Planar graphs producing knotted projections with three double points

* Cowork with Ryo Nikkuni

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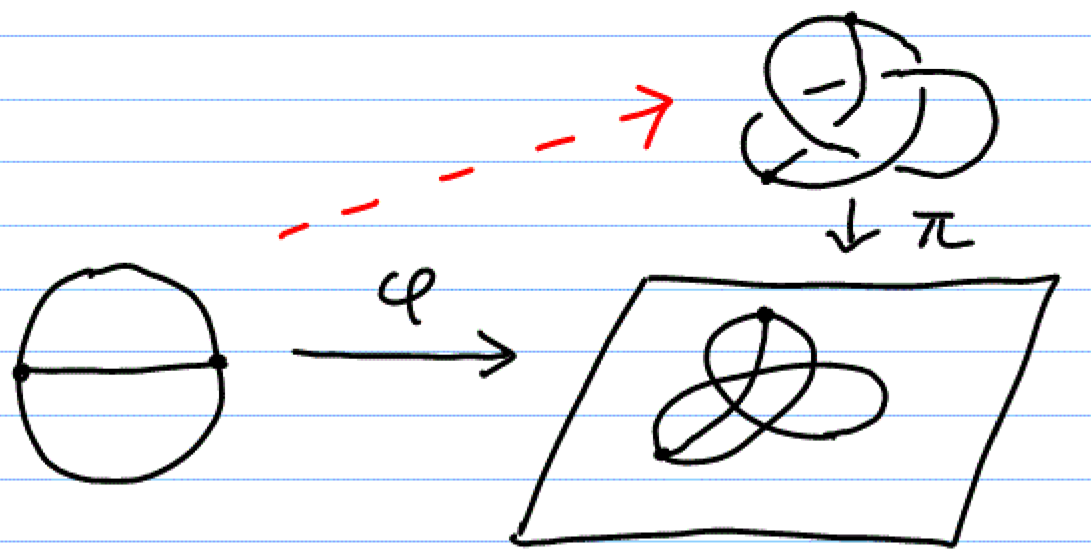
Hanyang Univ, 2010. 08

* G : a finite graph

$\varphi : G \to S^2$, a generic immerison (called a projection of G)

* Selecting over/under passing at each double point,

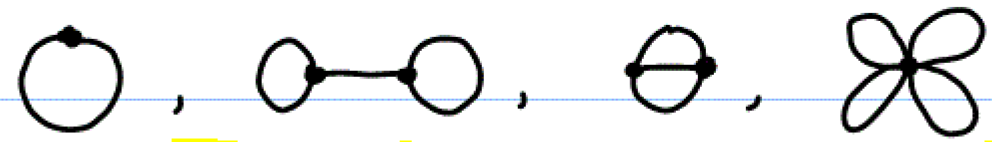
a diagram representing an embedding of G into S^3 is obtained

