# Multiplicity distance of spatial graphs

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§1. Multiplicity in a category

§2. Multiplicity category of knots and its distance

§3. Multiplicity category of spatial graphs and its distance

## §1. Multiplicity in a category

$\mathcal{C} = (\mathcal{O}, \{\mathrm{Hom}(X, Y)\}_{X,Y\in\mathcal{O}}, \circ)$ : category

${}^{\forall}\varphi \in \mathrm{Hom}(X, Y), {}^{\forall}\psi \in \mathrm{Hom}(Y, Z)$, $\psi \circ \varphi \in \mathrm{Hom}(X, Z)$ is defined.

Notation: $\varphi \in \mathrm{Hom}(X, Y) \Leftrightarrow \varphi : X \to Y$

(1) ${}^{\forall}\varphi : X \to Y$, ${}^{\forall}\psi : Y \to Z$, ${}^{\forall}\tau : Z \to W$,

$(\varphi \circ \psi) \circ \tau = \varphi \circ (\psi \circ \tau)$

(2) ${}^{\forall}X \in \mathcal{O}$, ${}^{\exists}\mathrm{id}_X : X \to X$ s.t.

$\forall\varphi : X \to Y$, $\varphi \circ \mathrm{id}_X = \varphi$

$\forall\psi : Y \to X$, $\mathrm{id}_X \circ \psi = \psi$

(3) $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(Z, W) \neq \emptyset$

$\Rightarrow X = Z$ and $Y = W$

**Definition.**

$$m : \bigcup_{(X,Y)\in\mathcal{O}\times\mathcal{O}} \mathrm{Hom}(X,Y) \to \mathbb{N}\cup\{\infty\}$$

is a multiplicity on $\mathcal{C}$

$\overset{\text{def}}{\Longleftrightarrow}$ (1) ${}^{\forall}X \in \mathcal{O}$, $m(\mathrm{id}_X) = 1$, and

(2) ${}^{\forall}\varphi : X \to Y$, ${}^{\forall}\psi : Y \to Z$, $\;m(\psi\circ\varphi) \le m(\varphi)m(\psi)$.

Here $\forall n \in \mathbb{N}$, $n \le \infty$, $n\cdot\infty = \infty\cdot n = \infty$,

$\infty \le \infty$, $\infty\cdot\infty = \infty$.

**Definition.** $X, Y \in \mathcal{O}$

$$m(X : Y) = \min\{m(\varphi) \mid \varphi \in \mathrm{Hom}(X, Y)\}$$

multiplicity of $X$ over $Y$ with respect to $m$

**Proposition.** *$m$: multiplicity on $\mathcal{C}$*

*(1) $^{\forall}X \in \mathcal{O}$, $m(X : X) = 1$*

*(2) $^{\forall}X, Y, Z \in \mathcal{O}$, $m(X : Z) \leq m(X : Y)m(Y : Z)$.*

$m$ has finite multiplicity property (FMP)
$\overset{\text{def}}{\Longleftrightarrow} \forall X, Y \in \mathcal{O},\ m(X : Y) < \infty$

**Definition.**

$$d_m(X, Y) = \log_e(m(X : Y)m(Y : X))$$

$m$-distance of $X$ and $Y$

**Proposition.** *$m$: multiplicity on $\mathcal{C}$ with FMP*

*$(\mathcal{O}, d_m)$ is a pseudo metric space i.e.*

(D1′) $d_m(X, Y) \geq 0$, $d_m(X, X) = 0$,

(D2) $d_m(X, Y) = d_m(Y, X)$,

(D3) $d_m(X, Z) \leq d_m(X, Y) + d_m(Y, Z)$.

