Knotting and Linking in the Petersen family

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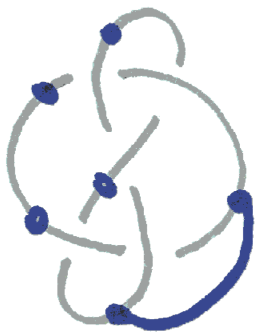
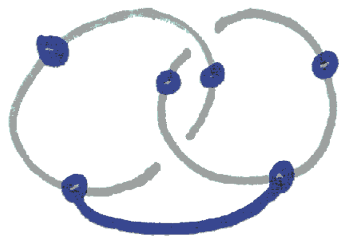
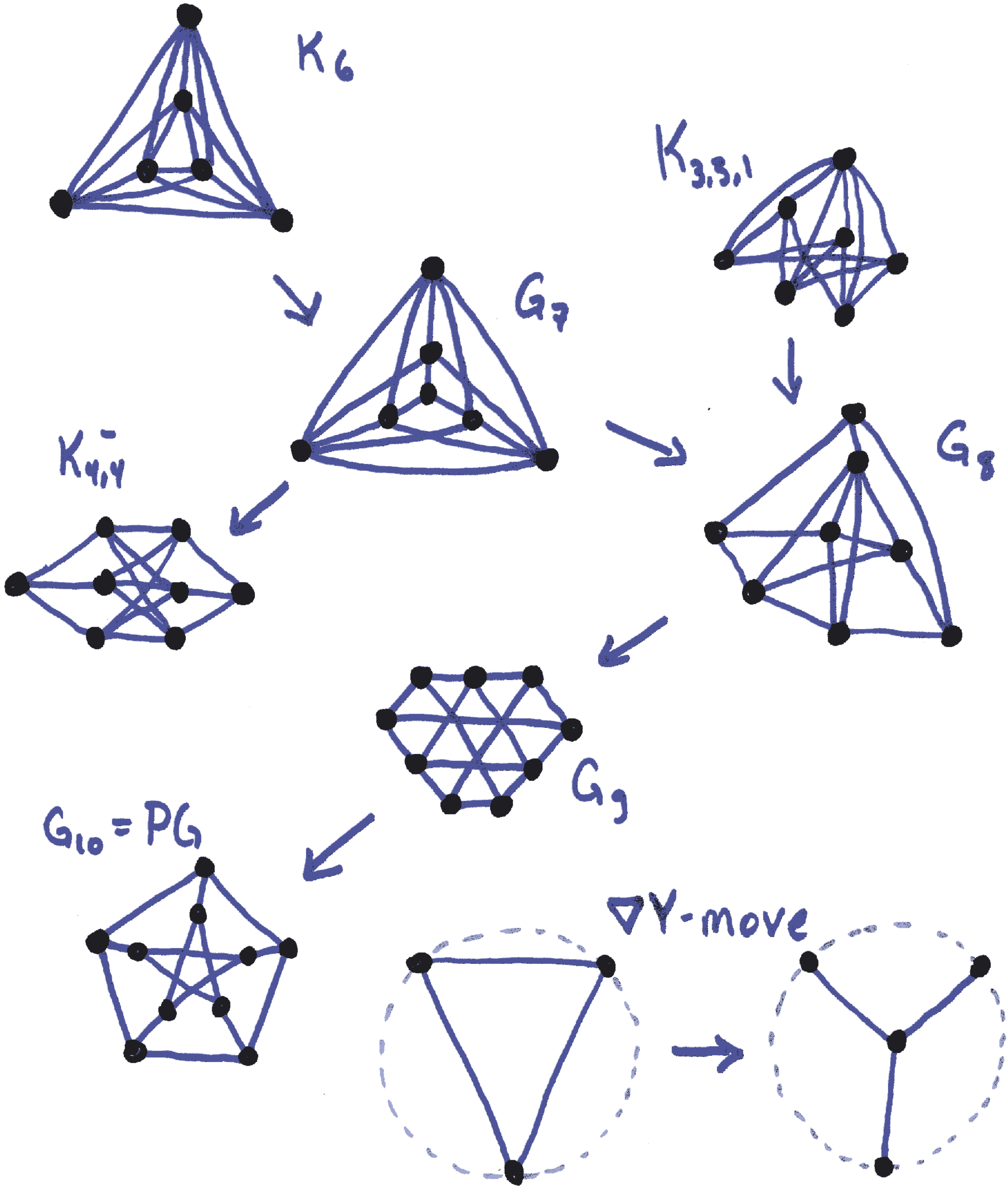
A spatial graph is <u>knotted</u> if it contains a nontrivial knot. A spatial graph is <u>linked</u> if it contains a nontrivial link.