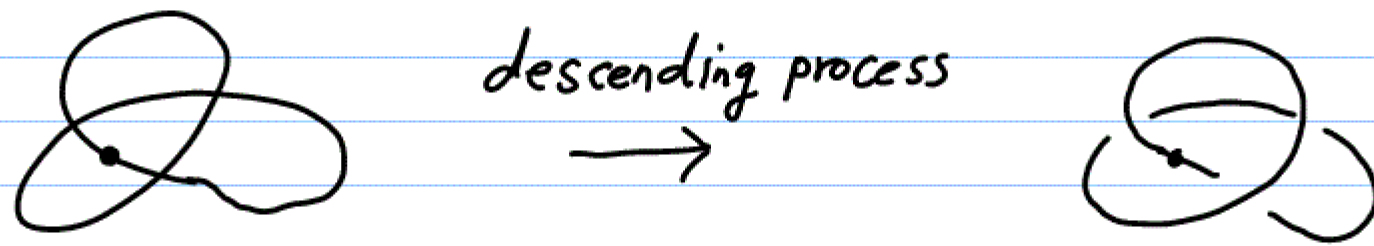


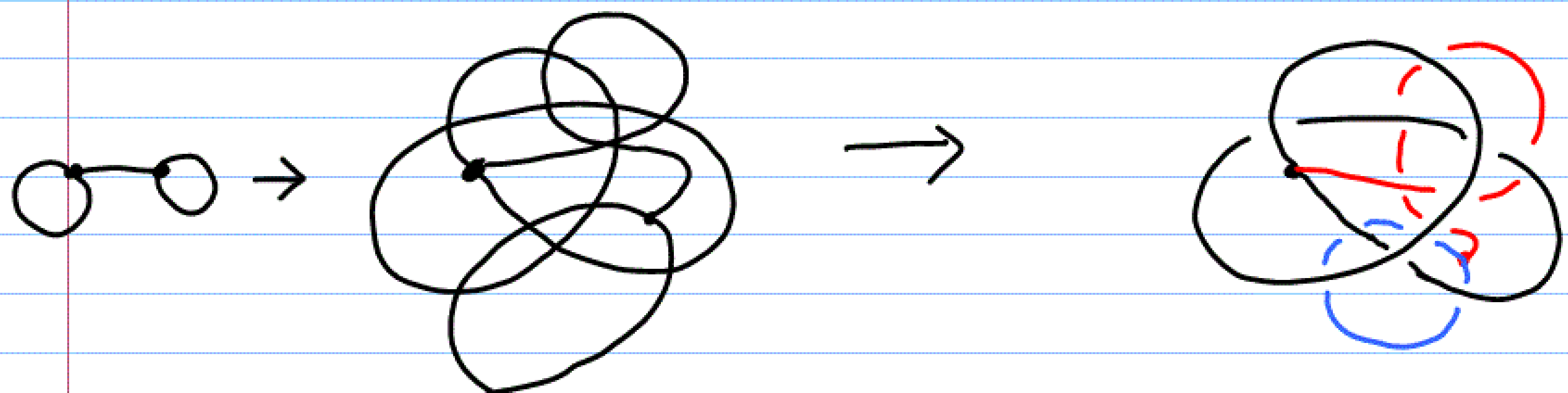
* $SE(\varphi) = \{$ isotopy classes of spatial embedding of G that are obtained from $\varphi \}$

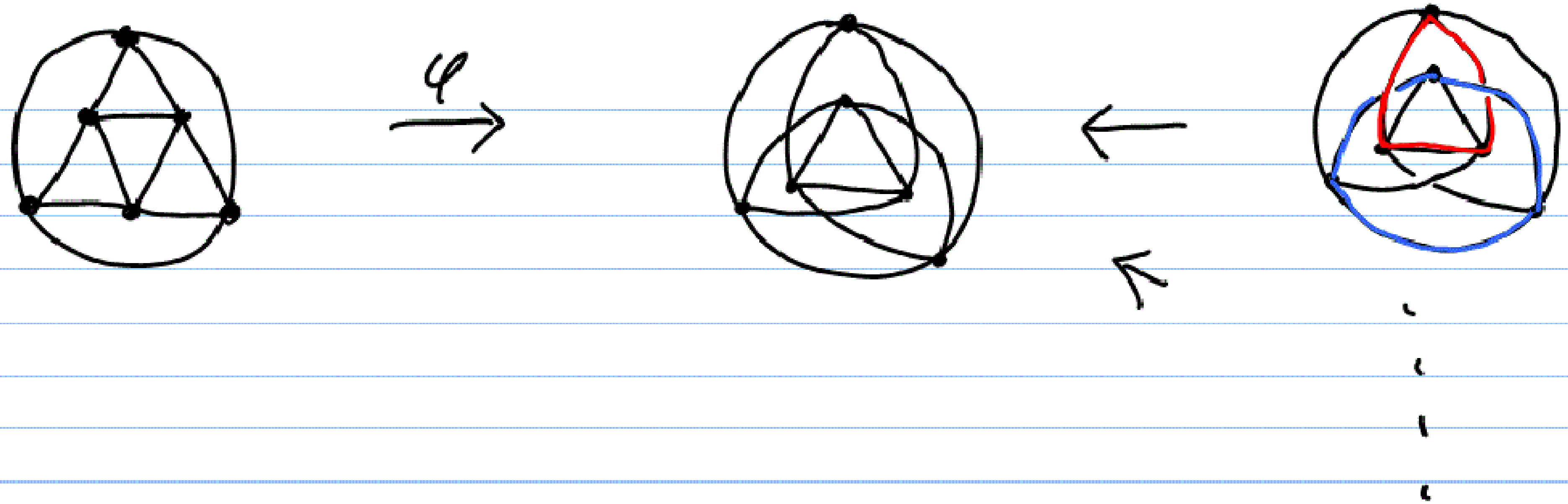
φ is knotted, if trivial embedding $\notin SE(\varphi)$.