**Example.** $\mathcal{C}$: subcategory of the category of sets and maps (SET)

$X, Y \in \mathcal{O}$, $f : X \to Y$

$$m(f) = \sup\{|f^{-1}(y)| | y \in Y\}$$

$m$: map multiplicity

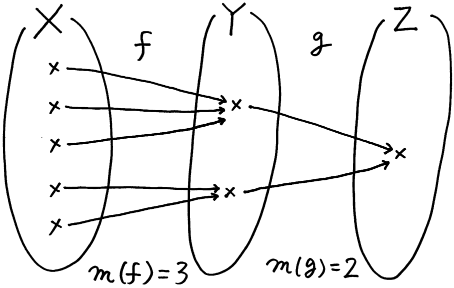
**Proposition.** *map multiplicity is a multiplicity, i.e.*

*(1)* $^{\forall}X \in \mathcal{O}$, $m(\mathrm{id}_X) = 1$, *and*

*(2)* $^{\forall}f : X \to Y$, $^{\forall}g : Y \to Z$,

$m(g \circ f) \leq m(f)m(g)$

**Example.**



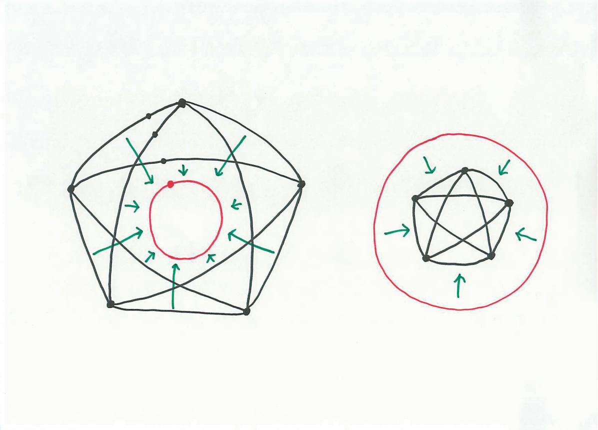

$$m(g \circ f) = 5 \leq 6 = 3 \cdot 2$$

**Remark.** There are canonical functors from TOP, GROUP, etc. to SET.

**Example.**

$m(K_5 : \mathbb{S}^1) = 3$ , $m(\mathbb{S}^1 : K_5) = 1$

$d_m(K_5, \mathbb{S}^1) = \log_e(3 \cdot 1) = \log_e 3$



**Example.** $R$: PID, $M, N$: $R$-modules finitely generated over $R$

$r(M)$: the minimal number of generators of $M$ over $R$

$f : M \to N$ : $R$-linear map

$$m_{\mathrm{ker}}(f) = e^{r(\mathrm{ker} f)}$$

$$m_{\mathrm{coker}}(f) = e^{r(\mathrm{coker} f)}$$

$(\mathrm{coker} f = N/f(M))$

**Proposition.** *(1) $m_{\mathrm{ker}}$ is a multiplicity.*

*(2) $m_{\mathrm{coker}}$ is a multiplicity.*

**Proposition.** *(1) $d_{m_{\mathrm{ker}}}(M, N) = 0 \Leftrightarrow M \overset{R}{\cong} N$.*
*(2) $d_{m_{\mathrm{coker}}}(M, N) = 0 \Leftrightarrow M \overset{R}{\cong} N$.*

When $R$ is a field,

$M$ and $N$ are finite dimensional vector spaces over $R$, and

$d_{m_{\mathrm{ker}}}(M, N) = d_{m_{\mathrm{coker}}}(M, N) = |\mathrm{dim} M - \mathrm{dim} N|$.

## §2. Multiplicity category of knots and its distance

$\mathcal{K}$: oriented knot types in $\mathbb{S}^3$, $K_1, K_2 \in \mathcal{K}$

$f = (V_2, k_1)$ is a morphism from $K_1$ to $K_2$

$\overset{\text{def}}{\Longleftrightarrow}$ $V_2 \subset \mathbb{S}^3$ : 1-oriented knotted solid torus with knot type $K_2$,

$k_1 \subset \mathrm{int}\, V_2$ : oriented knot in $\mathbb{S}^3$ with knot type $K_1$.

$(V, k) = (V', k')$ $\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists} h : \mathbb{S}^3 \to \mathbb{S}^3$ ori. preserving homeo. s.t.

