Heegaard Floer homology and Spatial graphs

Yuanyuan Bao (Univ. of Tokyo)

International Workshop on Spatial graphs 2016

Waseda University

§ Heegaard Floer (HF) for closed 3-mfd

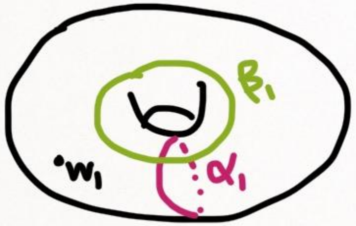
M : a closed ori. 3-mfd

<u>Def</u>: $(\Sigma_g, \alpha=\{\alpha_1, \alpha_2, \cdots, \alpha_d\}, \beta=\{\beta_1, \beta_2, \cdots, \beta_d\}, W=\{w_1, w_2, \cdots, w_n\})$ is an n-pointed Heegaard diagram of M if

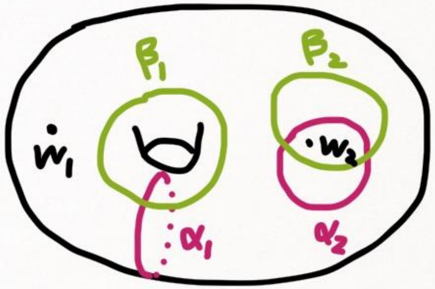
- Σ_g : genus g closed ori. surface. (Heegaard surface)
  α, β : two sets of pairwise disjoint s.c.c. on Σ_g (α-curve, β-curve)
  W : a set of n points on $\Sigma_g \setminus \alpha \cup \beta$. $n = d-g+1$ (base point)
- Attaching 2-handles along $\alpha \cup \beta$ to Σ_g gives $2n$-punctured M.
- Let $\{A_i\}_{i=1}^n = \Sigma_g \setminus \alpha$, $\{B_i\}_{i=1}^n = \Sigma_g \setminus \beta$.
  Then $w_i \in A_i \cap B_i$ (by relabelling $\{B_i\}_{i=1}^n$ if necessary).

Example:

Σ_1



Σ_1



$H = (\Sigma_g, \alpha, \beta, w)$: an n-pointed Heegaard diagram of M.

$\mathrm{Sym}^d(\Sigma_g) := \Sigma_g^{\times d} / S_d$

$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_d$

$\mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_d$

Fix j : a complex structure on Σ_g

η : a 2-form on Σ_g, which is Kähler w.r.t. j.

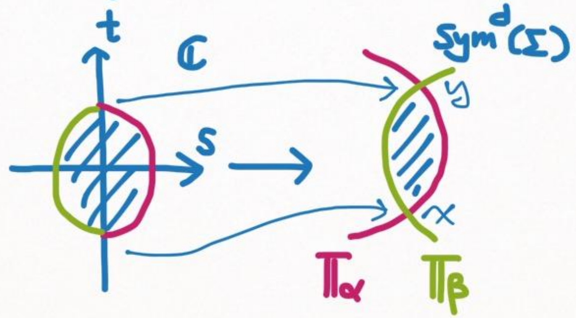
$\mathcal{J}(j, \eta, V_w)$: the set of nearly symmetric almost cpx str. over $\mathrm{Sym}^d(\Sigma_g)$ w.r.t. j, η, w.

Def: For a generic path $J_s \subset \mathcal{J}(j, \eta, V_w)$

$$CF^-(H, J_s) := \mathbb{F}[U_1, \cdots U_n]\langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle$$

$$\partial^-(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y)} \# \frac{\mathcal{M}(\phi)}{\mathbb{R}} \prod_{i=1}^{n} U_i^{n_{w_i}(\phi)} \cdot y, \quad \text{for } x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta.$$

Here

$\mathcal{M}(\phi) := \{ u :$  $\mid [u] = \phi, \ \partial_s u + J_s \partial_t u = 0 \}$