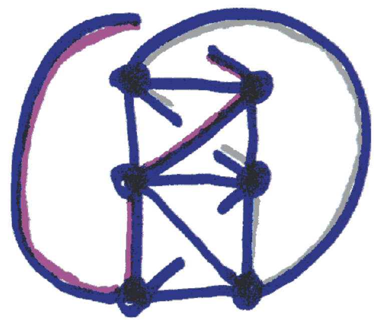


A spatial graph is <u>algebraically linked</u> if it contains a link L with $lk(L) \neq 0$. A spatial graph is <u>complexly algebraically linked</u> (CCA linked) if it contains a link L s.t. $|lk(L)| > 1$ or at least two links L, K s.t. $lk(L) \neq 0$ + $lk(K) \neq 0$.

The Petersen family PF



A graph G is __intrinsically linked__ if every embedding of G into $\mathbb{R}^3$ (or S^3) is linked.

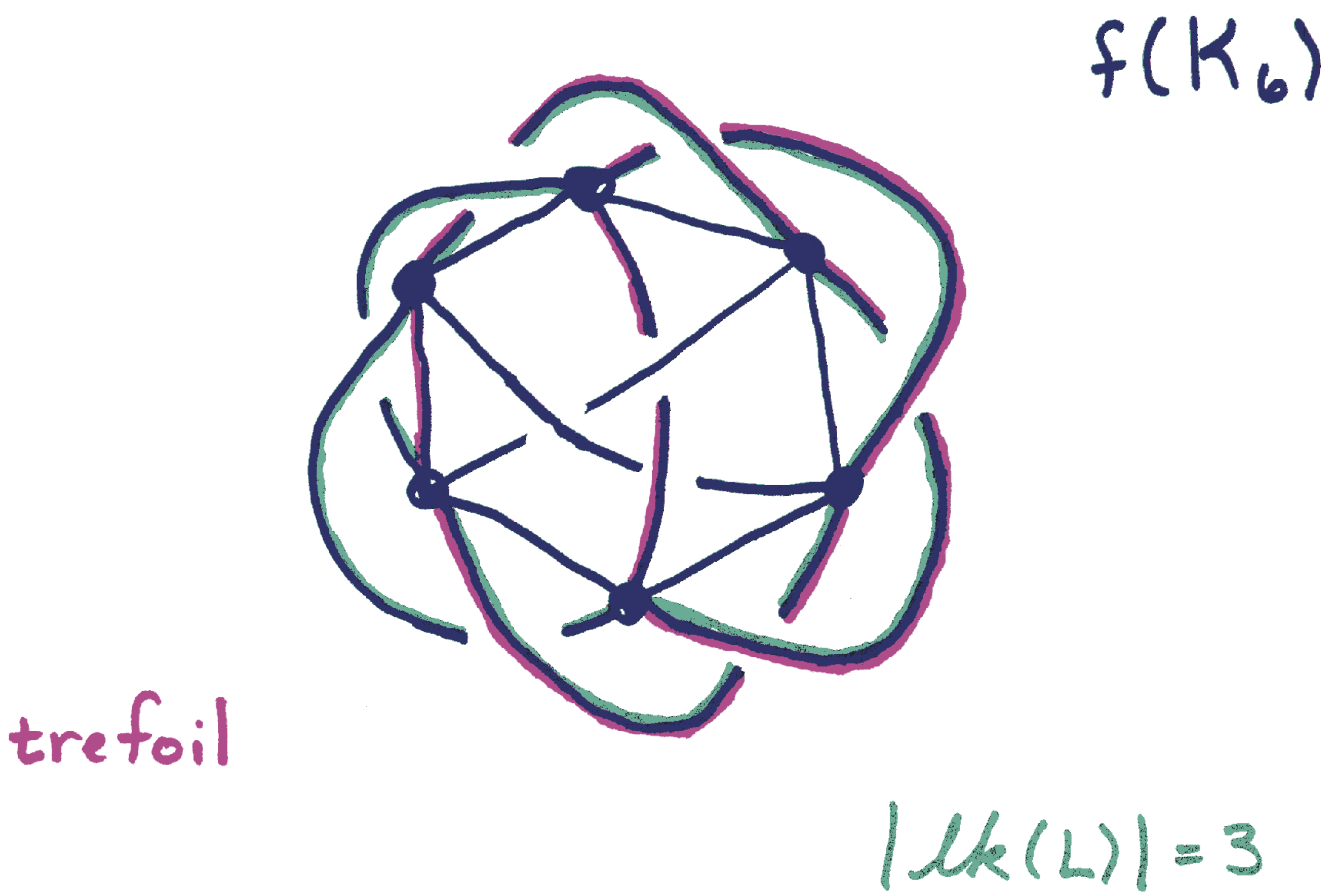


The $\mathcal{PF}$ is the complete set of minor minimal intrinsically linked graphs. [Conway, Gordon, Sachs, Robertson, Seymour, and Thomas]

$\forall$ $G \in \mathcal{PF}$ $\exists$ an embedding f s.t. $f(G)$ is knotless.

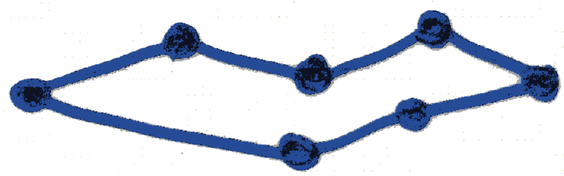