$h(V) = V'$ respecting 1-orientation, and

$h(k) = k'$ respecting orientation

$\mathrm{Hom}(K_1, K_2)$ : the set of morphisms from $K_1$ to $K_2$

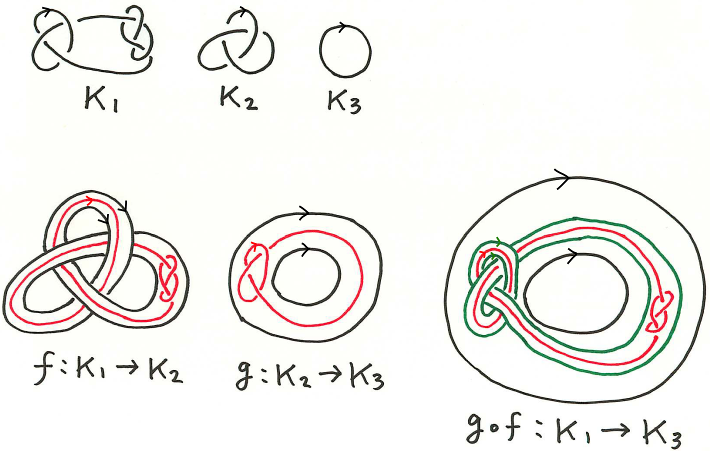
$f : K_1 \to K_2$, $g : K_2 \to K_3$, $f = (V_2, k_1)$, $g = (V_3, k_2)$

$V_2'$ : regular neighbourhood of $k_2$ with $V_2' \subset \mathrm{int} V_3$, $k_1'$ : knot in $V_2'$

s.t. $(V_2', k_1') = (V_2, k_1)$

$g \circ f = (V_3, k_1')$

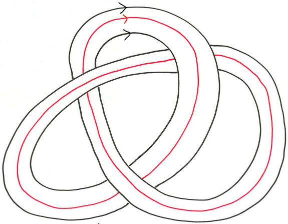
**Example.**



**Proposition.** $\mathcal{C}_{\mathcal{K}} = (\mathcal{K}, \{\mathrm{Hom}(K, J)\}_{K,J\in\mathcal{K}}, \circ)$ *is a category.*

$\mathcal{C}_{\mathcal{K}}$ : multiplicity category of Knots

$\mathrm{id}_K : K \to K$ is given by $\mathrm{id}_K = (V, k)$ where $V$ is a regular neighbourhood of a knot $k$ with knot type $K$.



**Definition.** $f : K_1 \to K_2$ : morphism, $f = (V_2, k_1)$,

$h : V_2 \to \mathbb{S}^1 \times \mathbb{D}^2$ : homeo. $\pi : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1$ : natural projection

$h$ is generic $\overset{\text{def}}{\Longleftrightarrow}$ $\pi \circ h|_{k_1} : k_1 \to \mathbb{S}^1$ is a Morse map,

i.e. it has only finitely many critical points

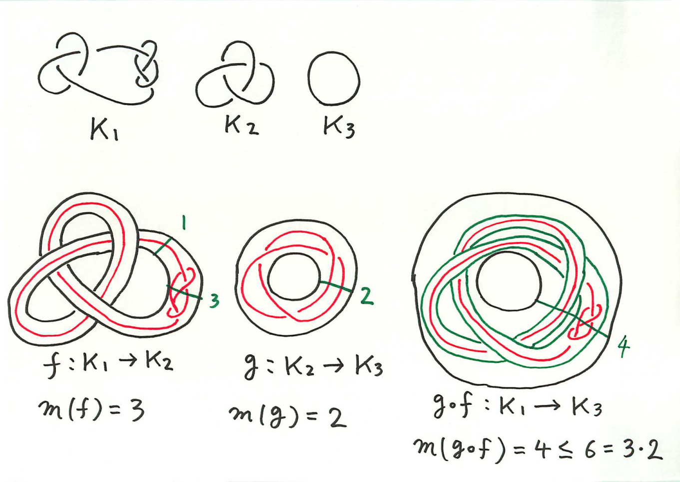
in different levels.

$m(h) = \max\{|(\pi \circ h|_{k_1})^{-1}(y)| \mid y \in \mathcal{S}^1\}$

$m(f) = \min\{m(h) \mid h : V_2 \to \mathbb{S}^1 \times \mathbb{D}^2 \text{ generic homeomorphism}\}$

**Proposition.** *$m$ is a multiplicity on $\mathcal{C}_{\mathcal{K}}$ with FMP.*

**Example.**



**Proposition.** *(1)* $f : K_1 \to K_2$, $m(f) = 1 \;\Rightarrow\; K_1 = \pm K_2$.

*(2)* $m(K_1 : K_2) = 1 \Leftrightarrow K_1 = \pm K_2$.

