On the Conway-Gordon theorems

Ryo Nikkuni
Department of Mathematics,
Tokyo Woman's Christian University

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Notations

For a finite graph G,

$$\Gamma(G) = \Gamma^{(1)}(G) \overset{\text{Def.}}{=} \{\text{all cycles of } G\}$$

$$\Gamma_k(G) \overset{\text{Def.}}{=} \{\text{all } k\text{-cycles of } G\}$$

$$\Gamma^{(n)}(G) \overset{\text{Def.}}{=} \left\{ \bigcup_{i=1}^{n} \gamma_i \;\middle|\; \gamma_i \cap \gamma_j = \emptyset \ (i \neq j),\ \gamma_i \in \Gamma(G) \right\}$$

$$\Gamma^{(n)}_{m_1, m_2, \dots, m_n}(G) \overset{\text{Def.}}{=} \left\{ \bigcup_{i=1}^{n} \gamma_i \;\middle|\; \gamma_i \cap \gamma_j = \emptyset \ (i \neq j),\ \gamma_i \in \Gamma_{m_i}(G) \right\}$$

$$\mathrm{SE}(G) \overset{\text{Def.}}{=} \{\text{all spatial embeddings of } G\}$$

For an oriented knot K or 2-component oriented link L,

$\mathrm{lk}(L)$: linking number of L

$a_2(K)$: 2nd coefficient of the Conway polynomial $\nabla_K(z)$

§1. Introduction

Theorem 1.1. [Conway-Gordon '83]

(1) $\forall f \in \mathrm{SE}(K_6)$, $\displaystyle\sum_{\lambda \in \Gamma^{(2)}(K_6)} \mathrm{lk}(f(\lambda)) \equiv 1 \pmod 2$.

(2) $\forall f \in \mathrm{SE}(K_7)$, $\displaystyle\sum_{\gamma \in \Gamma(K_7)} \mathrm{Arf}(f(\gamma)) \equiv 1 \pmod 2$,

where Arf denotes the Arf invariant.

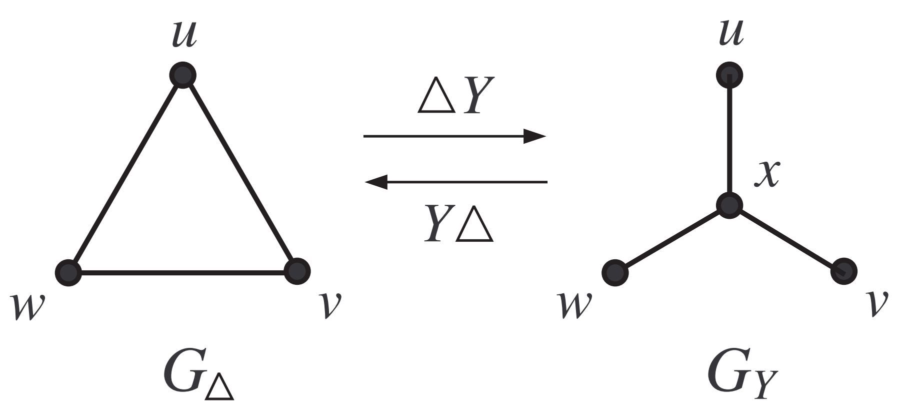
Note. For a knot K, $\mathrm{Arf}(K) \equiv a_2(K) \pmod 2$.

Corollary 1.2.

(1) [Sachs '84] K_6 is intrinsically linked. (IL)

(2) K_7 is intrinsically knotted. (IK)

$\triangle Y$-exchange
and $Y\triangle$-exchange



Proposition 1.3. [Motwani-Raghunathan-Saran '88]
$G_\triangle$ is IL (resp. IK) $\Longrightarrow$ G_Y is IL (resp. IK).

$$\mathrm{PF} \overset{\text{Def.}}{=} \left\{ G \;\middle|\; K_6 \xrightarrow{\triangle Y \ \&\ Y\triangle' s} G \right\} : \text{ Petersen family}$$

$$\mathrm{HF} \overset{\text{Def.}}{=} \left\{ G \;\middle|\; K_7 \xrightarrow{\triangle Y \ \&\ Y\triangle' s} G \right\} : \text{ Heawood family}$$

$$\mathrm{PF}_\triangle \overset{\text{Def.}}{=} \left\{ G \;\middle|\; K_6 \xrightarrow{\triangle Y' s} G \right\}, \quad \mathrm{HF}_\triangle \overset{\text{Def.}}{=} \left\{ G \;\middle|\; K_7 \xrightarrow{\triangle Y' s} G \right\}$$

Today's Topics:

§2. Conway-Gordon type theorems for graphs in $\mathrm{PF}_\triangle$ or $\mathrm{HF}_\triangle$, and its integral lifts.
(partially joint work with K. Taniyama)

§3. Intrinsic knottedness of graphs in $\mathrm{HF} \setminus \mathrm{HF}_\triangle$.
(joint work with R. Hanaki, K. Taniyama and A. Yamazaki)

§2. $\triangle Y$-exchanges and the Conway-Gordon theorems

Theorem 2.1. [N '09]

(1) $\forall f \in \mathrm{SE}(K_6)$,

$$2\left\{\sum_{\gamma\in\Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma\in\Gamma_5(K_6)} a_2(f(\gamma))\right\} = \sum_{\lambda\in\Gamma^{(2)}(K_6)} \mathrm{lk}(f(\lambda))^2 - 1.$$

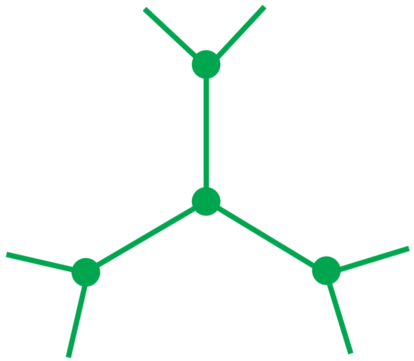
(2) $\forall f \in \mathrm{SE}(K_7)$,

$$7\sum_{\gamma\in\Gamma_7(K_7)} a_2(f(\gamma)) - 6\sum_{\gamma\in\Gamma_6(K_7)} a_2(f(\gamma)) - 2\sum_{\gamma\in\Gamma_5(K_7)} a_2(f(\gamma))$$
$$= 2\sum_{\lambda\in\Gamma^{(2)}_{3,4}(K_7)} \mathrm{lk}(f(\lambda))^2 - 21.$$

Remark. Original Conway-Gordon theorems can be obtained from Thm. 2.1 by taking the mod 2 reduction on both sides.

$$\Phi^{(n)} : \left\{ \lambda' \in \Gamma^{(n)}(G_\triangle) \mid \lambda' \not\supset \triangle \right\} \longrightarrow \Gamma^{(n)}(G_Y) \quad \text{surjective map}$$

Note. $\sharp {\Phi^{(n)}}^{-1}(\lambda) \leq 2 \ \text{ for } \ \forall \lambda \in \Gamma^{(n)}(G_Y)$.