Knotting $\longleftrightarrow$ linking

Thm: (O'D) If f is a CA linked embedding of $G \in \mathcal{PF}$, then $f(G)$ is knotted.



Notation:

A <u>k-cycle</u> γ is a subgraph homeo. to S^1 that contains exactly k edges.



$\Gamma_k(G)$ is the set of all k-cycles in G.

$\Gamma_{a,b}(G)$ is the set of all disjoint pairs of cycles $\alpha \cup \beta$, in G, where α is an a-cycle and β is a b-cycle.

<u>Thm</u>: (Nikkuni) For any spatial embedding f of K_6, that

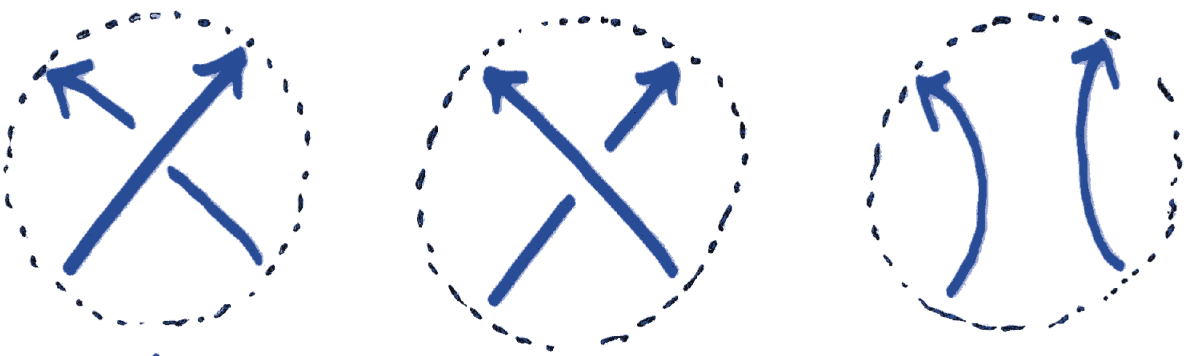
$$\sum_{\lambda \in \Gamma_{3,3}(K_6)} lk(f(\lambda))^2 =$$

$$2\left\{ \sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) \right\} + 1$$

where a_2 is the second coefficient of the Conway polynomial.

