Minor theory for framed 4-valent graphs

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Outline

Wagner's Conjecture

Some years ago, a milestone in graph theory was established: as a result of series of papers by Robertson, Seymour (and later joined by Thomas) [3] proved the celebrated Wagner conjecture [5] which stated that if a class of graphs (considered up to homeomorphism) is **minor-closed** (i.e., it is closed under **edge deletion**, **edge contraction** and **isolated node deletion**), then it can be characterized by a **finite number** of excluded minors.

This conjecture was motivated by various evidences for concrete natural minor-closed properties of graphs, such as knotless or linkless embeddability in $\mathbb{R}^3$, planarity or embeddability in a standardly embedded $S_g \subset \mathbb{R}^3$.

Here we say that a property P is *minor-closed* if for every graph X possessing this property every minor Y of G possesses P as well. Later, we shall define the notion of *minor* in a way suitable for **framed** 4-**graphs.**

The most famous evidence of this conjecture is the *Pontrjagin-Kuratowski* planarity criterion which states (in a slightly different formulation) that a graph is not planar if and only if it contains one of the two graphs shown in Fig. 1 **as a minor.**

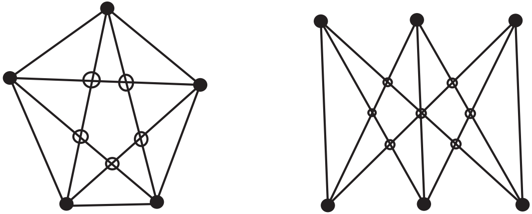
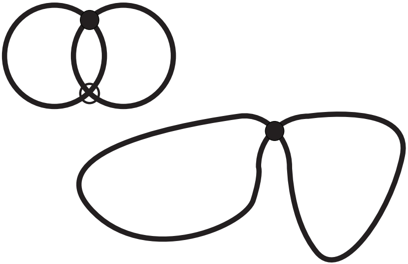


Figure: The two Kuratowski graphs, K_5 and $K_{3,3}$

Definition

A *framed* 4-valent graph is a regular 4-graph where at each vertex, the four edges incident to it are split into two pairs; vertices from the same pair are called *opposite*.

We allow *loops* and *multiple edges*; we also allow *circular components* homeomorphic to a 1-circle. Circular components themselves are called *circular edges*.



Framed 4-graphs

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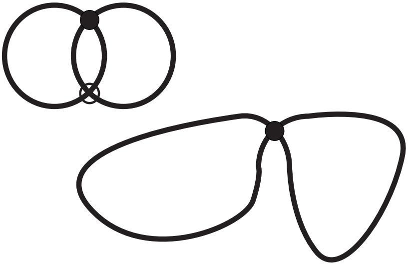

Framed graphs appear as medial graphs for arbitrary graphs embedded in 2-surfaces, they also appear as singular links or knot projections; by using Euler tours they can be encoded by chord diagrams.

In 2004, the author proved Vassiliev's conjecture on planarity of singular links saying that a framed 4-graph is non-planar if it contains a pair of cycles having **exactly one transverse** intersection.

Later, Nikonov [Nikonov] proved that this theorem was equivalent to the Pontrjagin-Kuratowski theorem, so, we have a first evidence of the following fact: classical problems for framed 4-graphs are easier as those for arbitrary graphs, and they are deeply connected.

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Besides that, the author proved several criteria for non-planarity and non-embeddability into $\mathbb{R}P^2$ for different classes of framed 4-graphs. It is important that for framed 4-graphs one can consider "**minor-closed properties**", where by "**minor**" we take a graph which can be obtained from the initial one by a finite sequence of **smoothings** at vertices (and deletions of separate components).

Similar criterion for $*$-graphs (Friesen).

As another evidence of the minor-closed property we shall take the planarity problem for framed 4-graphs admitting a **source-sink** structure; for such graphs, the non-planarity is equivalent to the existence of a **non-trivial "sublink"** for each embedding in $\mathbb{R}^3$ and to an existence of a non-trivial 2-component "sublink" with an **odd linking number**.

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Theorem ([VasConj])

A framed 4-valent graph K is planar if it does not contain two cycles with a unique **transverse** *intersection point.*

Later on, we call this graph the *Vassiliev obstruction* and denote it by Γ.