$f(G_Y)$

$$\varphi : \mathrm{SE}(G_Y) \longrightarrow \mathrm{SE}(G_\triangle) \quad \text{map}$$

Proposition 2.2. For $f \in \mathrm{SE}(G_Y)$ and $\lambda \in \Gamma^{(n)}(G_Y)$,

$$f(\lambda) \cong \varphi(f)(\lambda') \quad \text{for} \quad \forall \lambda' \in {\Phi^{(n)}}^{-1}(\lambda).$$

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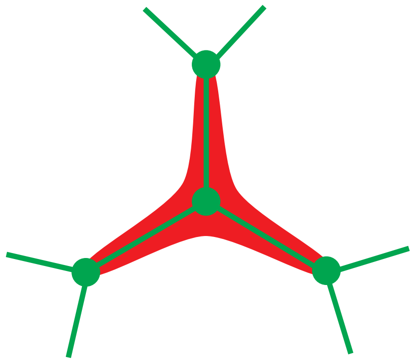

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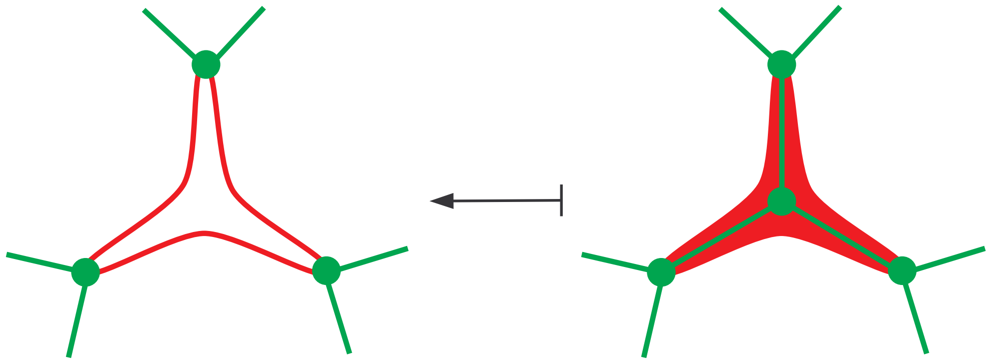

$\varphi(f)(G_{\triangle})$ $\quad$ $f(G_Y)$

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$$\begin{array}{ccccccc}\omega : & \Gamma^{(n)}(G_\triangle) & \longrightarrow & \mathbb{Z} & , & & \\ \tilde{\omega} : & \Gamma^{(n)}(G_Y) & \longrightarrow & \mathbb{Z} & & \text{maps} \\ & \cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid & & \\ & \lambda & \longmapsto & \displaystyle\sum_{\lambda' \in \Phi^{(n)^{-1}}(\lambda)} \omega(\lambda') & & \end{array}$$

Proposition 2.3.

$f \in \mathsf{SE}(G_Y)$

v : link invariant with $v(\text{trivial knot}) = v(\text{split link}) = 0$

$$\Longrightarrow \sum_{\lambda \in \Gamma^{(n)}(G_Y)} \tilde{\omega}(\lambda) v(f(\lambda)) = \sum_{\lambda' \in \Gamma^{(n)}(G_\triangle)} \omega(\lambda') v(\varphi(f)(\lambda')).$$

Theorem 2.4. [N-T]

(1) $\forall G \in \mathsf{PF}_\triangle$, $\exists \omega : \Gamma(G) \to \mathbb{Z}$ map s.t.

$$2 \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) = \sum_{\lambda \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\lambda))^2 - 1$$

for $\forall f \in \mathsf{SE}(G)$.

(2) $\forall G \in \mathsf{HF}_\triangle$, $\exists \omega : \Gamma(G) \to \mathbb{Z}$, $\exists \xi : \Gamma^{(2)}(G) \to \mathbb{Z}$ maps s.t.

$$\sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) = 2 \sum_{\lambda \in \Gamma^{(2)}(G)} \xi(\lambda) \mathrm{lk}(f(\lambda))^2 - 21$$

for $\forall f \in \mathsf{SE}(G)$.

(Outline of the proof of Thm. 2.4) $K_6 \xrightarrow{\triangle Y} Q_7$

$$\omega : \Gamma(K_6) \longrightarrow \mathbb{Z} \qquad \xi : \Gamma^{(2)}(K_6) \longrightarrow \mathbb{Z}$$

$$\gamma \mapsto \begin{cases} 1 & (\gamma \in \Gamma_6(K_6)) \\ -1 & (\gamma \in \Gamma_5(K_6)) \\ 0 & (\text{otherwise}) \end{cases} \qquad \lambda \longmapsto 1$$

$$\begin{aligned} 2\sum_{\gamma \in \Gamma(Q_7)} \tilde{\omega}(\gamma) a_2(f(\gamma)) &\overset{\text{Prop. 2.3}}{=} 2\sum_{\gamma' \in \Gamma(K_6)} \omega(\gamma') a_2(\varphi(f)(\gamma')) \\ &\overset{\text{Thm. 2.1 (1)}}{=} \sum_{\lambda' \in \Gamma^{(2)}(K_6)} \xi(\lambda') \mathrm{lk}(\varphi(f)(\lambda'))^2 - 1 \\ &\overset{\text{Prop. 2.3}}{=} \sum_{\lambda \in \Gamma^{(2)}(Q_7)} \tilde{\xi}(\lambda) \mathrm{lk}(f(\lambda))^2 - 1. \end{aligned}$$

Note. $\tilde{\xi}(\lambda) = 1$ for $\forall \lambda \in \Gamma^{(2)}(Q_7)$. $\square$

Corollary 2.5.

(1) [Taniyama-Yasuhara '01] $\forall G \in \mathsf{PF}_\triangle$,

$$\sum_{\lambda \in \Gamma^{(2)}(G)} \mathrm{lk}(f(\lambda)) \equiv 1 \pmod 2 \quad \text{for} \quad \forall f \in \mathsf{SE}(G).$$

(2) $\forall G \in \mathsf{HF}_\triangle$, $\exists \omega : \Gamma(G) \to \mathbb{Z}_2$, s.t.

$$\sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) \equiv 1 \pmod 2 \quad \text{for} \quad \forall f \in \mathsf{SE}(G).$$

Remark. Recently, integral version of the Conway-Gordon type theorem for $K_{3,3,1} \in \mathsf{PF} \setminus \mathsf{PF}_\triangle$ was given by O'Donnol.

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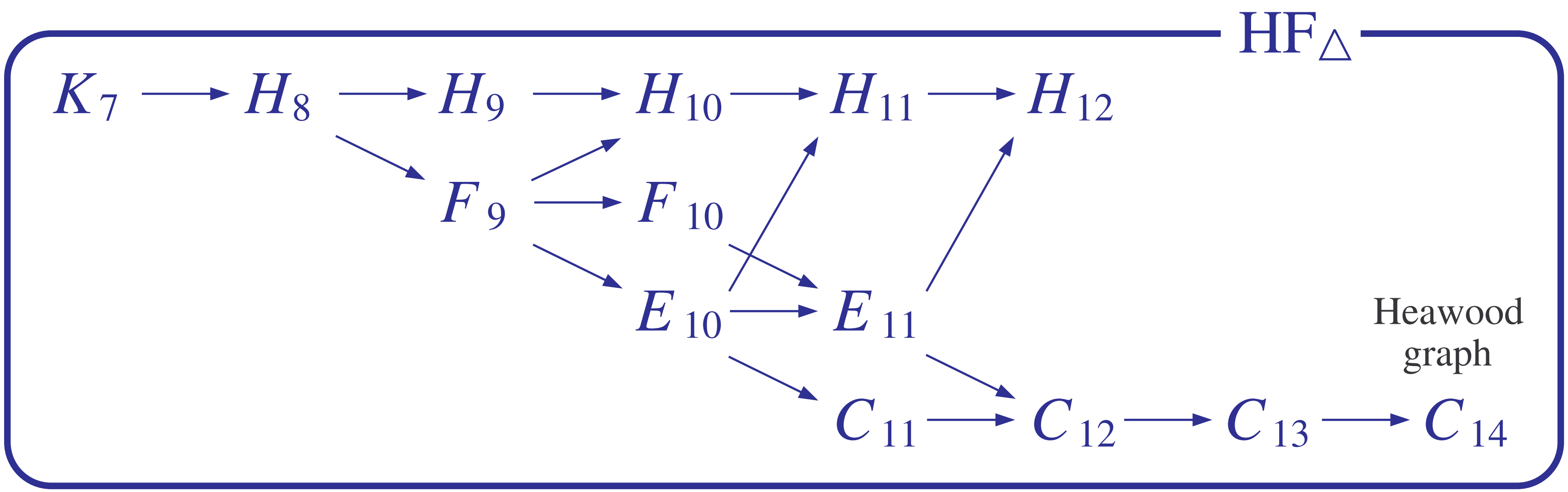
$$\sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) \equiv 1 \pmod 2 \quad \text{for} \quad \forall f \in \mathsf{SE}(G).$$

In other words, $\exists \Gamma \subset \Gamma(G)$ s.t.

$$\sum_{\gamma \in \Gamma} \mathrm{Arf}(f(\gamma)) \equiv 1 \pmod 2 \quad \text{for} \quad \forall f \in \mathsf{SE}(G).$$

Remark. Recently, integral version of the Conway-Gordon type theorem for $K_{3,3,1} \in \mathsf{PF} \setminus \mathsf{PF}_{\triangle}$ was given by O'Donnol.