The second coefficient of the Conway Polynomial a_2

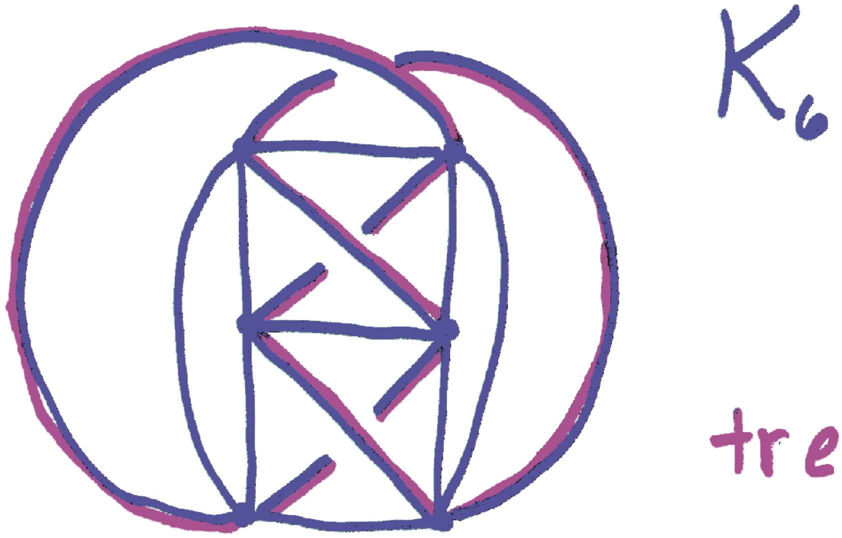
$$\nabla(K) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$



K_+ K_- L_0

- $a_2(K_+) - a_2(K_-) = lk(L_0)$
- $a_2(L) = 0$ L link
- $a_2(\bigcirc) = 0$
- $a_2($$) = 1$

Cor: If f is a CA linked embedding of K_6 then f is knotted.