* A planar graph is trivializable, if it does not have any knotted projection.

e.g.)  are trivializable.







Each of 8 embeddings from φ contains a Hopf-lk.

$\therefore$ φ is knotted.

* $\mathcal{T} = \{ \text{trivializable graphs} \}$

Note that $\mathcal{T}$ is minor-closed. [$\cup \cap$]

($\because H \underset{\text{minor}}{\triangleleft} G$. H has a knotted proj φ.

$\Rightarrow$ Extending φ to G, we obtain a knotted proj of G)

* By the Wagner conjecture (proved by [R-S]),

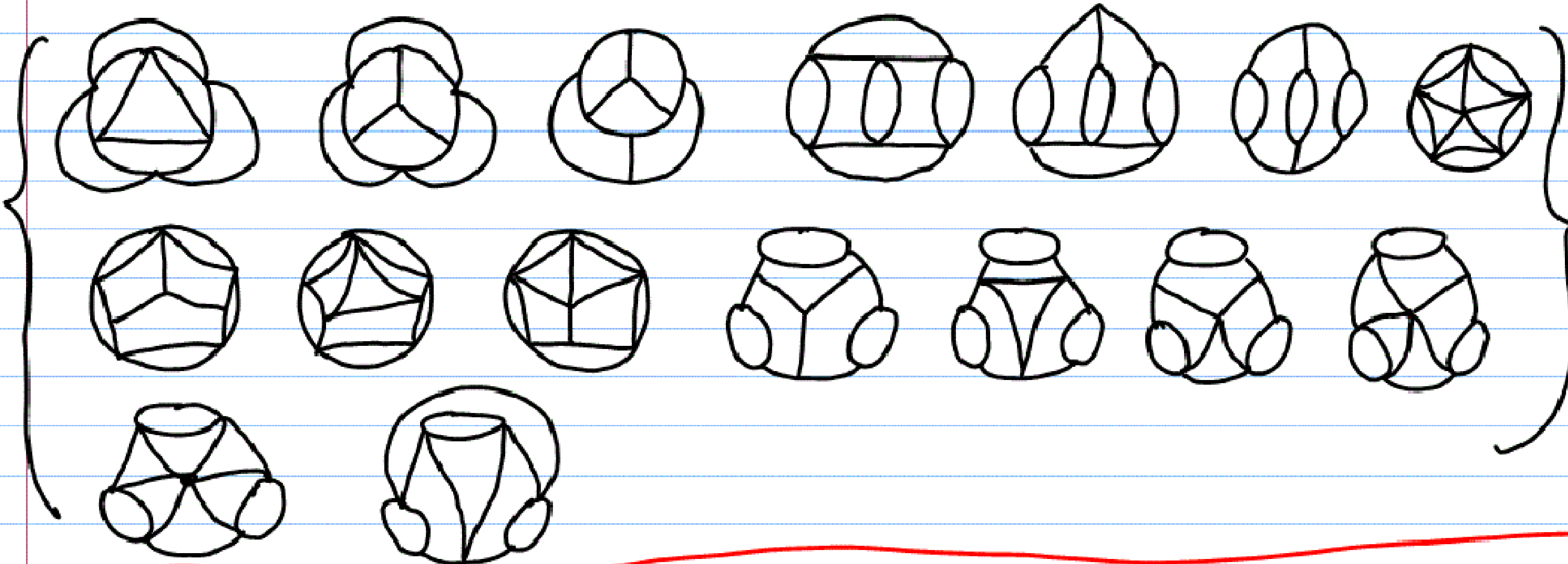
$\Omega(\mathcal{T}) = \{ G \notin \mathcal{T} \mid \text{every minor of } G \in \mathcal{T} \}$ is finite.

* We may challenge to determine $\Omega(\mathcal{T})$.

* In this talk, we are interested in $\Omega(\mathcal{T})$.

* Until now 16 elts of $\Omega(\mathcal{T})$ have been found out.

