The restoring argument and the new intrinsically knotted graphs with 22 edges

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Joint work with Thomas Mattman and Seungsang Oh

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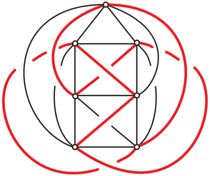
- A *graph* $G = (V, E)$ is a set of vertices and edges.
- A *spatial embedding* of a graph G is an embedding of G in $\mathbb{R}^3$.
- A *complete graph* on n vertices(K_n) is a graph with n vertices all possible edges connecting those n vertices.
- A graph G is *intrinsically knotted* (IK) if every embedding of the graph contains a non-trivially knotted cycle.

$K_{3,3,1,1}$

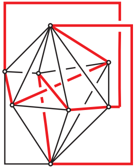
- Conway and Gordon (1983)

  Every embedding of K_7 contains a knotted cycle.



- Foisy (2002)

  $K_{3,3,1,1}$ is an intrinsically knotted graph.



Introductions

- A graph H is *minor* of another graph G if H can be obtained from G by edge contracting or edge deleting some edges.
- A graph G is intrinsically knotted and has no proper minor which is intrinsically knotted, G is said to be *minor minimal intrinsically knotted*.
- Robertson and Seymour (2003)

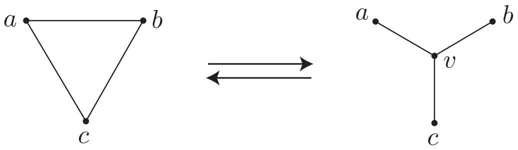
  There are only finite number of minor minimal intrinsically knotted graphs.

Open Problem

Finding the complete set of minor minimal intrinsically knotted graphs.

Introductions

- ∇-Y *move* and Y-∇ *move*



- Motwani, Raghunathan and Saran (1988)

  ∇-Y move preserves intrinsic knottedness.

- Flapan and Naimi (2008)

  Some Y-∇ moves do not preserve intrinsic knottedness.

- If G' is obtained from G by some ∇-Y or Y-∇ moves then G and G' are *cousin*. The set of all cousins of G is called the G *family*.

Intrinsically knotted graphs with at most 21 edges

A graph is *n-apex* if it can obtain a planar graph by removing n vertices.

- Blain, Bowlin, Fleming, Foisy, Hendricks and Lacombe (2007), Ozawa and Tsutsumi (2007)

  If G is a 2-apex, then G is not intrinsically knotted.

- Johnson, Kidwell and Michael (2010)

  Any intrinsically knotted graph consists at least 21 edges.

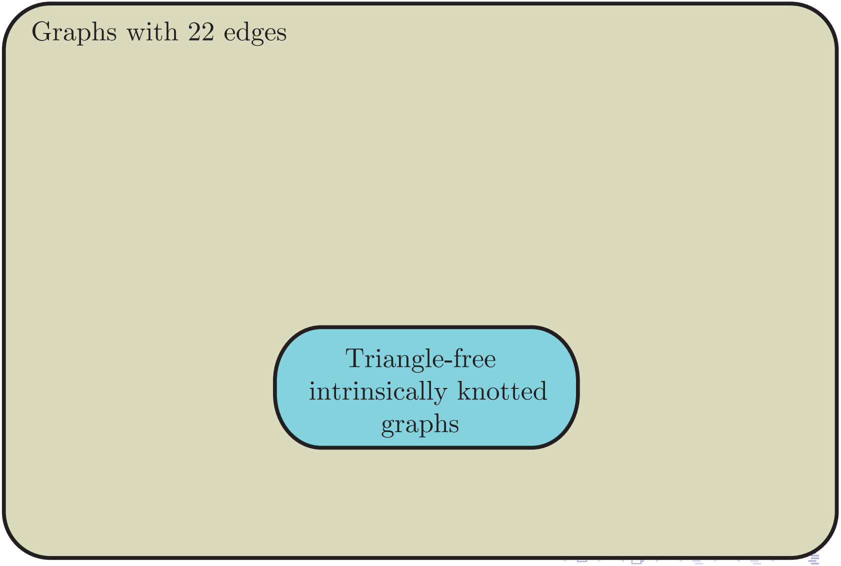
- Lee, Kim, Lee and Oh (2015), Barsotti and Mattman (2016)

  The only triangle-free intrinsically knotted graphs with 21 edges are H_{12} and C_{14}.