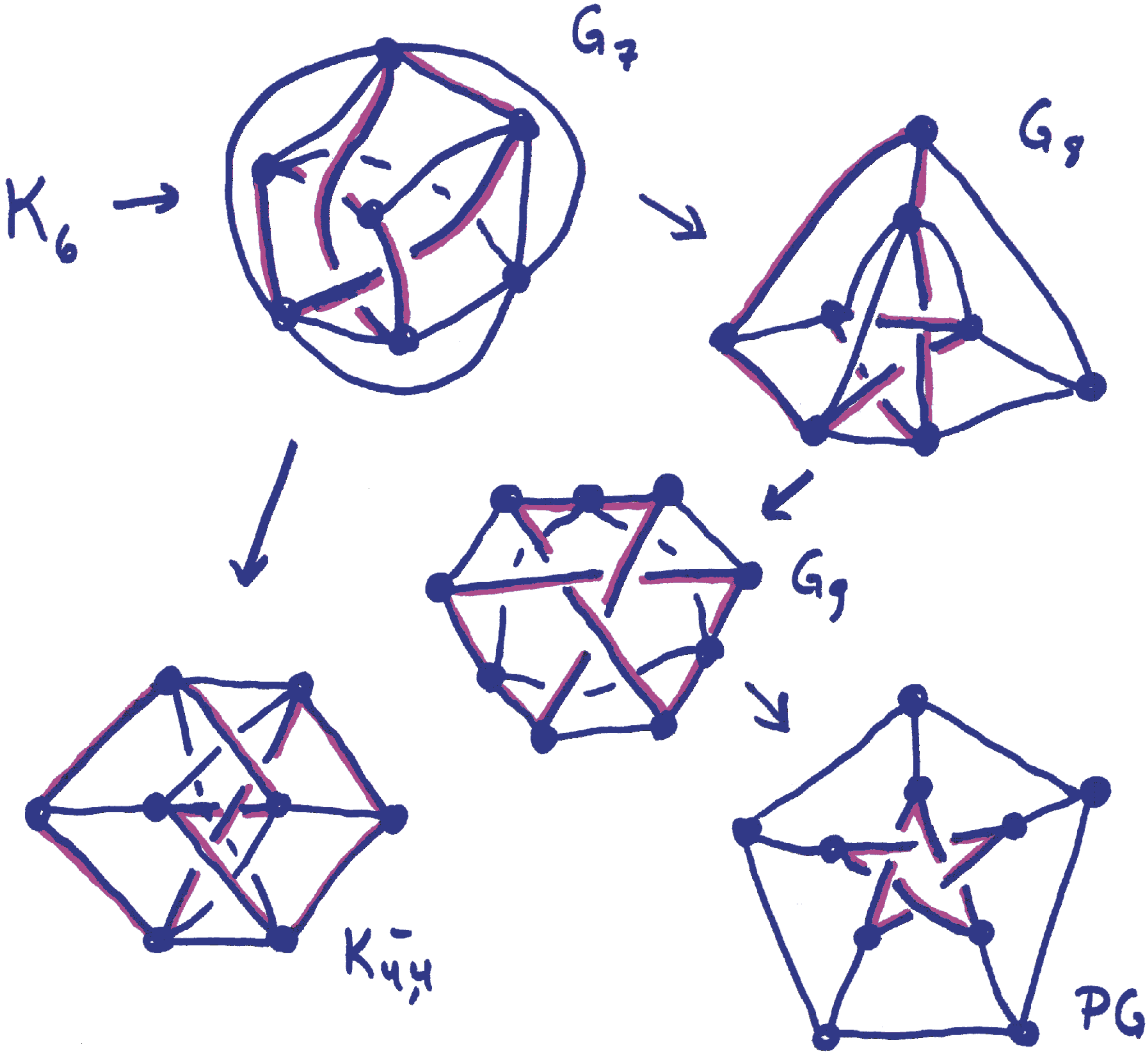


- $$\sum_{\lambda \in \Gamma_{3,3}(K_6)} lk(f(\lambda))^2 = 2\left[\sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma))\right] + 1$$
- $a_2(K) \neq 0 \Rightarrow K$ nontrivial

<u>Prop</u> (O'D) The property

f CA linked $\Rightarrow$ f Knotted

is preserved under ∇Y moves.

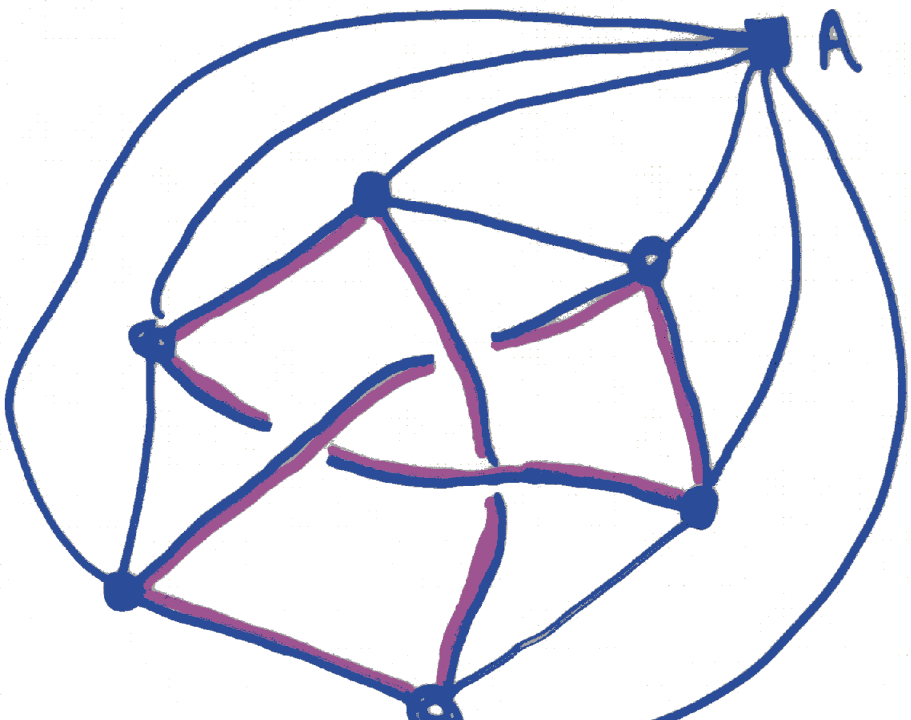


Thm: (O'D) For any spatial embedding f of $K_{3,3,1}$

$$\sum_{\lambda \in \Gamma_{3,4}(K_{3,3,1})} lk(f(\lambda))^2 =$$

$$2\left\{ \sum_{\gamma \in \Gamma_7} a_2 - \sum_{A \in \gamma \in \Gamma_5} a_2 - 2 \sum_{A \notin \gamma \in \Gamma_6} a_2 \right\} + 1$$

where a_2 is the second coefficient of the Conway Polynomial.