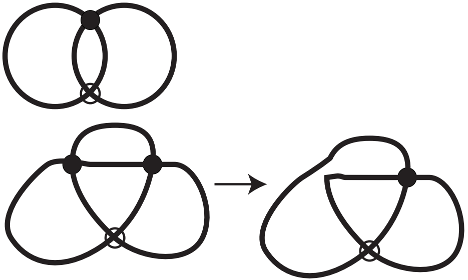

Figure: A non-planar framed 4-graph and its Vassiliev's obstruction

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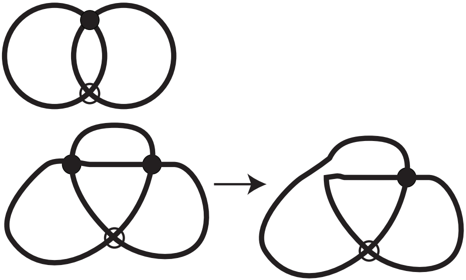

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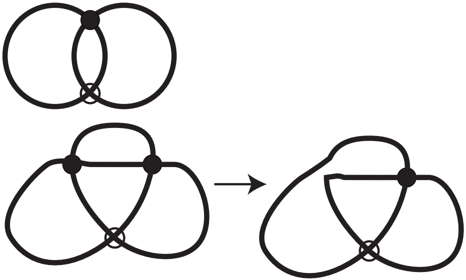

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Minors for framed 4-graphs

Definition

A framed 4-valent graph G' is a *minor* of a framed 4-valent graph G if G' can be obtained from G by a sequence of *smoothing* operations ( $\to$ ) and **deletions of connected components.**

Source-sink structure

Definition

We say that a framed 4-graph Γ admits a *source-sink structure* if there is an orientation of all edges Γ such that at every vertex of Γ some two opposite edges are incoming, and the other two are emanating. Certainly, for every connected framed four-valent graph, if a source-sink structure exists, then there are exactly two such structures.

Statement

If Γ admits a source-sink structure then every minor Γ' of Γ admits a source-sink structure as well. Indeed, the smoothing operation can be arranged to preserve the source-sink structure.

Outline

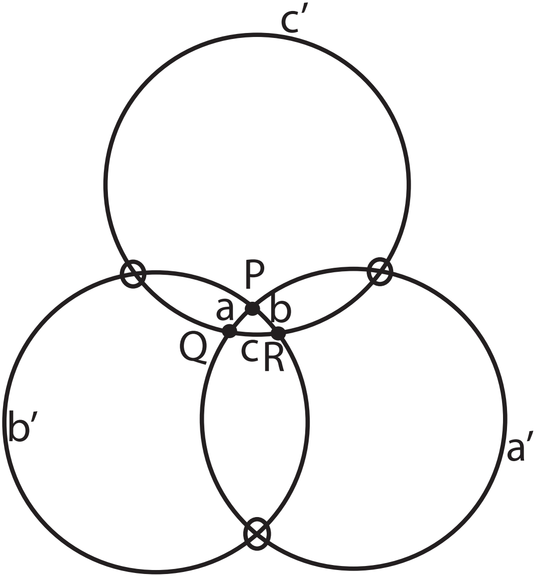

Figure: The graph Δ

Orientable Case

Theorem

Let Γ be a framed 4-graph admitting a source-sink structure. Then the following four conditions are equivalent:

1. *Every generic immersion of Γ in $\mathbb{R}^2$ requires at least 3 additional crossings;*
2. *For every embedding of Γ, there exists a pair of rotating loops with odd linking number;*
3. *Γ has no* **linkless** *(see further) embedding in $\mathbb{R}^3$;*
4. *Γ is not planar;*
5. *Γ contains Δ as a minor.*

Vassiliev's Conjecture and Pontrjagin-Kuratowski Planarity Criterion

In [Nikonov], I.M.Nikonov found a new proof of the Pontrjagin-Kuratowski planarity by using framed 4-graphs and vice versa.

He converted **arbitrary graph** into **planar** 4-**graphs**.
(Let Γ be a graph with rotation. Then its *medial graph* $M(G)$ is a framed 4-graph.)
Nikonov's theorem that the Vassiliev conjecture is equivalent to Pontrjagin-Kuratowski planarity criterion leads us to the following goal:

Understand all graphs by using framed 4-graphs.

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s-minors

Definition

A framed 4-graph Γ' is an *s-minor* of a framed 4-graph Γ if it can be obtained from Γ by a sequence of two operations:

1. Passing to a subgraph with all vertices of even valency and deleting all vertices of valency 2;
2. Removing connected components.

Remark

By definition, if X is a minor of Y, then X is an s-minor of Y; the inverse statement is wrong: Γ is an s-minor of Δ, but is not a minor of Δ.

Besides, we can see in Fig. 5 that Γ sits inside Δ: in Fig. 5 Γ is drawn as a subgraph of Δ in *red*.

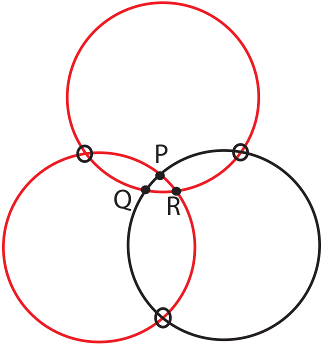

Figure: The Graph Γ inside the graph Δ

The Main Theorem (unorientable case)

Theorem

A graph P is non-planar if and only if it admits either Γ or Δ as a minor. Alternatively, P is non-planar if and only if it admits Γ as an s-minor.