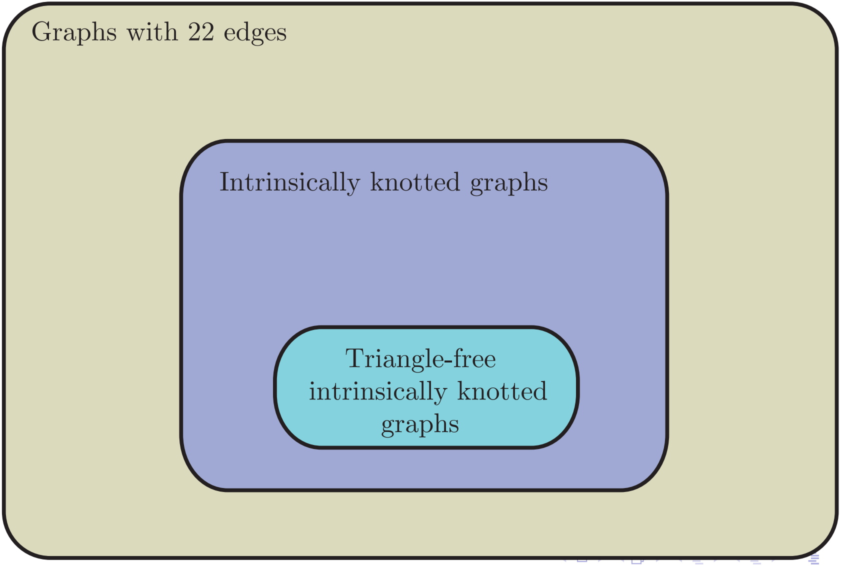
Intrinsically knotted graphs with 22 edges

Graphs with 22 edges

Intrinsically knotted graphs with 22 edges



Intrinsically knotted graphs with 22 edges



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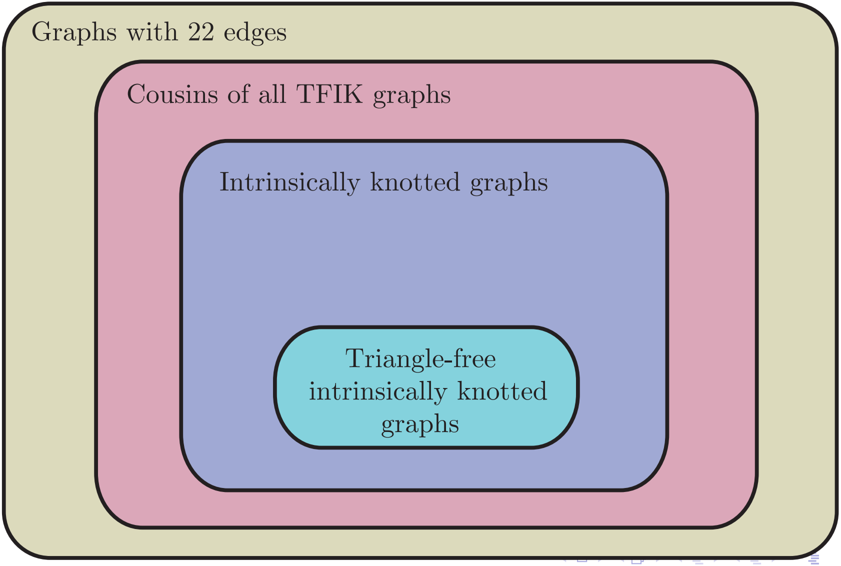
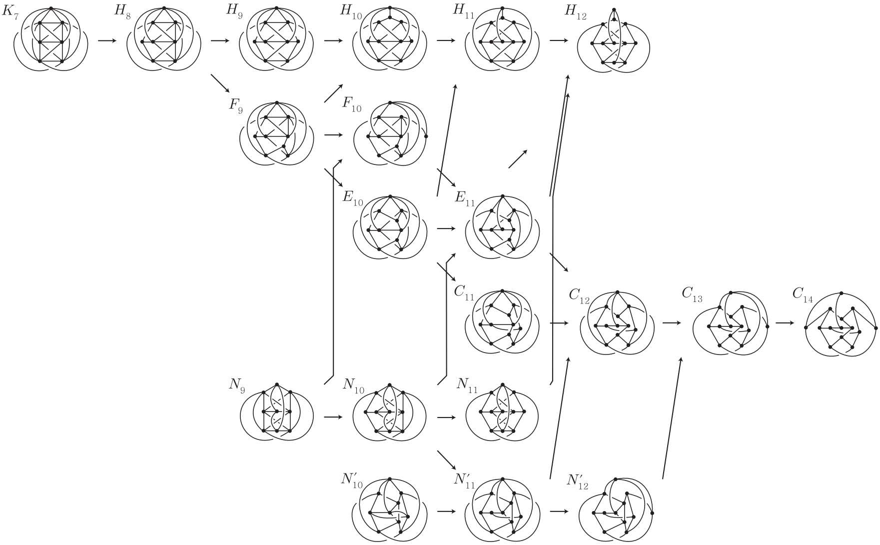


