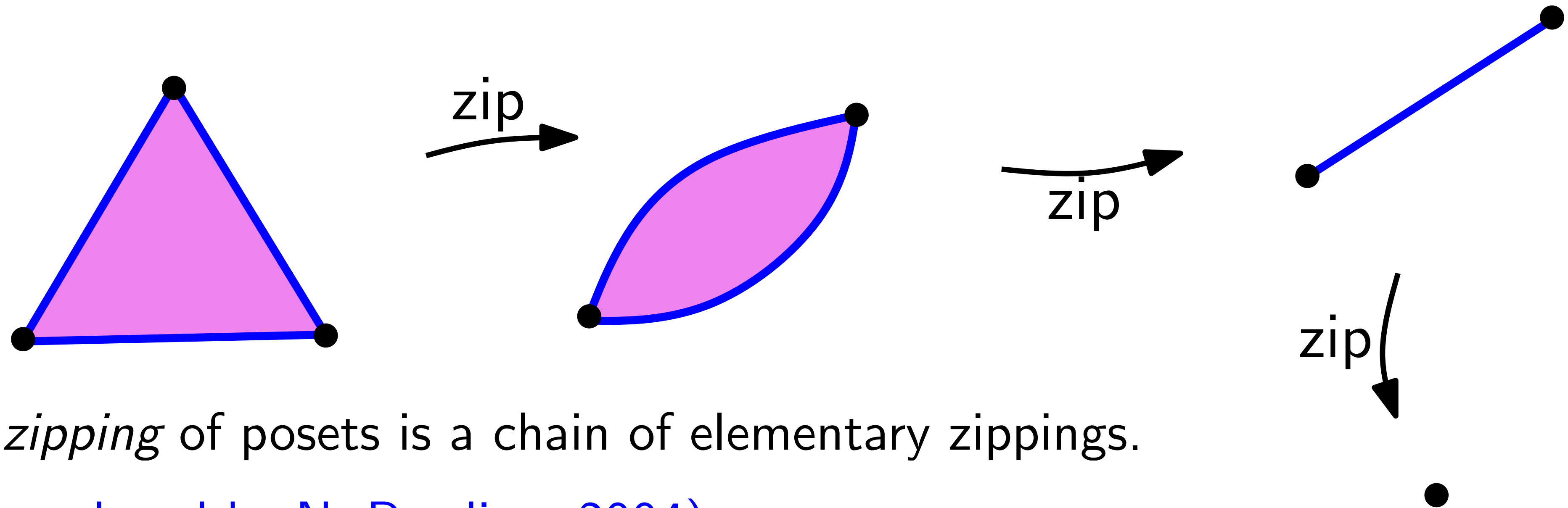

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Theorem 7. *A cell complex K zips onto a point if and only if the dual poset K^* is constructible.*

PART 4

GRAPHS AND MINORS IN HIGHER DIMENSIONS

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Lemma. Suppose that a compact n-polyhedron P PL embeds in $\mathbb{R}^m$, and $f : P \to Q$ is a PL map. Then Q embeds in $\mathbb{R}^m$ if either
(a) f is collapsible (=has collapsible point-inverses), or
(b) f is cell-like (=has contractible point-inverses) and $m - n \geq 3$.

Remark. It is obvious that Q embeds in $\mathbb{R}^m$ if $Q = P/C$, where C is a collapsible subpolyhedron of P. This can be iterated: $(P/C_1)/C_2$, etc.

Lemma. Collapsible maps preserve Cohen–Macaulayness.

Example. For each $n > 2$ there exist infinitely many Cohen–Macaulay pairwise non-homeomorphic n-polyhedra $K_1, K_2, \dots$ such that each K_r does not embed in $\mathbb{R}^{2n}$ but every its proper subpolyhedron does.

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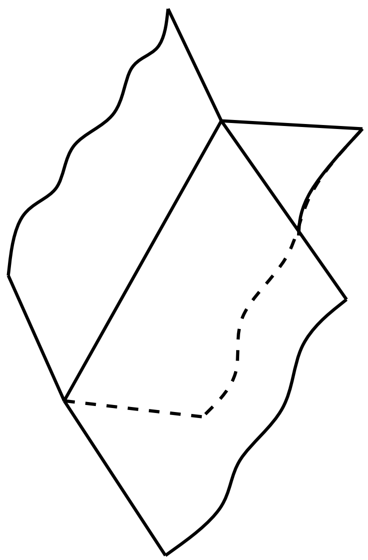
Let K be the n-skeleton of $\Delta^{(2n+2)}$, say. It is Cohen-Macaulay. Pick an $(n-1)$-cell A in K whose neighborhood in K is a book with at least 3 pages.