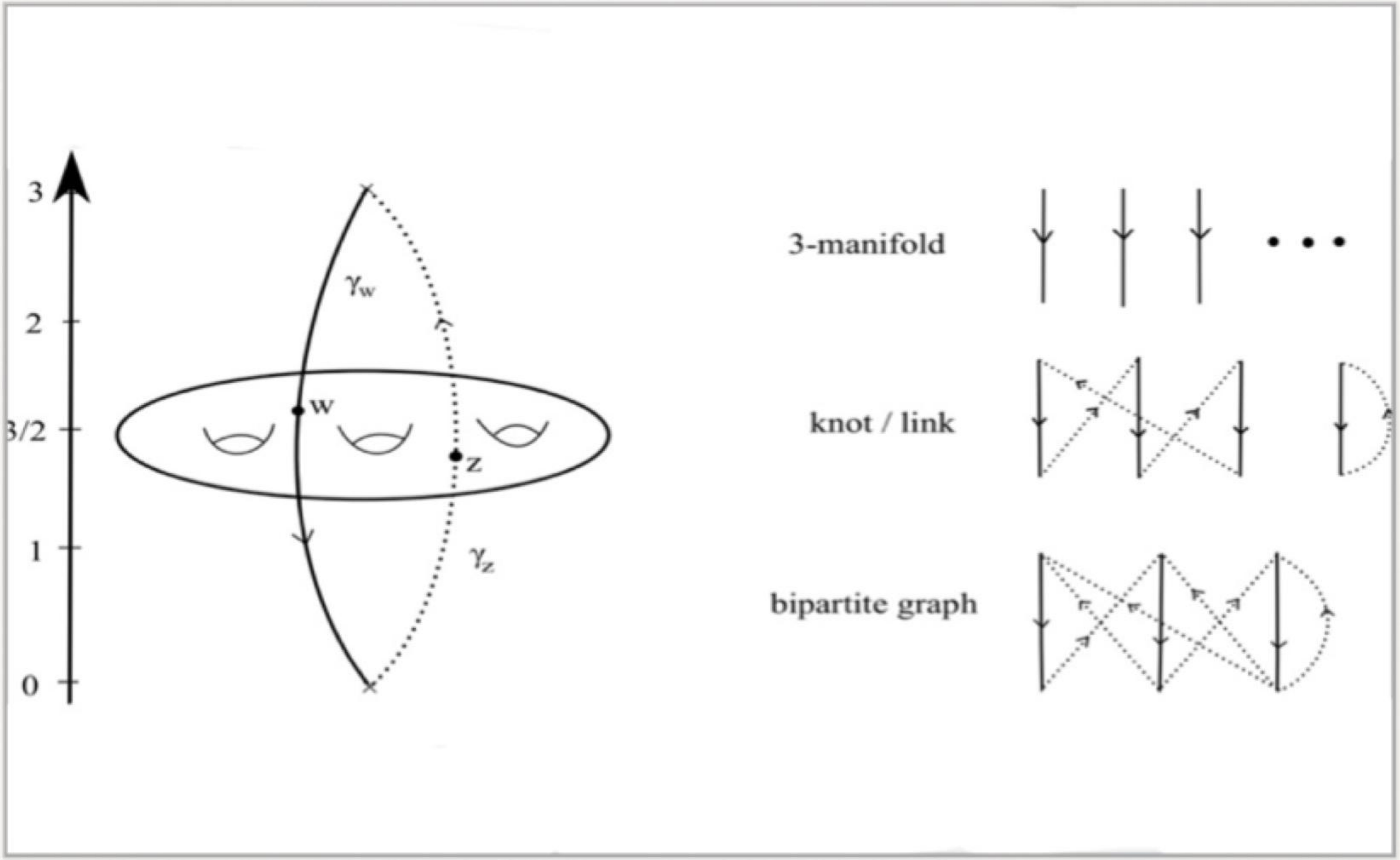
$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{3-mfd } M \arrow[d, rightsquigarrow] & & HF^-(M) \arrow[ll, "\text{A topological inv.}"'] \\
\text{Heegaard diagram } H \arrow[rr, rightsquigarrow] \arrow[rr, phantom, "\text{(Ozsváth–Szabó)}", shift left=12ex] & & \begin{array}{c}\text{Lagrangian Intersection Floer} \\ (CF^-, \partial^-)\end{array} \arrow[u, rightsquigarrow]
\end{tikzcd}
$$

<u>Question</u>:

- Invariant for a 3-mfd with boundary?
- Invariant for the embedding of a submanifold?