$E_9 + e$



K_7
H_8
H_9
H_{10}
H_{11}
H_{12}
F_9
F_{10}
E_{10}
E_{11}
C_{11}
C_{12}
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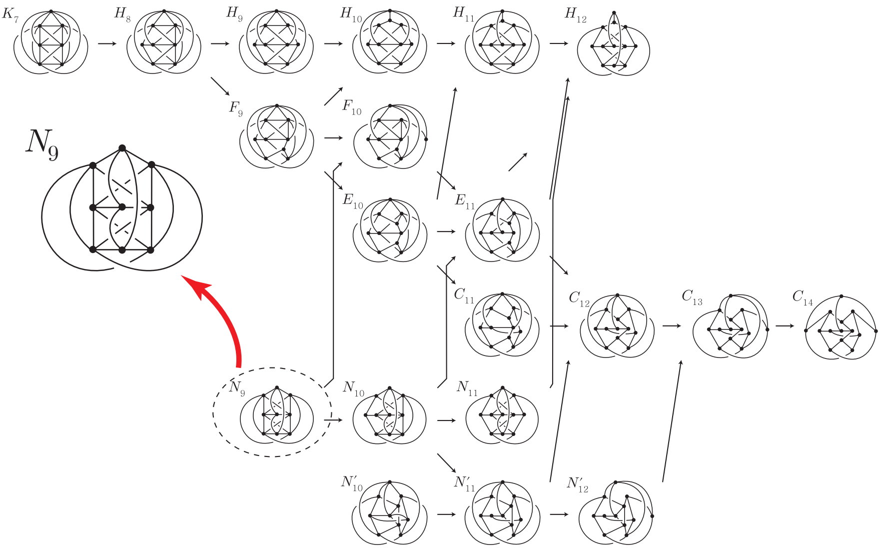

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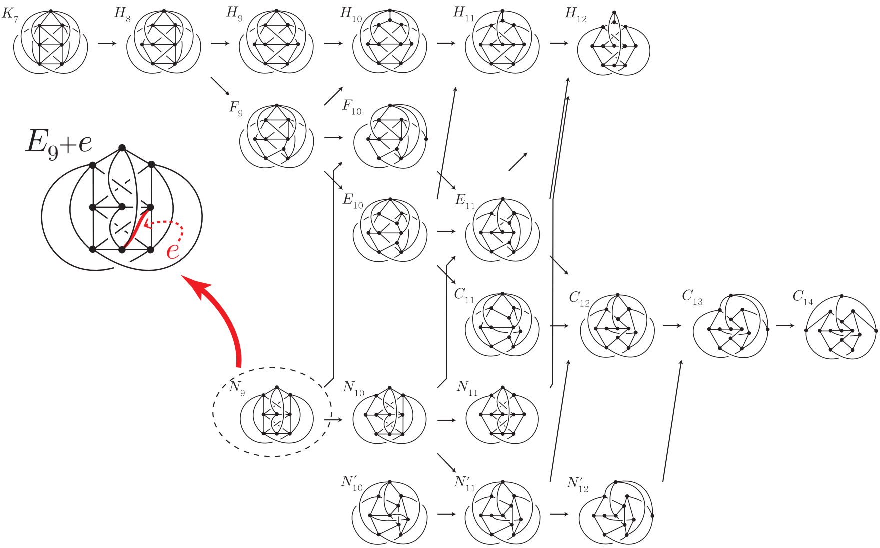

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E_9+e
e
E_{10}
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C_{11}
C_{12}
C_{13}
C_{14}
N_9
N_{10}
N_{11}
N'_{10}
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N'_{12}

Intrinsically knotted graphs with 22 edges

- Goldberg, Mattman and Naimi (2014)

  There are 58 intrinsically knotted graphs in the $K_{3,3,1,1}$ family and 110 intrinsically knotted graphs in the $E_9 + e$ family.