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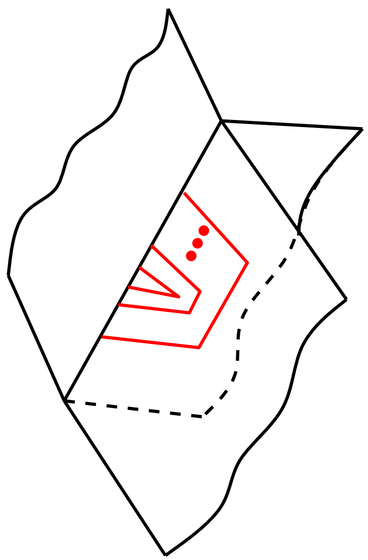

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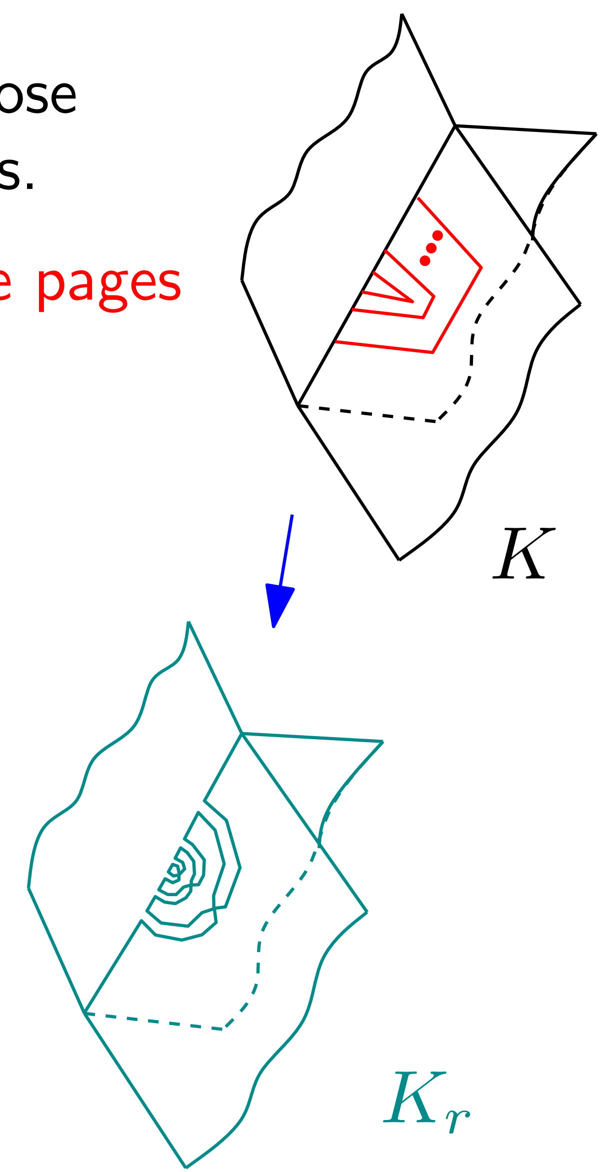
Now shrink each of these r balls to a point. The result is a new polyhedron K_r, where C is shrunk onto the wedge of $r + 1$ balls $C_0, \ldots, C_r$.