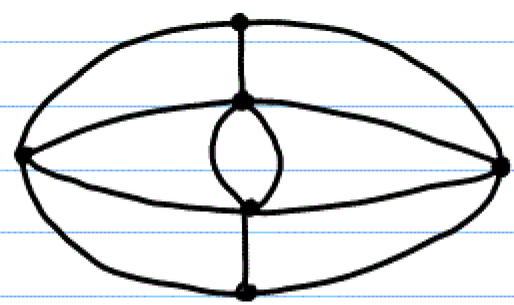
[T, S-S, T, N-O-T-T]



$\subset \Omega(\mathcal{T})$

* Seems to be Not easy to determine $\Omega(\mathcal{J})$.
* Not easy to determine whether a specific graph is trivializable or not.

e.g.)



Is it trivializable or not?

(The answer is not known yet)

Main Results

Thm 1 [許-新国 08]

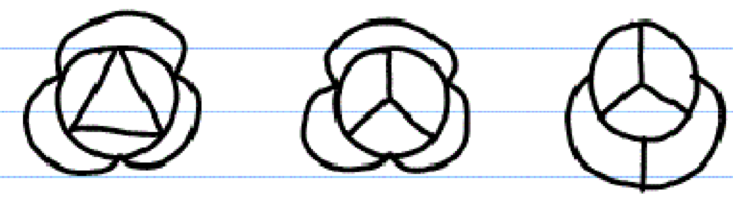
$\varphi : G \to S^2$ a knotted projection.

$\Rightarrow$ # of db pts of $\varphi \geq 3$

Motivated by Thm 1, we have proved :

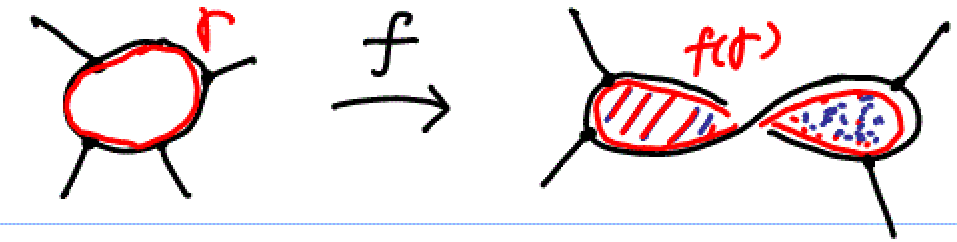
Thm 2 [許-新国 10]

G : an elt of $\Omega(\mathcal{T})$ which has a knotted proj with 3 db pts

$\Rightarrow$ $G \in \{$  $\}$

G_1 G_2 G_3

Pf of Thm 1 - Idea



$\boxed{\text{Key Lemma}}$ [Wu]

$f: G \to S^3$ an embedding of a planar graph G

Then, f is trivial $\Leftrightarrow$ $^\forall$cycle γ of G, $^\exists$a disk D_γ s.t. $\begin{cases} \partial D_\gamma = f(\gamma) \\ \mathrm{Int}\, D_\gamma \cap f(G) = \emptyset \end{cases}$

Let $\varphi: G \to S^2$ be a proj with # db pts < 3.

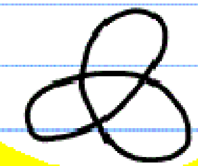
It's enough to show: For any embedding obtained from φ, the triviality condition holds. //

Pf of Thm 2 - idea

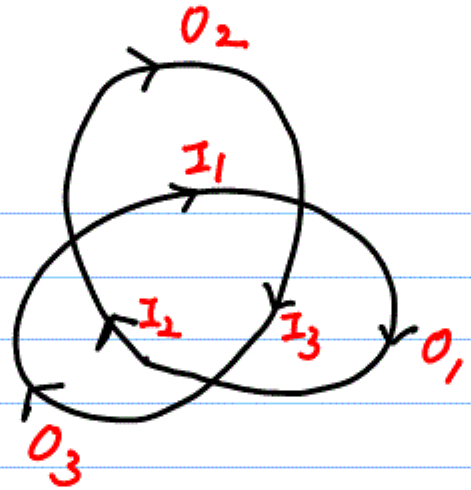
$\varphi : G \to S^2$ a knotted proj with three db pts.

Key Lemma

① Every embedding from φ contains a Hopf-link.