Family	Total graphs	IK graphs	TFIK graphs
$K_{3,3,1,1}$	58	58	4
$E_9 + e$	110	110	10

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Intrinsically knotted graphs with 22 edges

Theorem [Kim, Lee, Lee, Mattman and Oh]

Exactly 3 triangle-free intrinsically knotted graphs with 22 edges which has at least two vertices with degree 5.

Furthermore, there does not exist any triangle-free intrinsically knotted graph with 22 edges which has a vertex with degree 6 or more.

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Terminology

For any two distinct vertices a and b,

- $G = (V, E)$: a graph with 22 edges and a single degree 5 vertex.
- $\widehat{G}_{a,b} = (\widehat{V}_{a,b}, \widehat{E}_{a,b})$: the graph obtained from $G \backslash \{a, b\}$ by contracting one edge incident to a vertex of degree 1 or 2 repeatedly until no vertices of degree 1 or 2 exist.
  (removing a vertex means deleting interiors of all edges incident to it.)
- $\mathrm{dist}(a, b)$: the distance between two vertices a and b says the number of edges in the shortest path connecting them.
- $\deg(a)$: the degree of a vertex a.

How to prove the Main Theorem

- Construction all possible such triangle-free graph G with 22 edges and a single degree 5 vertex.
- Deleting two suitable vertices a and b of G.
- Check that the result graph is planar or not.

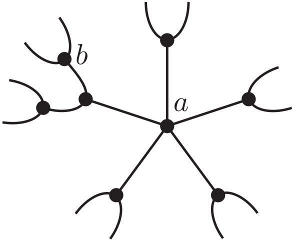
When we find the graph which is not 2-apex, we will find same graph in the $K_{3,3,1,1}$ or $E_9 + e$ family.

If we can not find the same graph in those family, we will check that it has minor which is intrinsically knotted.

Sketch of the Proof

We say that G is of $(|V_4|,|V_3|)$-type according to the degree of vertices. Since $|E(G)| = 22$, G has four types, (0,13), (3,9), (6,5) and (9,1).

Example of (0,13)-type.



$$|\widehat{E}_{a,b}| \leq 22$$

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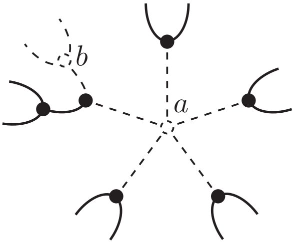

$$|\widehat{E}_{a,b}| \leq 22 - 8 = 14$$

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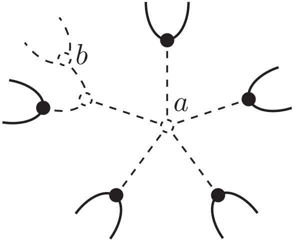

$$|\widehat{E}_{a,b}| \leq 22 - 8 - 1 = 13$$

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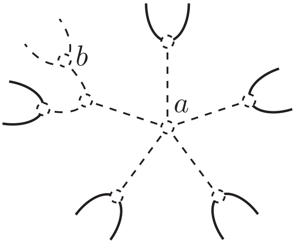
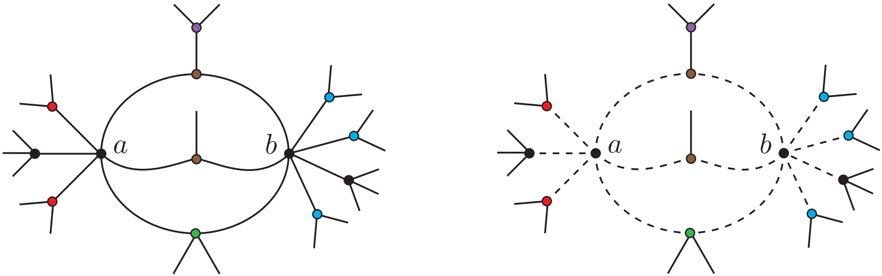


$$|\widehat{E}_{a,b}| \leq 22 - 8 - 1 - 5 = 8$$

Count Equation

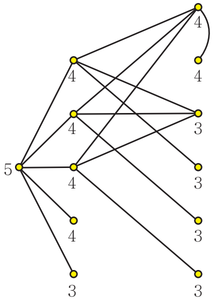
We have a *count equation* in $\widehat{G}_{a,b}$;

$$|\widehat{E}_{a,b}| = 22 - |E(a) \cup E(b)| - \{|V_3(a)| + |V_3(b)| - |V_3(a,b)| + |V_4(a,b)| + |V_Y(a,b)|\}$$