§3. Intrinsic knottedness of graphs in $\mathrm{HF} \setminus \mathrm{HF}_\triangle$



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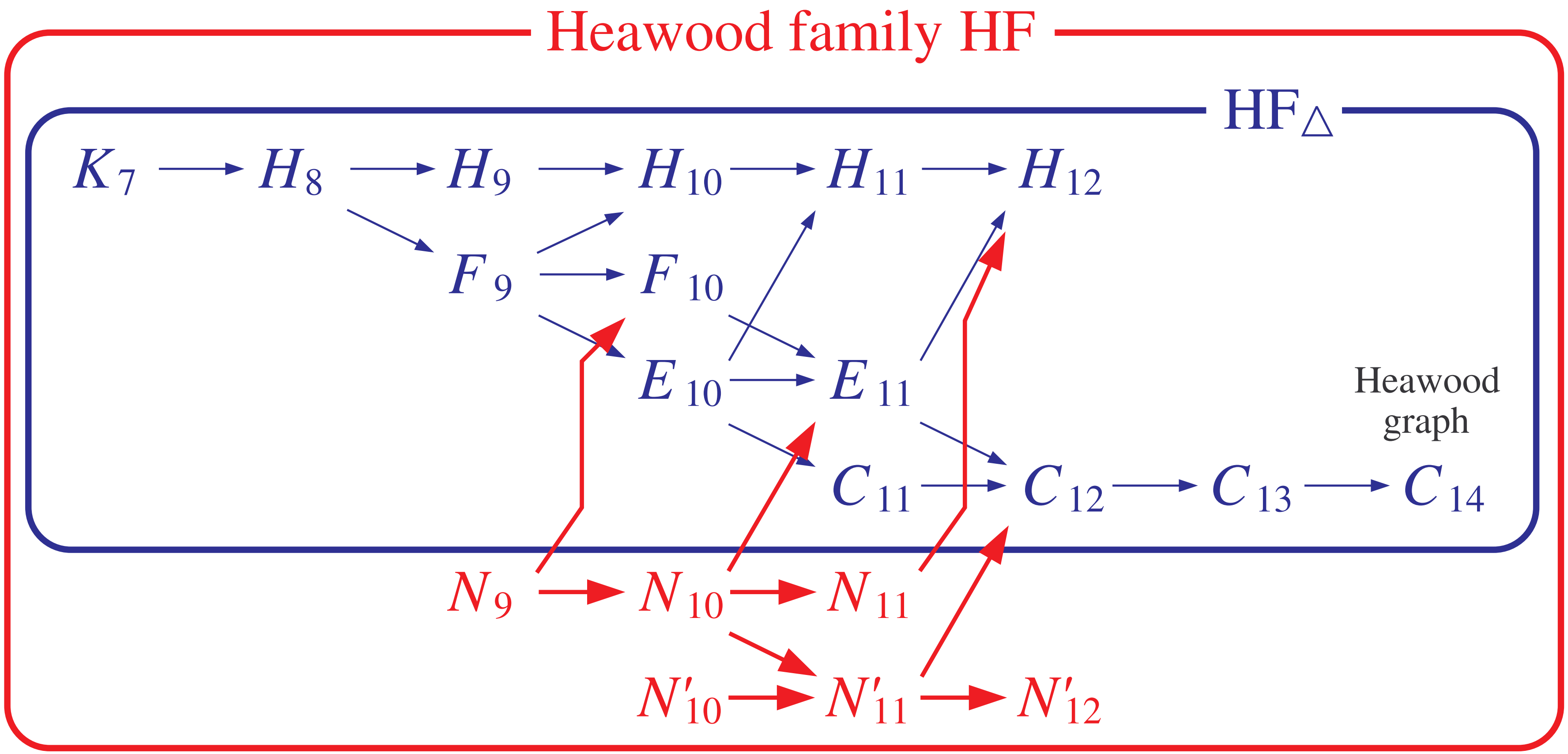