*(3)* $d_m(K_1, K_2) = 0 \Leftrightarrow K_1 = \pm K_2$.

**Proposition.** $(\mathcal{K}, d_m)$ *is an unbounded pseudo-metric space.*

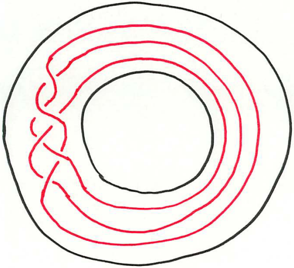
$d_m$ : multiplicity distance of knots

**Proposition.** $K, J \in \mathcal{K}$

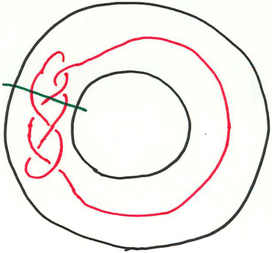
*(1)* $m(K : 0_1) \leq \mathrm{braid}(K)$

*(2)* $m(K : 0_1) \leq 2\mathrm{bridge}(K) - 1$

*(3)* $m(K : J) = 2 \Rightarrow K$ *is a* $(2, p)$*- cable of* $J$*, or* $K = 0_1$.



$m(K:O_1) \leqq \mathrm{braid}(K)$



$m(K:O_1) \leqq 2\,\mathrm{bridge}(K)-1$

**Definition.** $K \in \mathcal{K}$, $m(K) = m(K : 0_1)$

multiplicity of $K$, $m(K) \in \mathbb{N}$

**Remark.** $m(K) \leq n \Leftrightarrow d_m(K, 0_1) \leq \log_e 2n$.

**Proposition.** $K \in \mathcal{K}$

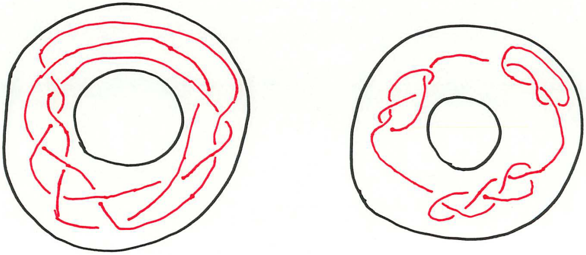
*(1)* $m(K) = 1 \Leftrightarrow K = 0_1$.

*(2)* $m(K) = 2 \Leftrightarrow K$ *is a* $(2, p)$*-torus knot.*

*(3)* $m(K) = 3 \Leftrightarrow K \neq 0_1$, $K$ *is not a* $(2, p)$*-torus knot, and*

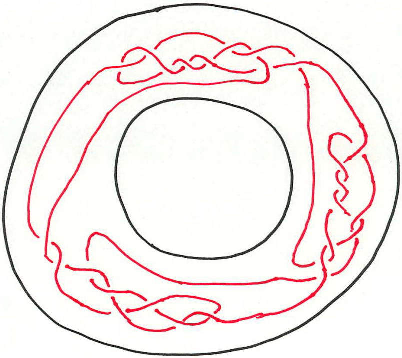
*(a)* $K$ *is a closed* 3*-braid, or*

*(b)* $K$ *is a connected sum of some* 2*-bridge knots.*



$m(K) = 3$

**Proposition.** *$K$:Montesinos knot $\Rightarrow$ $m(K) \leq 4$.*



$m$ (Montesinos knot) $\leqq 4$

## §3. Multiplicity category of spatial graphs and its distance

$\mathrm{SG}$: finite graphs embedded in $\mathbb{S}^3$ , $G_1, G_2 \in \mathrm{SG}$

$f : \mathbb{S}^3 \to \mathbb{S}^3$: degree 1 map such that $f(G_1) \subset G_2$

$(f, G_1, G_2)$ is a morphism from $G_1$ to $G_2$

$G_1, G_2, G_3 \in \mathrm{SG}$

$f : \mathbb{S}^3 \to \mathbb{S}^3$: degree 1 map such that $f(G_1) \subset G_2$

$g : \mathbb{S}^3 \to \mathbb{S}^3$: degree 1 map such that $g(G_2) \subset G_3$

$g \circ f : \mathbb{S}^3 \to \mathbb{S}^3$: degree 1 map such that $g \circ f(G_1) \subset G_3$