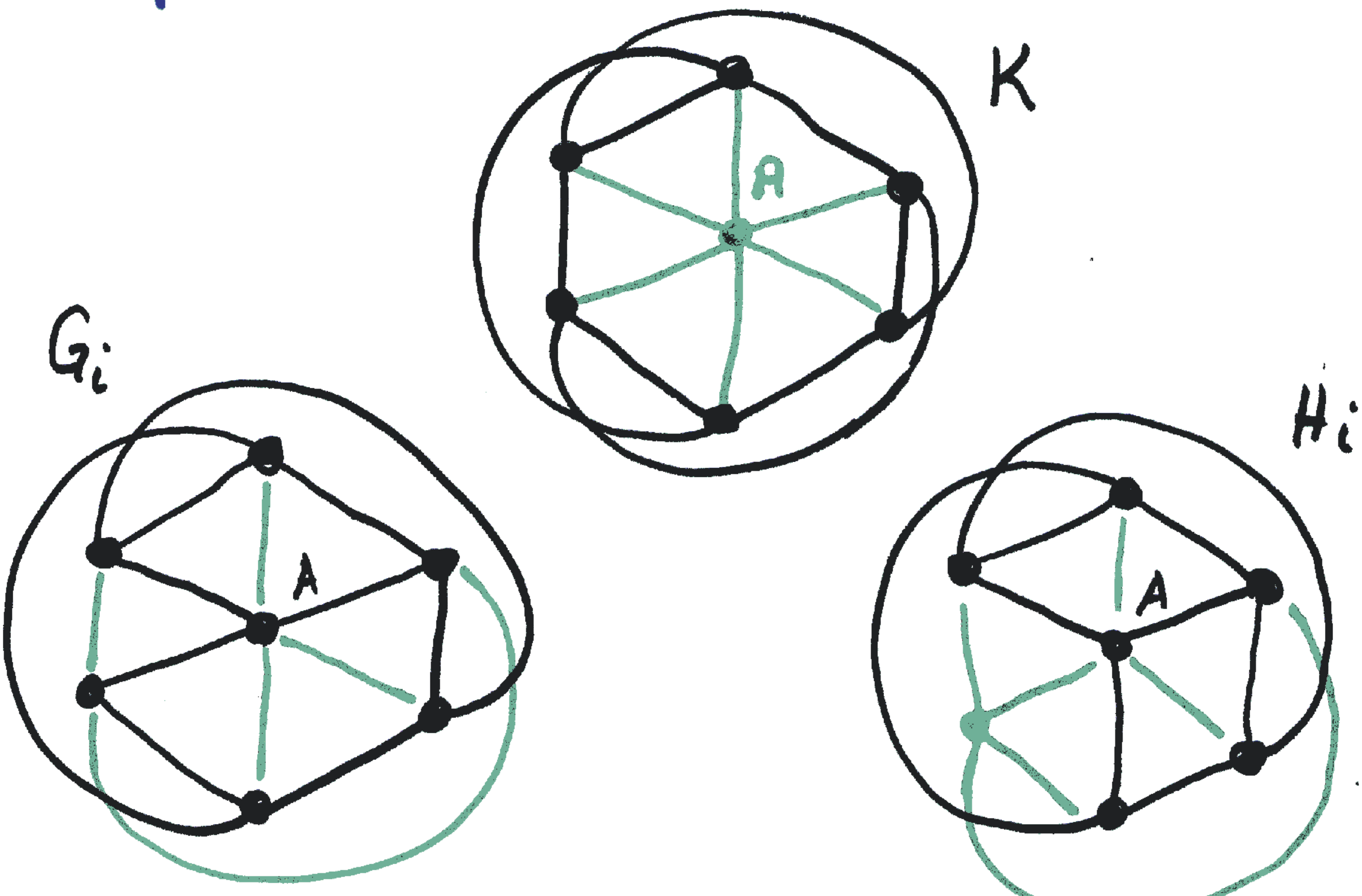


$f(K_{3,3,1})$

First, working with the generators of $L(K_{3,3,1})$ showed

$$\sum_{\lambda \in \Gamma_{3,4}(K_{3,3,1})} lk(f(\lambda))^2 =$$

$$\frac{1}{8}\sum_{G_i} \mathcal{L}(f|_{G_i})^2 - \frac{1}{2}\mathcal{L}(f|_K)^2 - \frac{1}{8}\sum_{H_i} \mathcal{L}(f|_{H_i})^2$$



where $\mathcal{L}(f(G)) \in L(G)$ is the Wu inv.

<u>Prop</u>: (Motohashi, Taniyama)

For $K_{3,3}$

$$\frac{\mathcal{L}(f)^2 - 1}{8} = \sum_{\gamma \in \Gamma_6} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_4} a_2(f(\gamma))$$

$$= \alpha(f(K_{3,3})).$$

So $\displaystyle\sum_{\lambda \in \Gamma_{3,4}} lk(f(\lambda))^2$

$$= \sum_{G_i} \alpha(f(G_i)) - 4\alpha(f(K)) - \sum_{H_i} \alpha(f(H_i)) + \frac{18-4-6}{8}.$$

Look at the cycles in $K_{3,3,1}$ ....