The Restoring Argument

The restoring argument is a construction operation that makes us obtain the graph G that avoids 2-apex as far as possible by the given incompletely connecting information.



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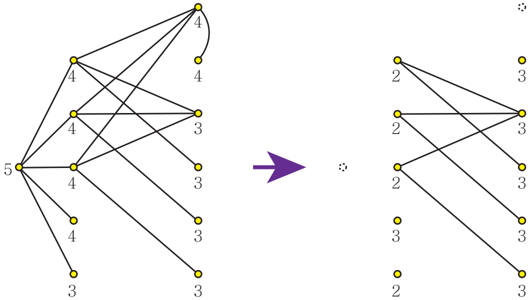

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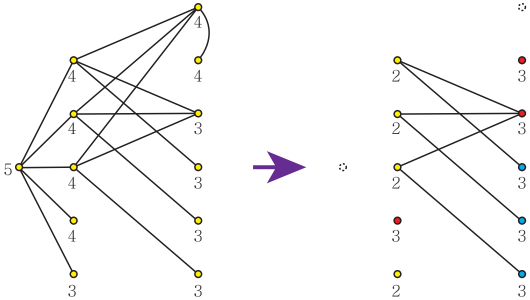

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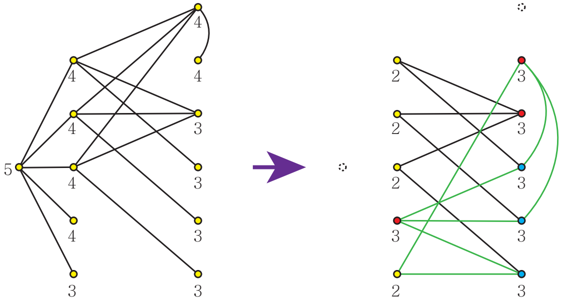

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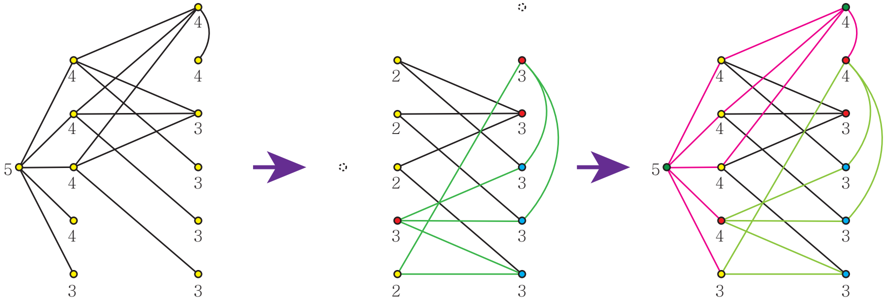

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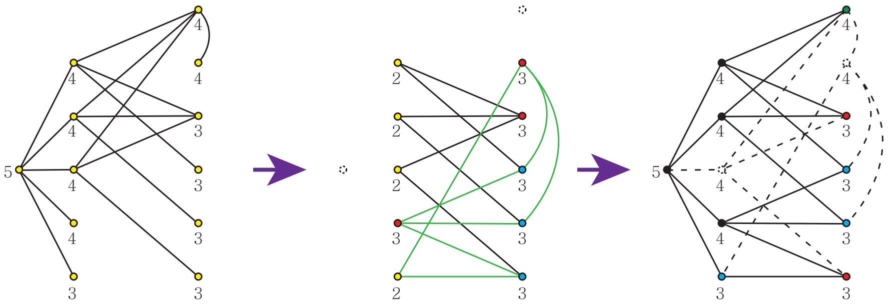

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Since $|E(G)| = 22$, G has four types, (0,13), (3,9), (6,5) and (9,1).