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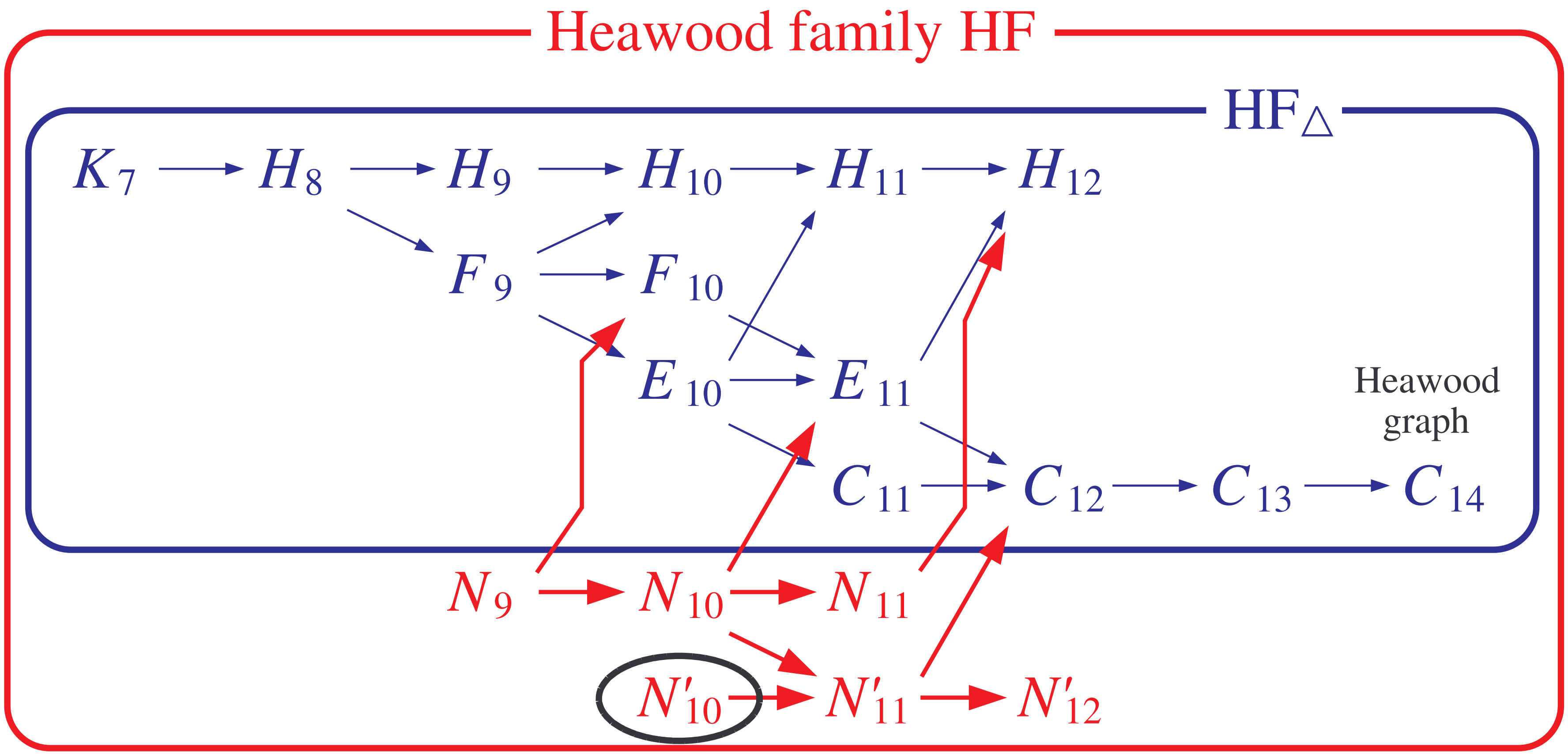
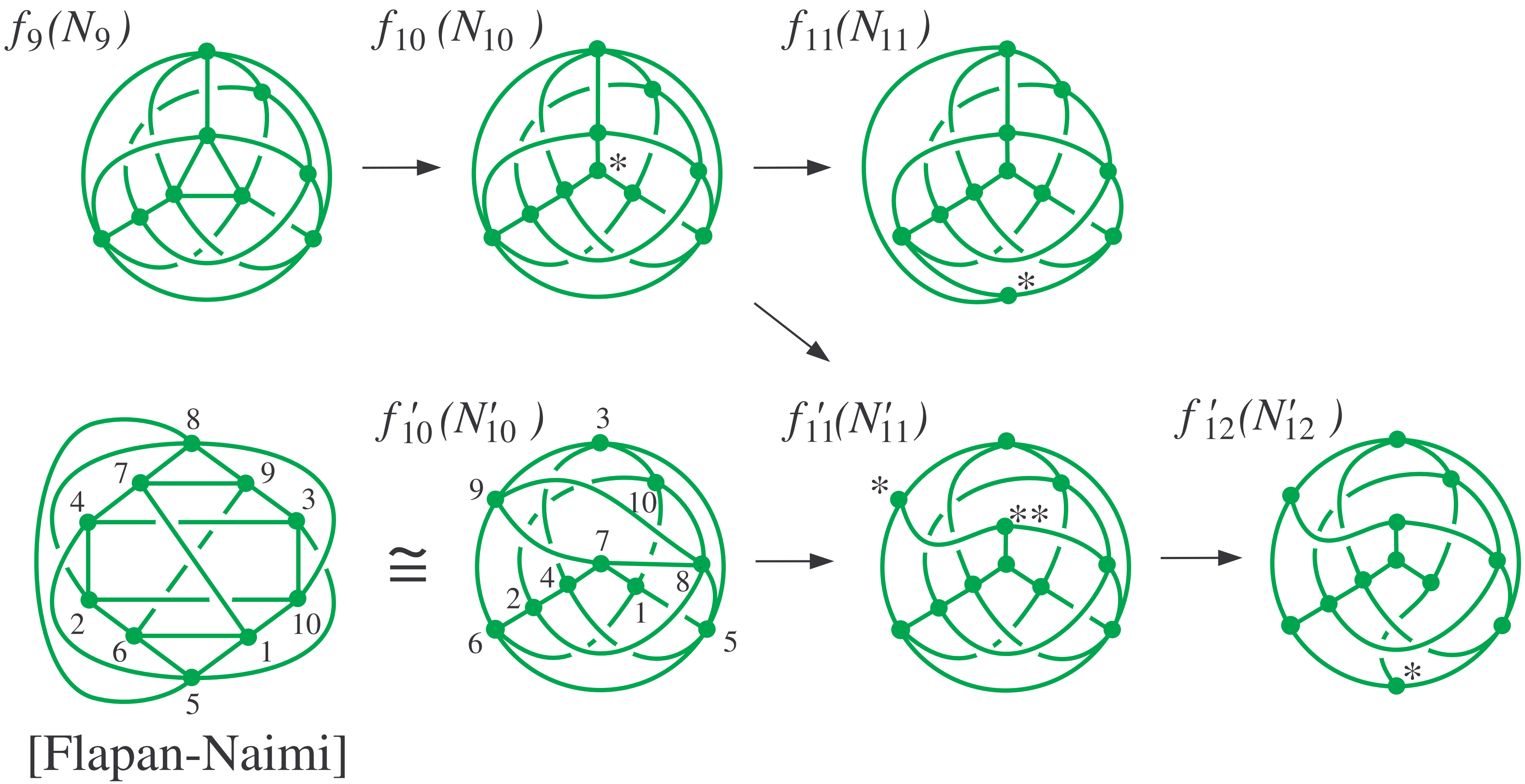


Theorem 3.1. [Flapan-Naimi '08] N'_{10} is not IK.

Theorem 3.2. [Hanaki-N-Taniyama-Yamazaki]
For $G \in \mathrm{HF}$, the following are equivalent:
(1) G is IK, (2) $G \in \mathrm{HF}_\triangle$, (3) $\Gamma^{(3)}(G) = \emptyset$.



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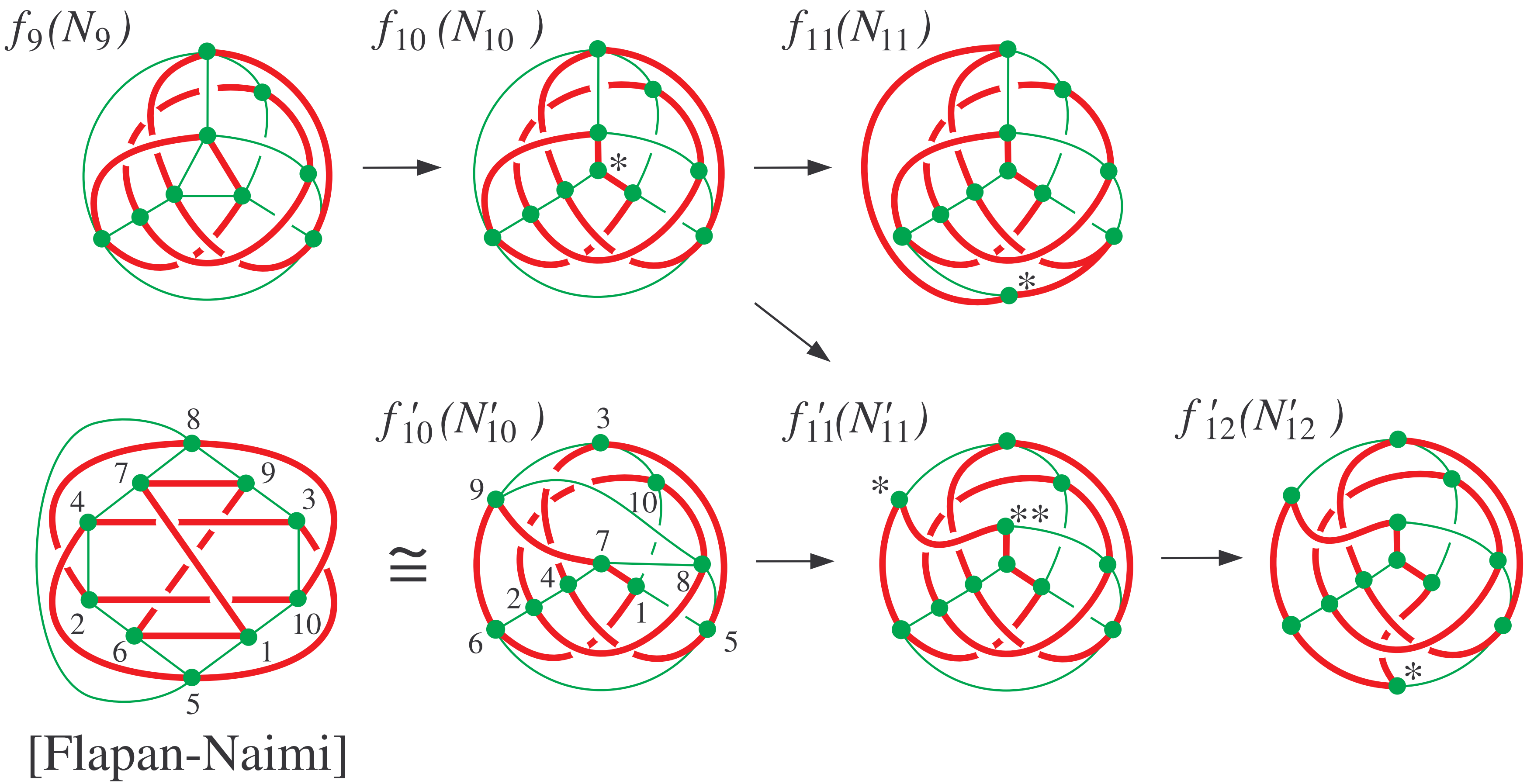