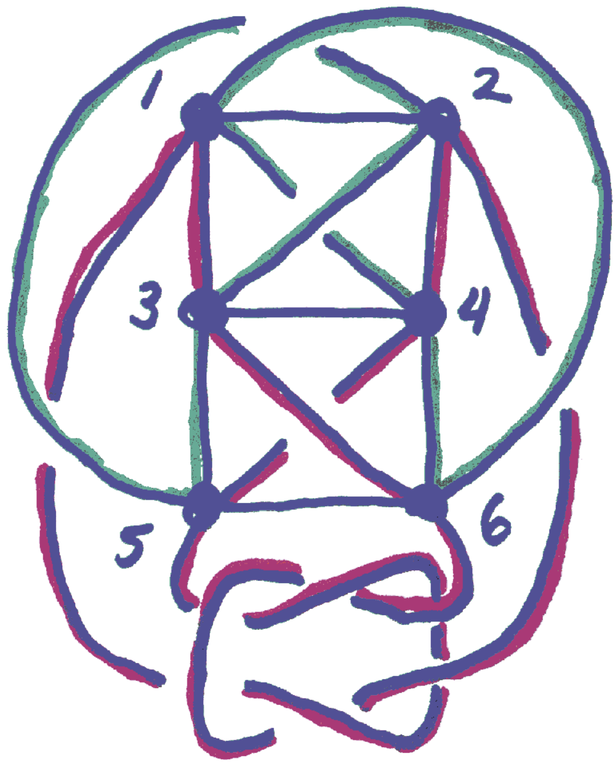
$$= 2\sum_{\gamma \in \Gamma_7} a_2(f(\gamma)) - 4\sum_{\substack{\gamma \in \Gamma_6 \\ A \notin \gamma}} a_2(f(\gamma)) - 2\sum_{\substack{\gamma \in \Gamma_5 \\ A \in \gamma}} a_2(f(\gamma)) + 1$$

For $G \in \mathcal{PF}$

$$\begin{tikzcd}[column sep=huge] \cancel{\text{CA}}\ \text{linked} \arrow[r] \arrow[r, bend left, squiggly, "?"] & \text{Knotted} \arrow[l, bend left, squiggly, "?"] \end{tikzcd}$$

If an embedding f of $G \in \mathcal{PF}$ is Knotted does this guarantee some more complicated linking?

No.