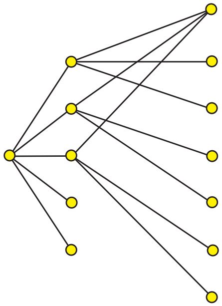
In (0,13) and (9,1) types, there does not exist any IK graph.

Now, we only need to check (3,9) and (6,5) types.

(3,9)-type

By using the restoring argument, we can obtain the following graph which is not 2-apex.

This graph is the cousin 29 of the $K_{3,3,1,1}$ family.



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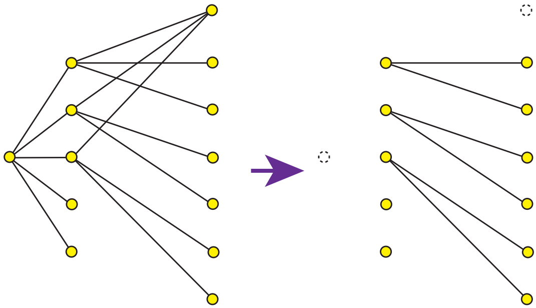

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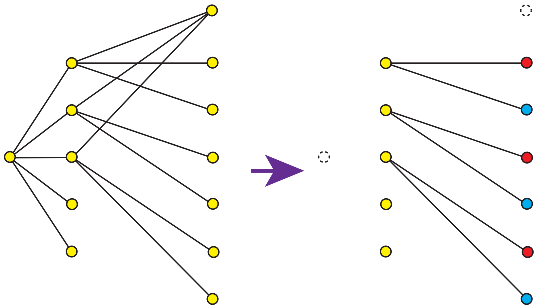

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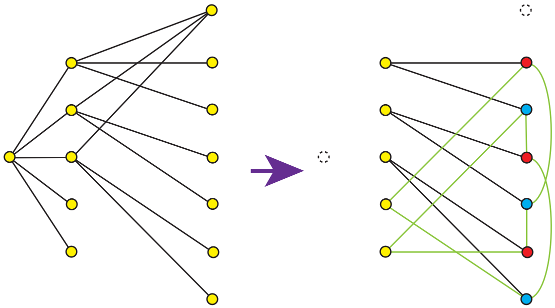

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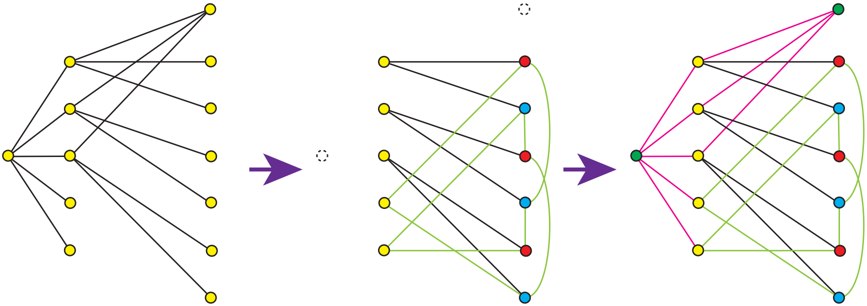
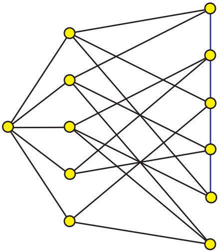


(6,5)-type

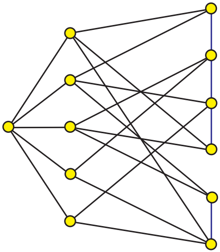
In this type, we will divide into three part according to $|V_4(a)|$.

If $|V_4(a)| \leq 2$ or $|V_4(a)| = 5$ then any graph can not be intrinsically knotted.

If $|V_4(a)| = 3$ then there are two intrinsically knotted graphs.



97th cousin of the $E_9 + e$ family



The new intrinsically knotted graph U_{12}

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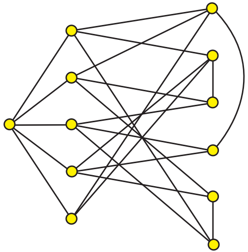
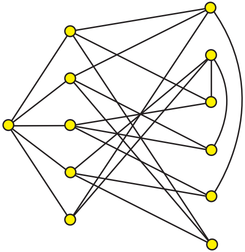


99th cousin of the $E_9 + e$ family



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Concluding remarks

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