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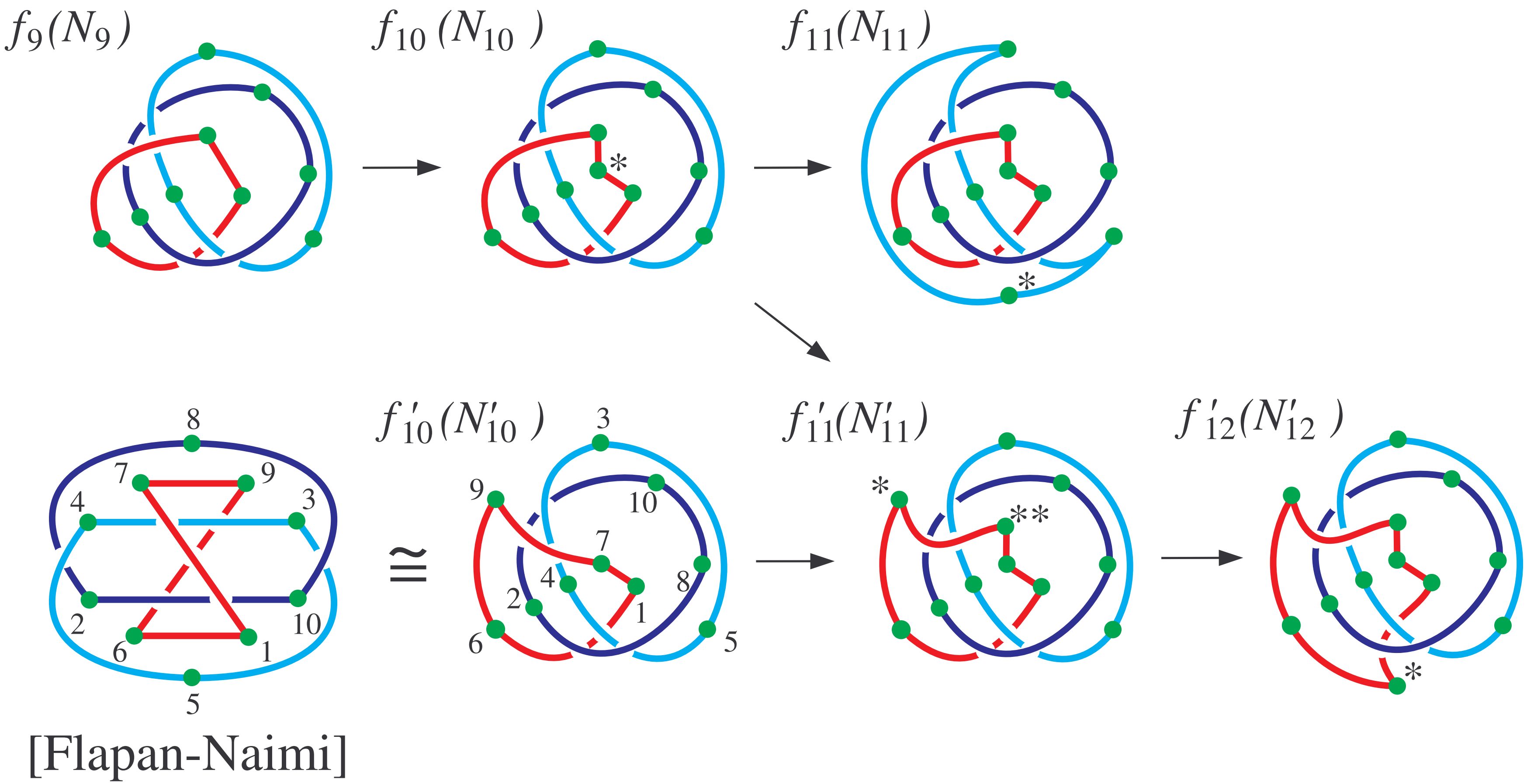

Definition 3.3. G : graph

(1) G is intrinsically 3-linked (I3L)

$\overset{\text{Def.}}{\iff} \forall f \in \mathrm{SE}(G),\ f(G) \supset$ nonsplittable 3-comp. link.

(2) [Foisy '06]

G is intrinsically knotted or 3-linked (I(K or 3L))

$$\overset{\text{Def.}}{\iff} \forall f \in \mathrm{SE}(G),\ f(G) \supset \begin{cases} \text{nontrivial knot} \\ \textbf{or} \text{ nonsplittable 3-comp. link.} \end{cases}$$

Note. I(K or 3L) $\neq$ "IK or I3L" .

Example.

(1) [Flapan-Naimi-Pommersheim '01] K_{10} is I3L.

(2) [Foisy '06] $\exists G$ s.t. G is I(K or 3L) but neither IK nor I3L.

Definition 3.4. [HNTY] G : graph

G is intrinsically knotted or completely 3-linked (I(K or C3L))

$\overset{\text{Def.}}{\Longleftrightarrow}$ $\forall f \in \mathrm{SE}(G)$, $f(G) \supset$ nontrivial knot **or** 3-comp. link any of whose 2-comp. sublink is nonsplittable.

Remark. IK $\Longrightarrow$ I(K or C3L) $\Longrightarrow$ I(K or 3L).

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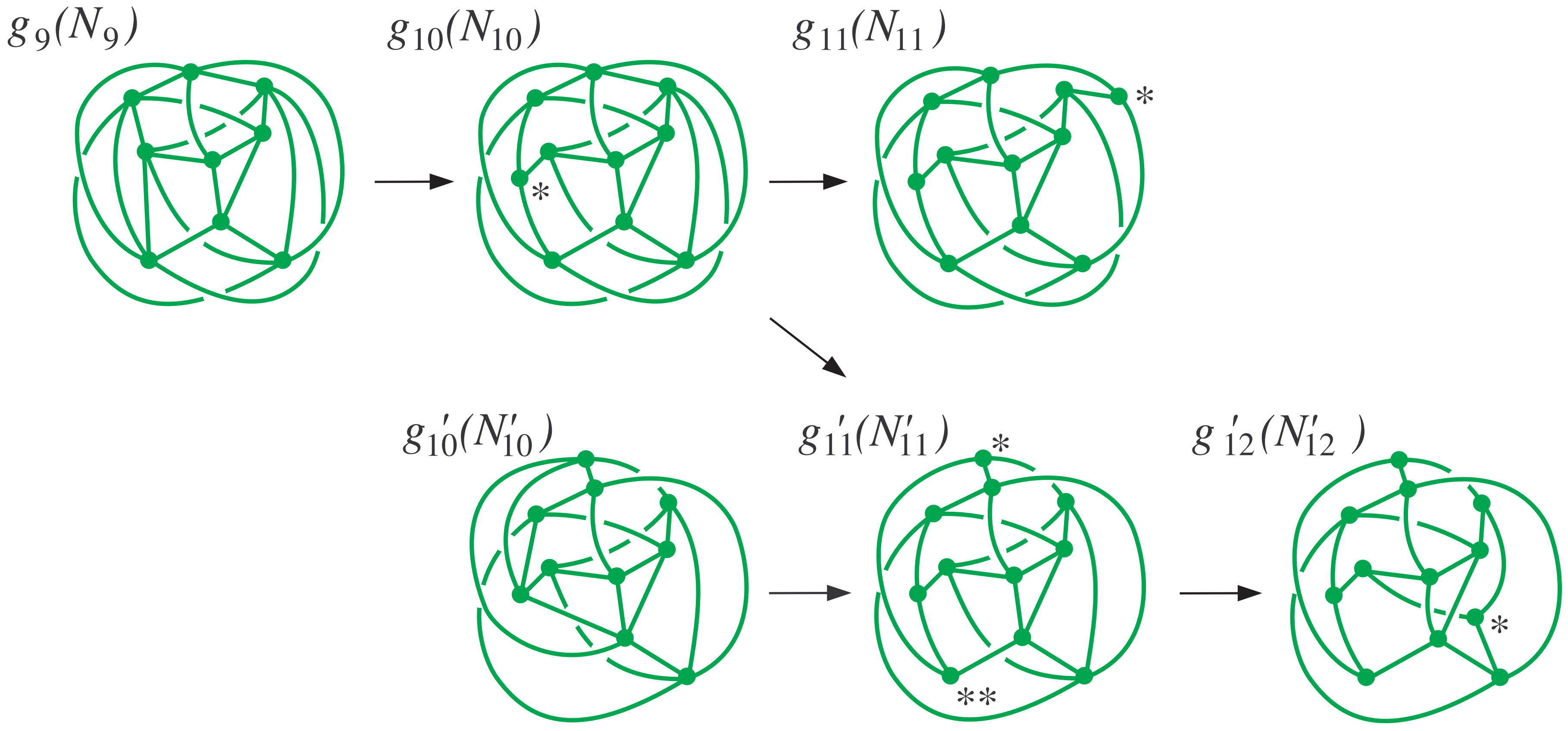
Remark. IK $\Longrightarrow$ I(K or C3L) $\Longrightarrow$ I(K or 3L).