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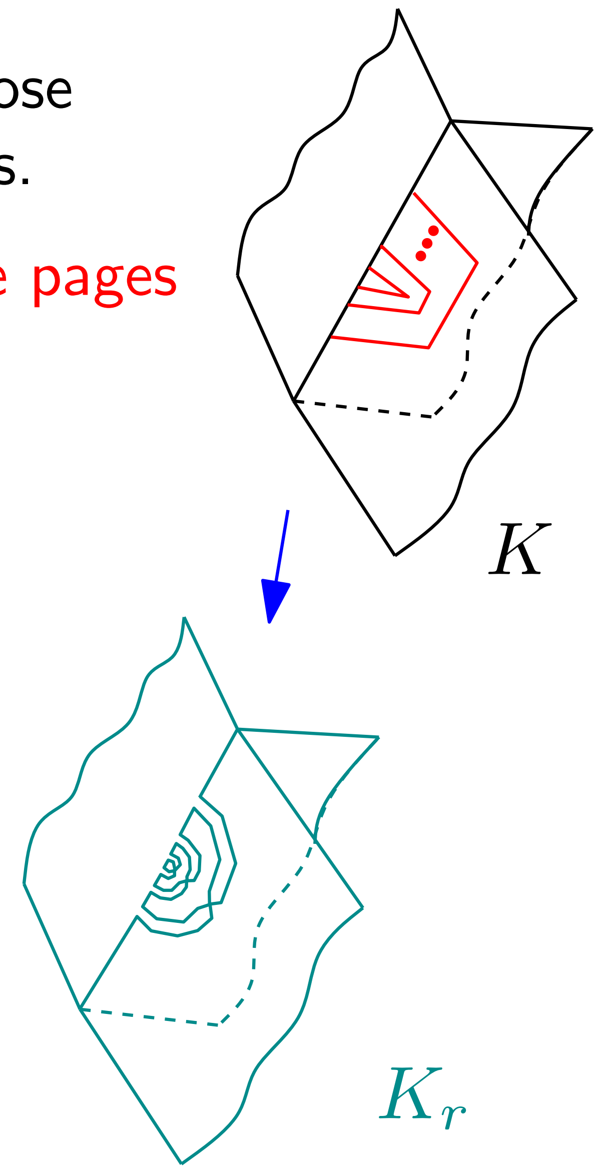

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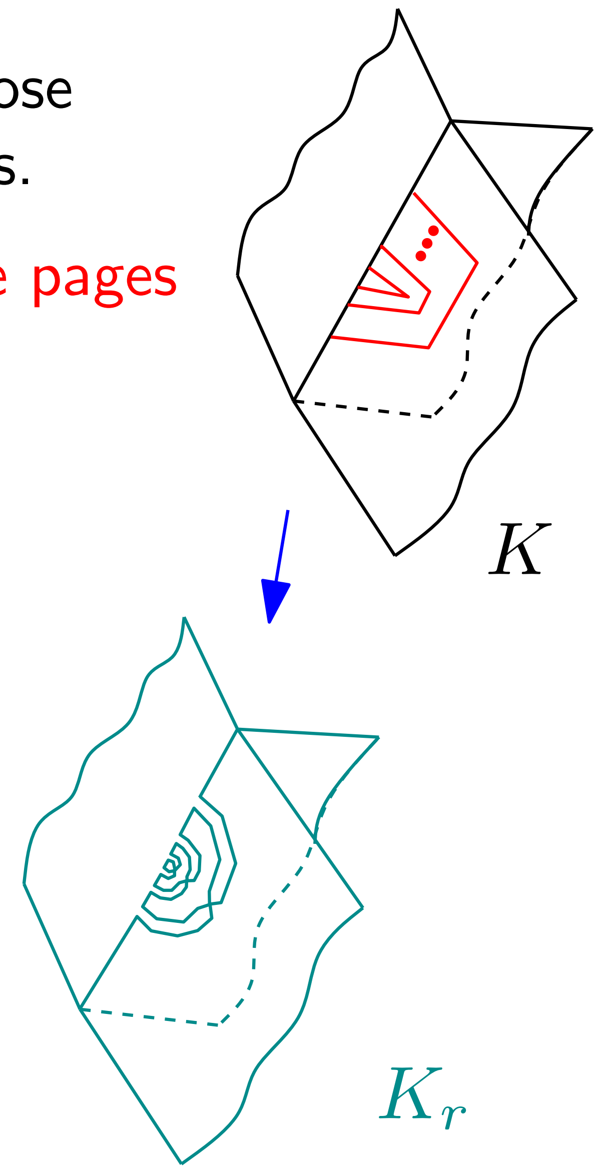
Also K_r does not embed in $\mathbb{R}^{2n}$ since it admits a collapsible map onto K: shrink the wedge of the balls $C_1, \dots, C_r$ to a point.



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Problem III. *Given an m, is the number of dichotomial m-spheres finite?*

References

- *Combinatorics of embeddings*, arXiv:1103.5457

  (to be updated)

- *The van Kampen obstruction and its relatives*,

Proc. Steklov Math. Inst. (2009) = arXiv:math.GT/0612082

3. Digression: Combinatorics of collapsing

- *Combinatorics of combinatorial topology*, arXiv:1208.6309