  (e.g. a link, or a graph)

§ Heegaard diagram for a link or a graph



Def: A graph (E, V) is a bipartite graph if

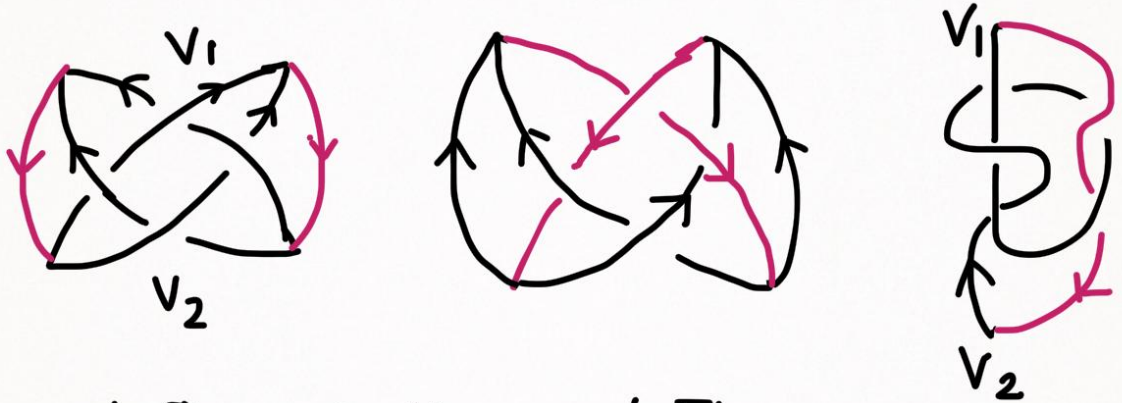
- $V = V_1 \amalg V_2 \quad V_i \neq \phi$
- $\forall e \in E$, e is incident to both V_1 & V_2.

If $|V_1| = |V_2|$, it is called a balanced bipartite graph.

Let G_{V_1, V_2} be a balanced bipartite graph. $|V_1| = |V_2| = n$

An orientation of G_{V_1, V_2} is called balanced if edges directing from V_1 to V_2 are disjoint and their endpoints occupy V_1 & V_2. Let them be $\{e_i\}_{i=1}^{n}$.

Example:

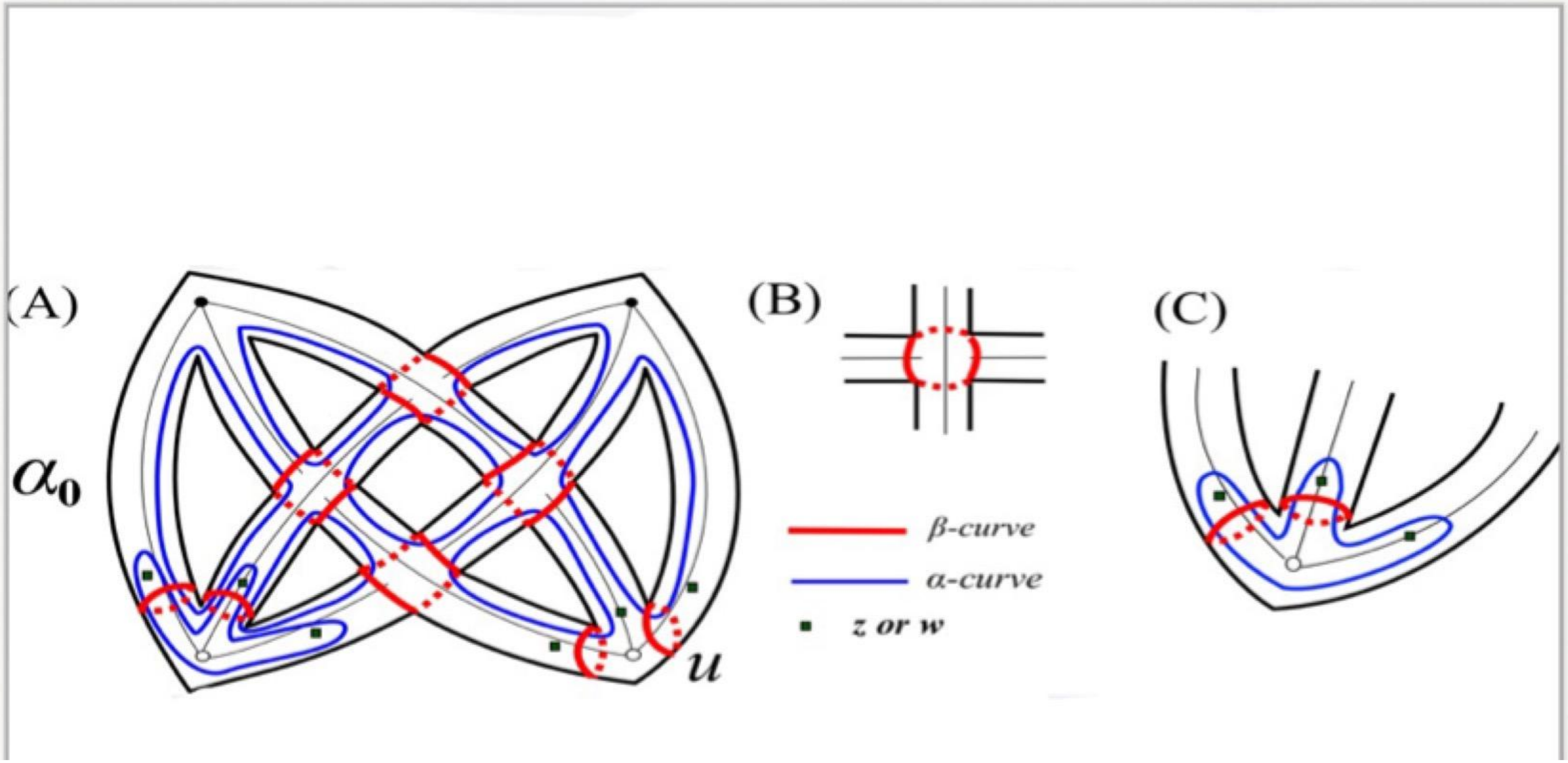


Note: We define the Heegaard Floer complex for a balanced bipartite graph with a balanced orientation.

<u>Def</u> (B.) $(\Sigma_g, \alpha, \beta, W, Z)$ is a Heegaard diagram of $(M, G_{v_1 v_2})$ if

- $(\Sigma_g, \alpha, \beta, W)$ is an n-pointed H.D. of M.
- $Z \subset \Sigma_g \setminus \alpha \cup \beta \cup W$ is a finite set of points.
- $G_{v_1, v_2} = \gamma_Z \cup \gamma_W$, and $\gamma_W = \{e_i\}_{i=1}^{n}$,

  where γ_Z and γ_W are the set of flow lines passing through Z and W resp.



Let $H=(\Sigma_g, \alpha, \beta, w, z)$ be a H.D. for (M, G_{v_1,v_2})

Def (B.) The Heegaard Floer complex is

$$CFG^-(H) = \mathbb{F}[U_1, U_2, \cdots, U_n]\langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle$$

$$\partial_G^-(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ n_z(\phi)=\{0\}}} \# \frac{M(\phi)^1}{\mathbb{R}} \prod_{i=1}^{n} U_i^{n_{w_i}(\phi)} \cdot y$$

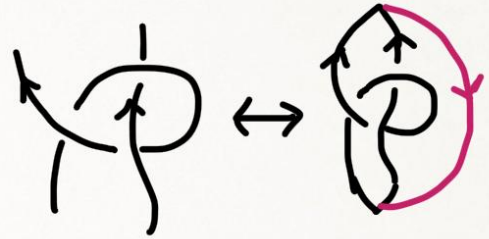
Rmk: • z is used to define a $H_1(M \backslash G_{v_1,v_2})$-grading to (CF^-, ∂^-).

• In general, it is not known if (CFG^-, ∂_G^-) is the associated graded complex of (CF^-, ∂^-).

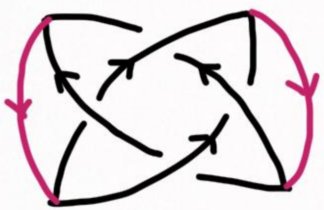
Thm (B.) $H_*(CFG^-, \partial_G^-)$ is an topological invariant of (M, G_{v_1,v_2}) with the given balanced orientation.