Theorem 3.5. [HNTY]

$\forall G \in$ HF is a minor-minimal I(K or C3L) graph.

Remark.

(1) $\forall G \in$ HF $\setminus$ HF$_\triangle$ is neither IK nor I3L.



(2) Foisy's I(K or 3L) graph is not I(K or C3L).

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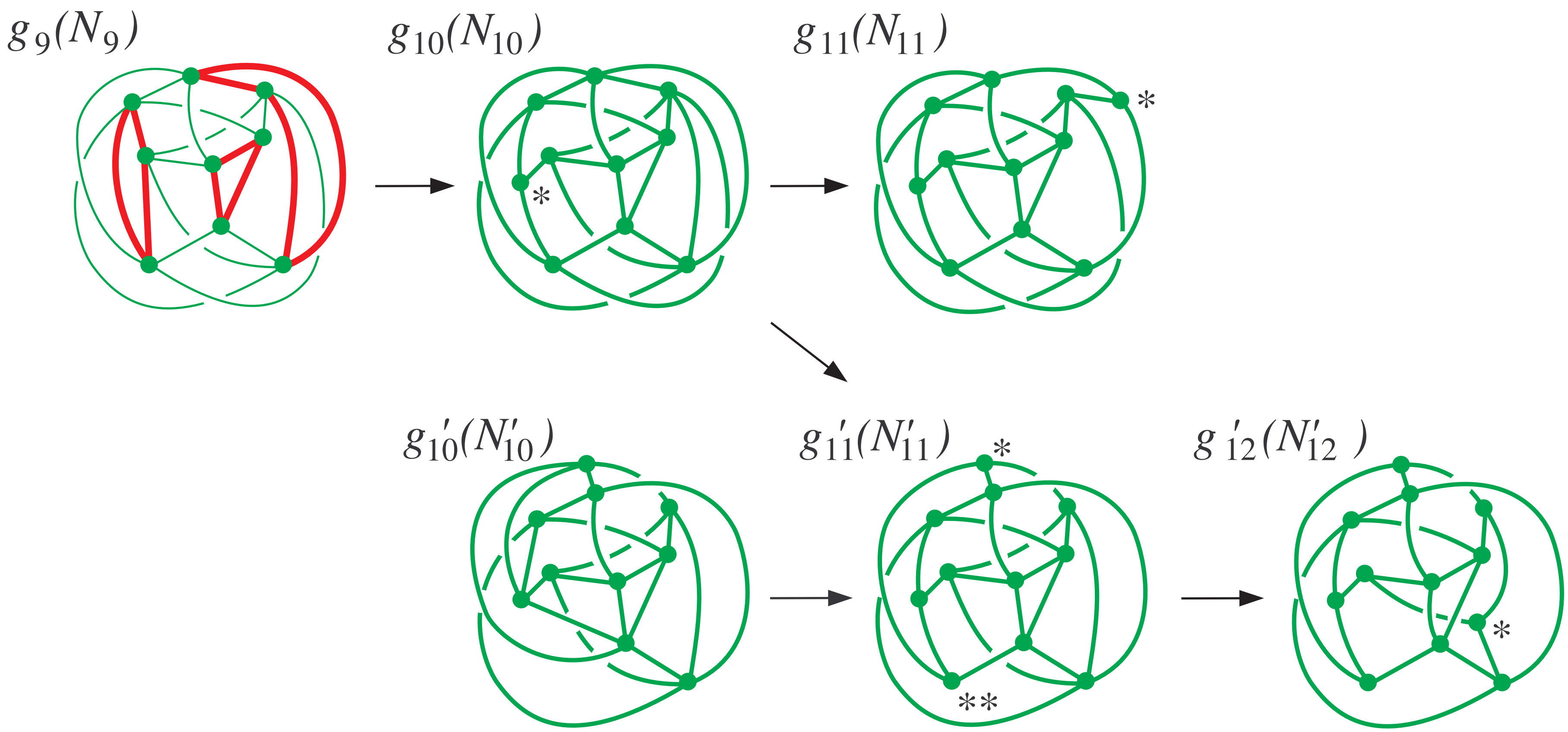

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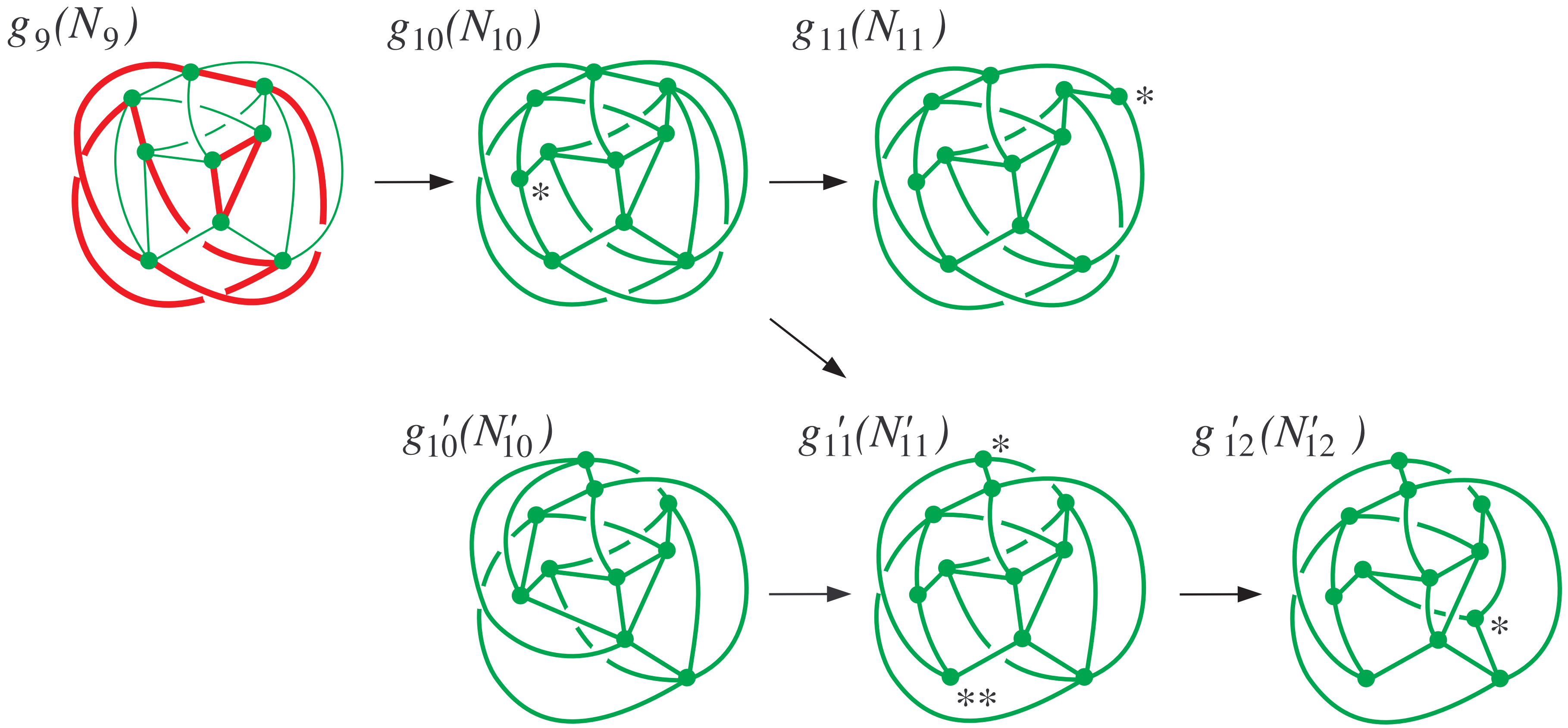