$f(K_6)$

Hopf link

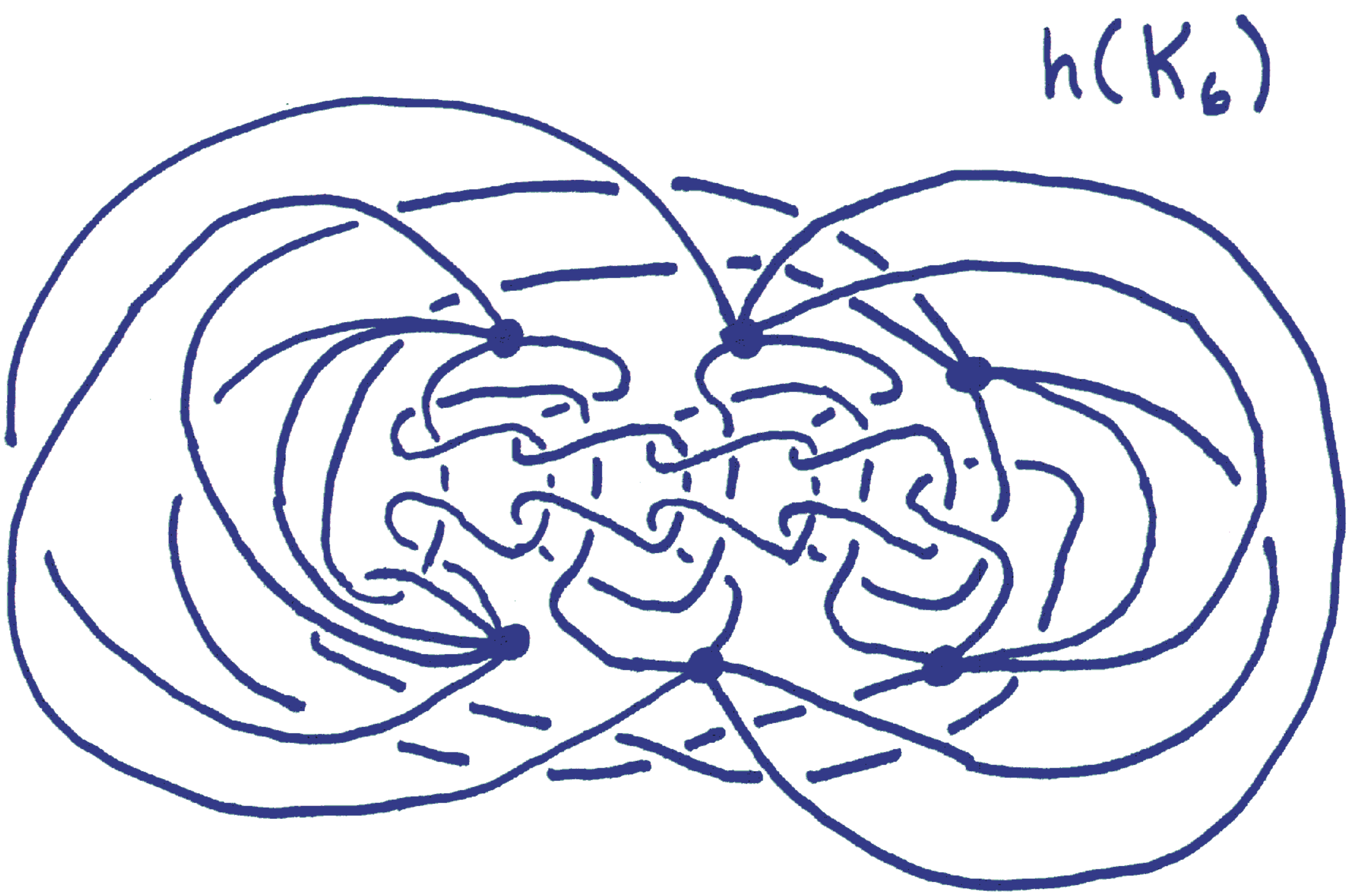
links w/ edges $\overline{15}$ + $\overline{26}$:

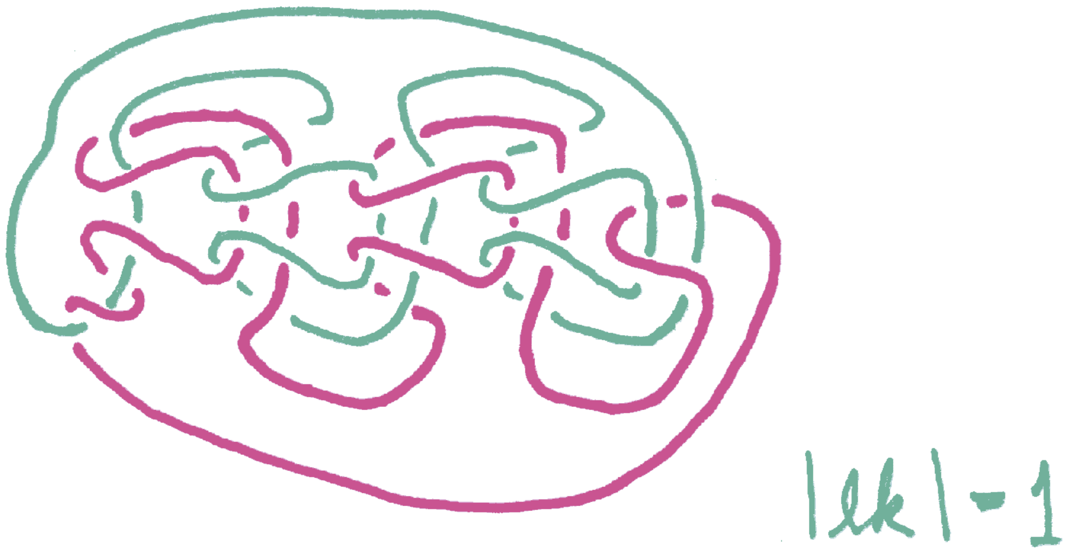
Is algebraic linking needed? Could other complexity give the same result?

- not CA linked
- must contain a link L with $|lk(L)| = 1$

Looking at embeddings that contain a link L with $|lk(L)| = 1$ where L is not a Hopf link or a link L with $|lk(L)| = 1$ and at least one additional L' with $|lk(L')| = 0$.

The embedding h is neither CA linked nor Knotted.





The embedding g is neither CA linked nor knotted.