§ Related Research

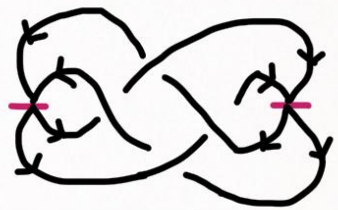
- Roberts (2009) HF for string link



- Alishahi & Eftekhary (2011) : A refinement of sutured Floer homology
- Harvay & O'Donnal (2015) : Combinatorial def for transverse graph.



bipartite

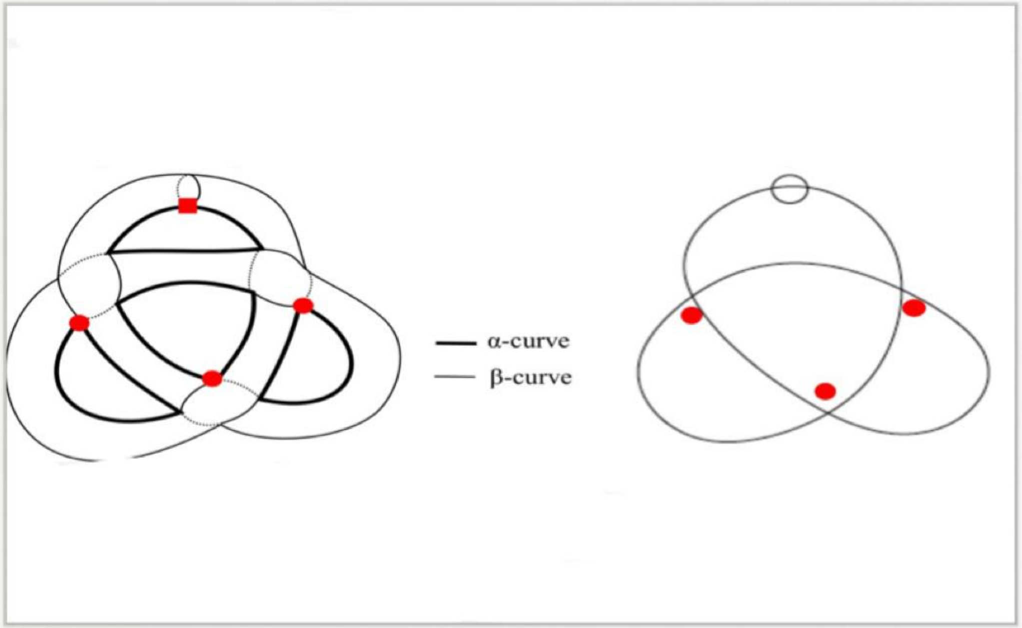


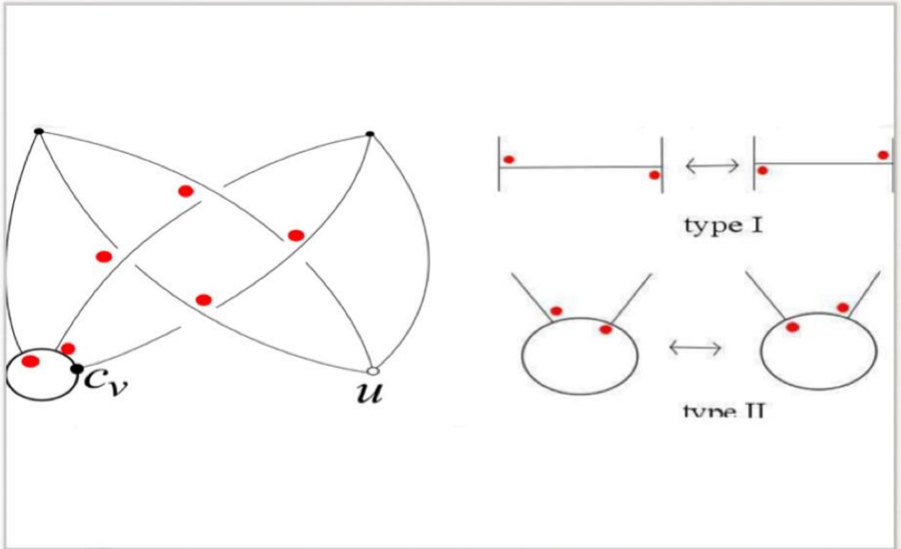
transverse

§ Next Plan 1

Each generator of CFG^- corresp. to a state of the graph projection.

- Study the transposition between states
- Study CFG^- for special graph projections.





§ Next Plan 2

Kronheimer & Mrowka (2015) $SO(3)$ Instanton Floer homology for trivalent spatial graph.

Thm (KM)

- $G \subset S^3$: trivalent graph $\rightsquigarrow J^{\#}(G)$
- $J^{\#}(G) = \{0\} \iff G$ has an embedded bridge

possibility to provide a new proof of the 4-color thm?

<u>Question</u>: How to define a Heegaard Floer homology for trivalent spatial graph?