Outline

Checkerboard Embeddings

Definition

A *checkerboard embedding* $f : P \to \Sigma$ is an embedding such that the connected components of the complement $\Sigma \backslash P$ can be colored in **black** and **white** in a way such that every two components sharing an edge have different colours.

- All embeddings into $\mathbb{R}^2$ are checkerboard.
- Checkerboard embeddability into any fixed 2-surface Σ is a **minor-closed** property: if the complement to the image of a framed 4-graph P is checkerboard colourable, then so is the complement to the image of P', where P' is obtained from P by a smoothing of a vertex.

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The Graph Γ_1

The framed 4-graph Γ_1 is the connected sum of two copies of Vassilev's graph.

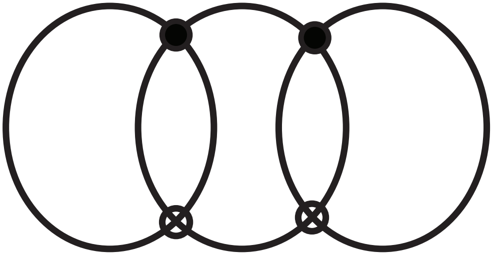

Figure: The graph Γ_1

Embeddings into $\mathbb{R}P^2$

Theorem

A framed 4-graph P is not checkerboard-embeddable in $\mathbb{R}P^2$ if and only if it contains one of the subgraphs Δ, Γ_1 as a minor.
More precisely, if P admits a source-sink structure then checkerboard-embeddability in $\mathbb{R}P^2$ is equivalent to checkerboard embeddability into $\mathbb{R}^2$. If P does not admit a source-sink structure then the only obstruction to checkerboard embeddability into $\mathbb{R}P^2$ is the existence of Γ_1 as a minor.

Outline

Main idea of the proof

To prove the main theorem, we use the concepts of **chord diagrams** and **rotating circuits**.

If a framed 4-graph is checkerboard-embeddable into a 2-surface Σ then it splits the surface into the "black part" and "white part".
The sum of genera of the black part and the white part equals the genus of the surface.

Rotating circuits

Definition

By a *rotating circuit* of a connected framed 4-graph Γ not homeomorphic to a circle we mean a surjective map $f : S^1 \to \Gamma$ which is a bijection everywhere except preimages of crossings of Γ such that at every crossing X the neighbourhoods $V(Y_1)$ and $V(Y_2)$ of the two preimages Y_1, Y_2 of X belong to unions of *adjacent* half-edges each. In other words, the circuit "passes" from a half-edge to a non-opposite half-edge.

For a framed 4-graph homeomorphic to the circle, the *circuit* is a homeomorphism of the circle and the graph.

A circuit f is called *good* at a vertex X of P if for the two inverse images $Y_1, Y_2 \in S^1$ of X, the neighbourhoods of the small segments $(Y_1 - \varepsilon, Y_1]$ and $(Y_2 - \varepsilon, Y_2]$ of the circle are taken by f to a pair of *opposite* half-edges at X.

Otherwise the rotating circuit is called *bad* at X.

The rotating circuit is *good* if it is good at every vertex.

Definition

By a *chord diagram* we mean either an oriented circle *(empty chord diagram)* or a cubic graph D consisting of an oriented cycle (the *core*) passing through all vertices of D such that the complement to it is a disjoint union of edges *(chords)* of the diagram.
A chord diagram is *framed* if every chord of it is endowed with a *framing* 0 or 1.

Definition

We say that two chords a, b of a chord diagram D are *linked* if the ends of the chord b belong to two different components of the complement $Co\backslash\{a_1, a_2\}$ to the endpoints of a in the core circle Co of D.

Having a circuit C of a framed connected 4-graph G, we define the framed chord diagram $D_C(G)$, as follows. If G is a circle, then $D_C(G)$ is empty. Otherwise, think of C as a map $f : S^1 \to D$; then we mark by points on S^1 preimages of vertices of G. Thinking of S^1 as a core circle and connecting the preimages by chords, we get the desired cubic graph. The framing of good vertices is set to be equal to 0, the framing of bad vertices is set to be equal to 1.

The opposite operation (of restoring a framed 4-graph from a chord diagram) is obtained by removing chords from the chord diagram and approaching two endpoints of each chord towards each other. For every chord, we create a crossing, and for every chord with framing zero, we create a small twist, as shown in Fig. 7.