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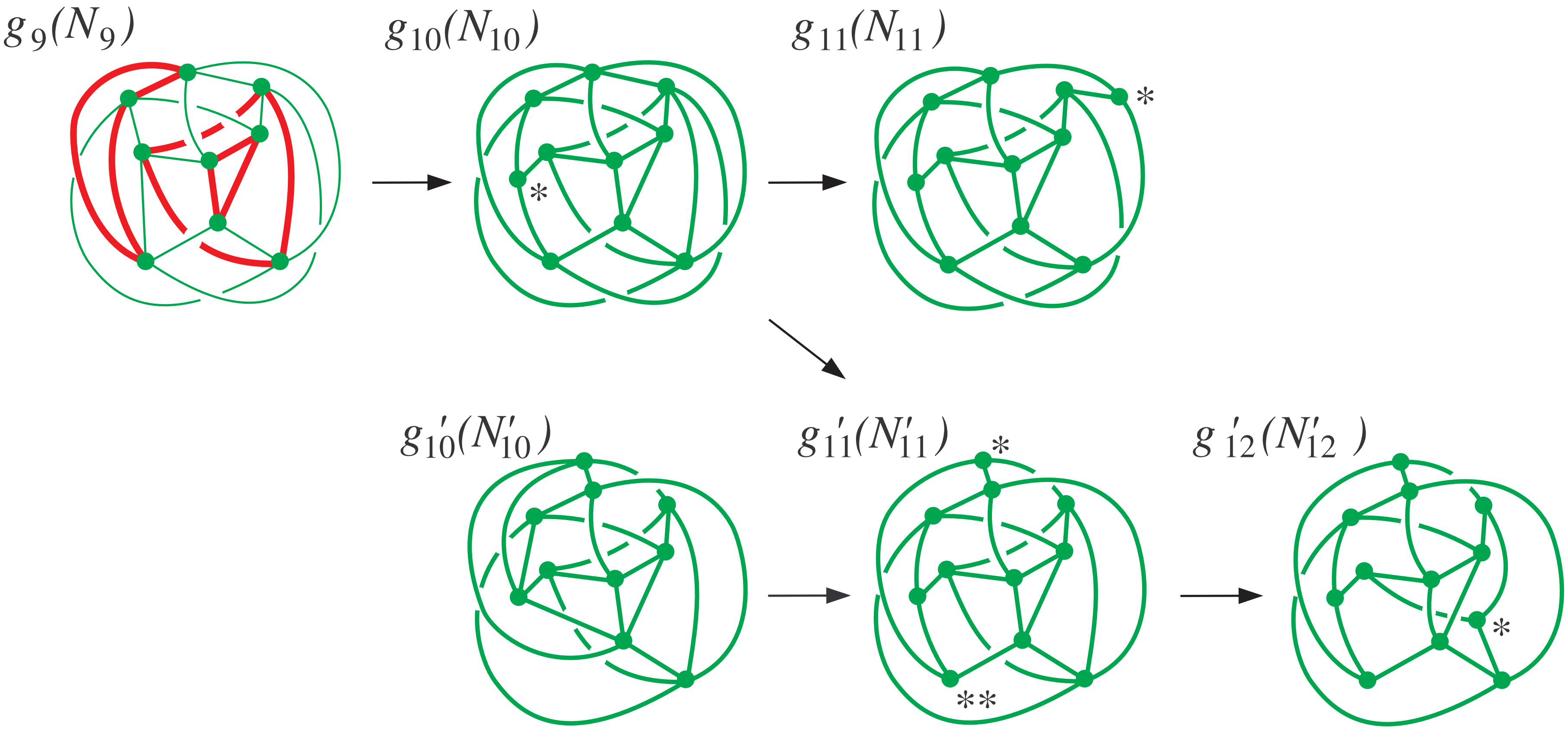

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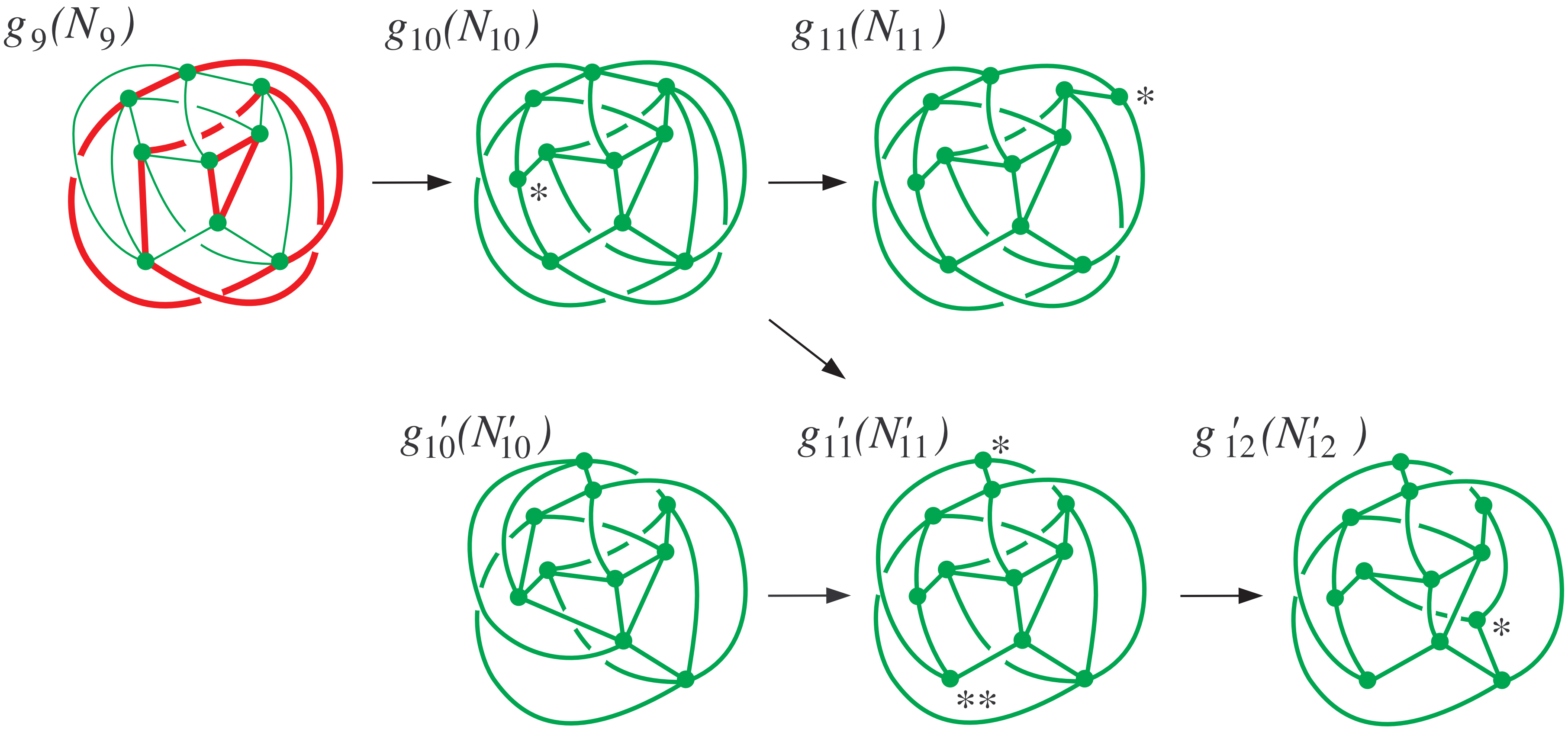