$(g, G_2, G_3) \circ (f, G_1, G_2) = (g \circ f, G_1, G_3)$

$\mathrm{Hom}(G_1, G_2)$ : the set of morphisms from $G_1$ to $G_2$

**Proposition.** *$\mathcal{C}_{\mathrm{SG}} = (\mathrm{SG}, \{\mathrm{Hom}(G_1, G_2)\}_{G_1, G_2 \in \mathrm{SG}}, \circ)$ is a category.*

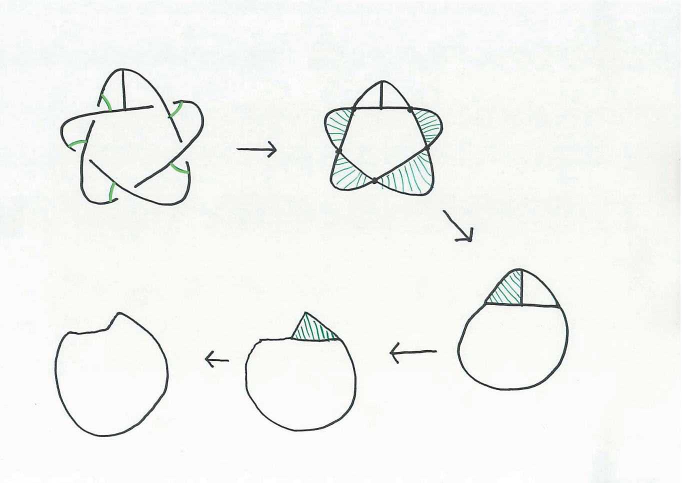
$\mathcal{C}_{\mathrm{SG}}$ : multiplicity category of spatial graphs

$(f, G_1, G_2) \in \mathrm{Hom}(G_1, G_2)$

$$m(f, G_1, G_2) = \sup\{|f^{-1}(y) \cap G_1| | y \in G_2\}$$

**Proposition.** *$m$ is a multiplicity on $\mathcal{C}_{\mathrm{SG}}$ with FMP.*

**Example.**



**Proposition.** *$(\mathrm{SG}, d_m)$ is a pseudo-metric space.*

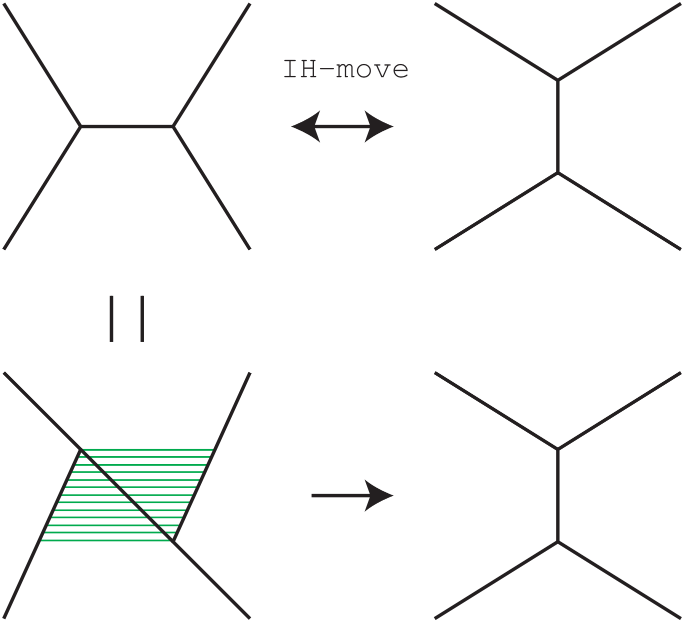
$d_m$ : multiplicity distance of spatial graphs

$G_1, G_2$: trivalent graphs in $\mathbb{S}^3$

$d_{IH}(G_1, G_2)$: IH-distance defined by A. Ishii and K. Kishimoto

**Proposition.** *$G_1, G_2$: trivalent graphs in $\mathbb{S}^3$,*

*$d_m(G_1, G_2) \leq 2(\log_e 3) d_{IH}(G_1, G_2)$.*


IH-move

Thank you