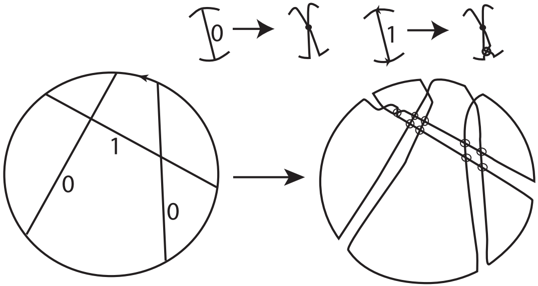

Figure: Restoring a framed 4-graph from a chord diagram

Definition

A (framed) chord diagram D' is called a *subdiagram* of a chord diagram D if D' can be obtained from D by deleting some chords and their endpoints (with framing respected).

It follows from the definition that the removal of a chord from a framed chord diagram results in a smoothing of a framed 4-graph. Consequently, if D' is a subdiagram of D, then the resulting framed 4-graph $G(D')$ is a *minor* of $G(D)$.

Vassiliev's conjecture $\to \Delta$ as a minor

Our goal is to prove that the non-planarity of a framed 4-graph with a source-sink structure yields the existence of Δ as a minor. After that, we see that every immersion of Δ requires at least 3 points, which is obvious, and prove that there for every embedding of Δ in $\mathbb{R}^3$, there exists a pair of rotating loops without crossing points having *odd linking number*. The latter automatically means that the embedding is not linkless.

We follow the proof of Vassiliev's conjectutre [4] from [VasConj]. Take a rotating circuit C for Γ; by assumption, Γ admits a source-sink structure, thus, the chord diagram $D_C(\Gamma)$ contains a $(2n+1)$-gon Δ_{2n+1} as a subdiagram, see Fig. 8.

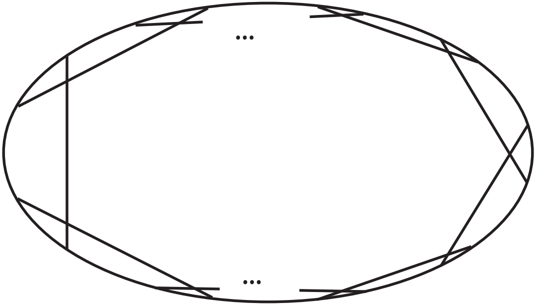

Figure: A $(2n+1)$-gon

Consequently, the initial graph will have a minor which corresponds to the chord diagram Δ_{2n+1}; we denote this framed 4-graph by Z_{2n+1}.
Now, we apply the following fact whose prove is left to the reader as an exercise: Δ is a minor of Z_{2n+1} for every natural n.

The proof for $\mathbb{R}P^2$ is **similar.**

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Linkless embeddability

Definition

An embedding i of a framed 4-graph P in $\mathbb{R}^3$ with a source-sink structure is called *linkless* if for every two rotating loops L_1, L_2 without transverse intersection the linking number of their images is 0.
Analogously, an embedding i of a framed 4-graph P in $\mathbb{R}^3$ with a source-sink structure is *knotless* if the image of the every rotating loop L is unknotted.

What is a link obtained from a graph?

For a crossing image, we *smooth* in some neighbourhood in $\mathbb{R}^3$ $\to$ or $\to$ with respect to the framing.

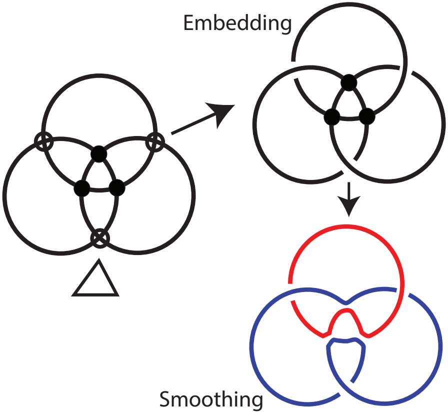

Figure: An embedding of Δ and a link obtained in its smoothing.

Outline

Elements of knot theory in the proof

Each embedding of Δ yields a link with an odd linking number. Just take the sum of four 2-component links modulo 2.

$$lk((a,b,c),(a',b',c')) + lk((a,b,c'),(a',b',c))$$

$$+lk((a,b',c),(a',b,c')) + lk((a',b,c),(a,b',c')).$$

is invariant.

Main Problems of the Stone Flower Project (Manturov-Nikonov)

1. Prove that the every minor-closed property of framed 4-graphs is defined by **finitely many** forbidden minors (Conjecture 1);
2. Prove that the first problem is equivalent to the Robertson-Seymour-Thomas proof of Wagner's conjecture.
   In particular,
   - Prove that Conjecture 1 yields the Wagner Conjecture.
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3. Understand the most important partial case: embeddability of framed 4-graphs into a surface of genus g.

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