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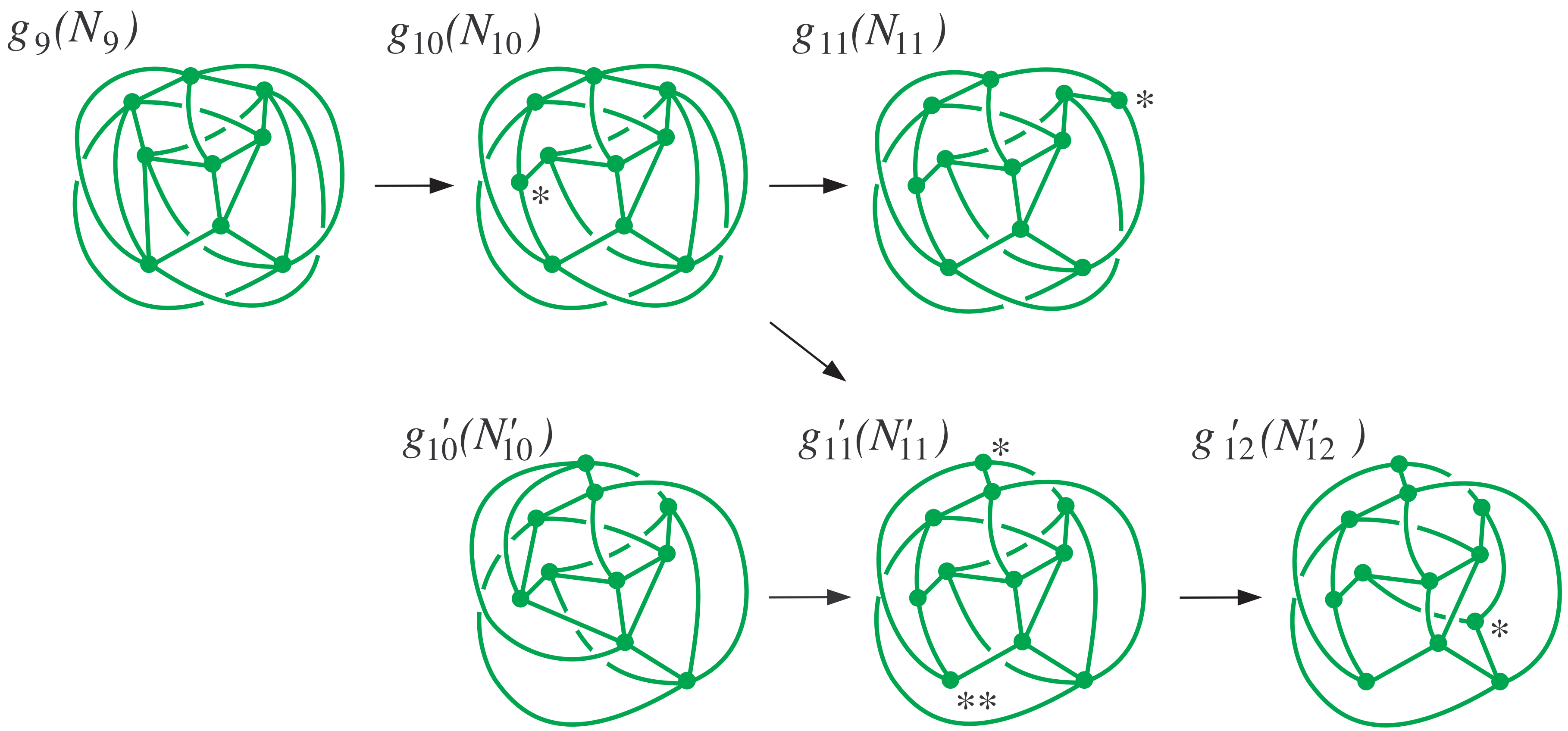

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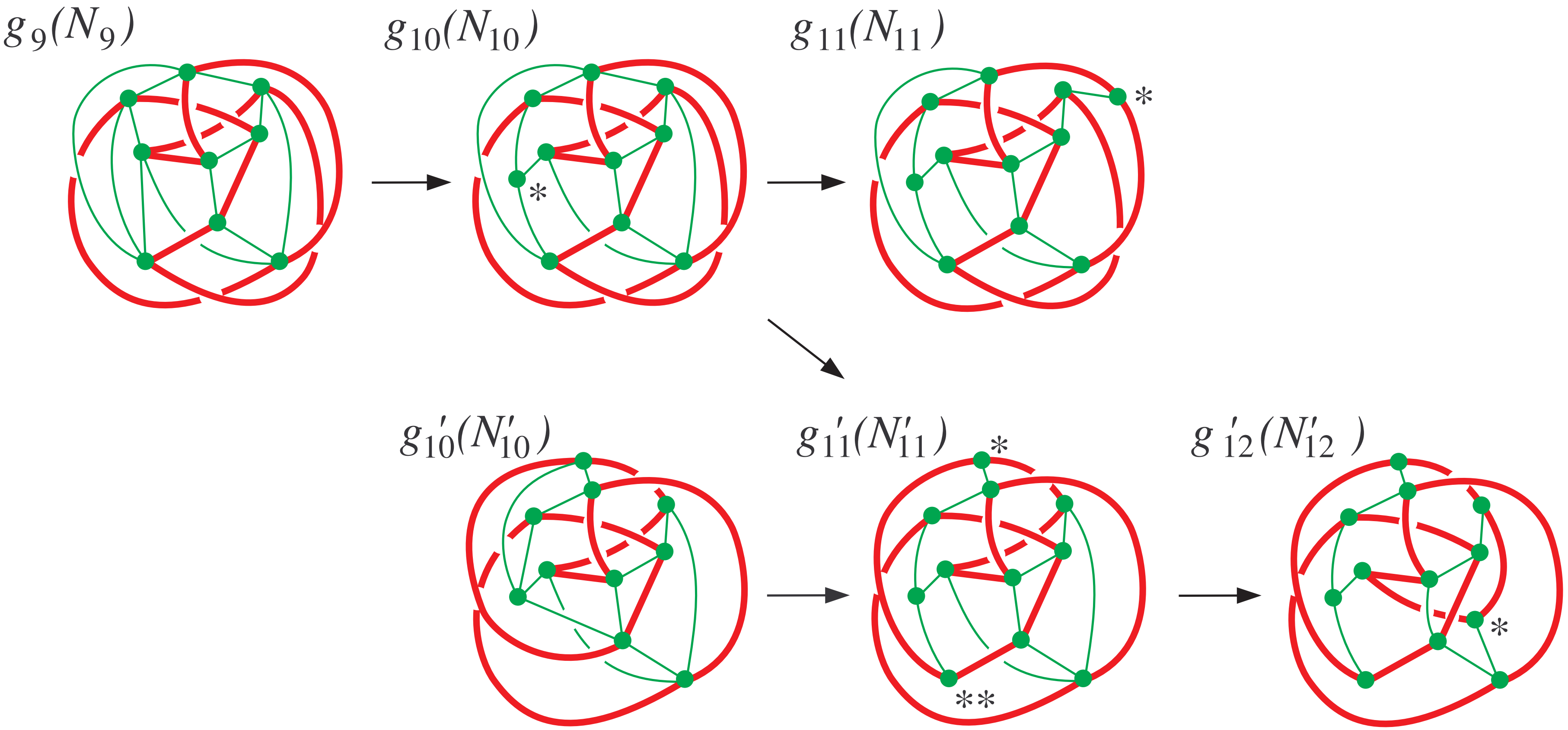

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