② $\exists$ a cycle γ of G s.t. $\varphi(\gamma) =$ 

(* Lemma ① is necessary for the proof of Lemma ②)

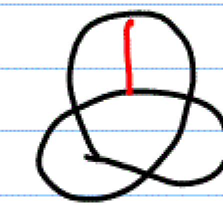
$\varphi(\gamma) =$ 

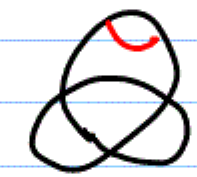
Connector := a simple subarc of $\varphi(G) - \varphi(\gamma)$ which connects different two subarcs among $I_1, I_2, I_3, O_1, O_2, O_3$.

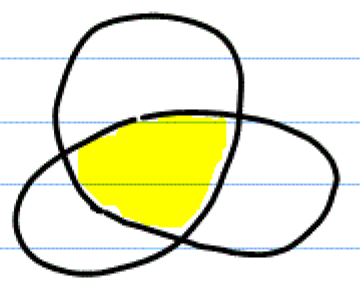
* Types of Connectors : distinguished by endpts pair

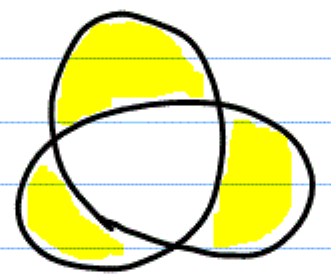
Three types of Regions

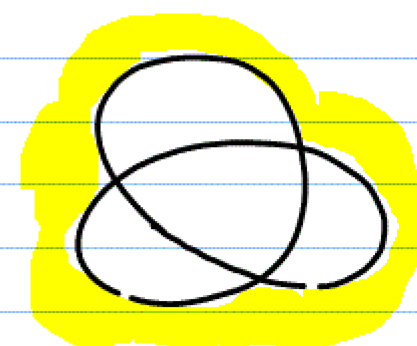
e.g.

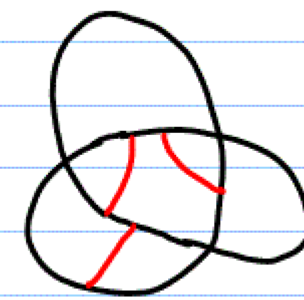

(o)

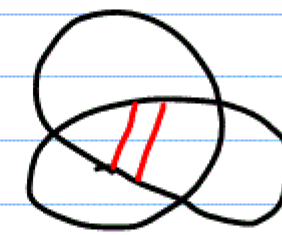

(x)


Inner Region


Intermediate "


Outer "

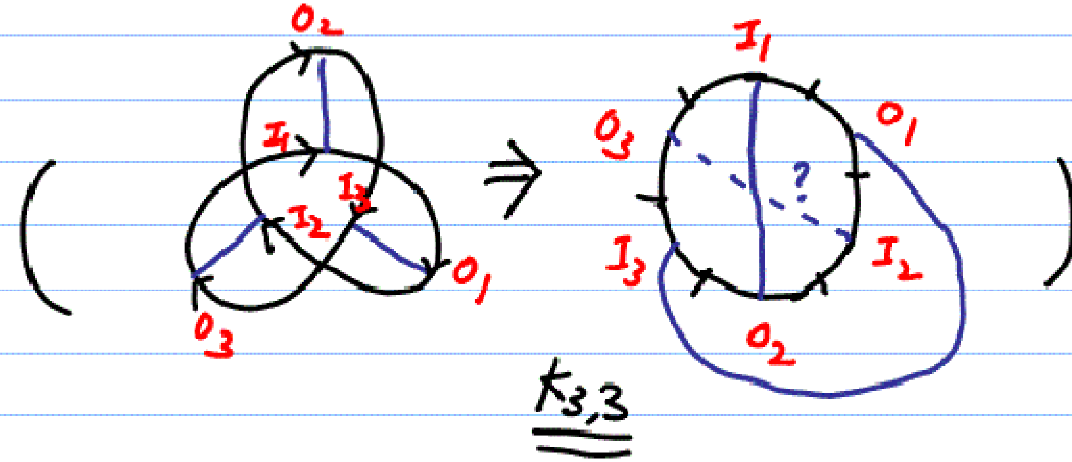

diff types


same type

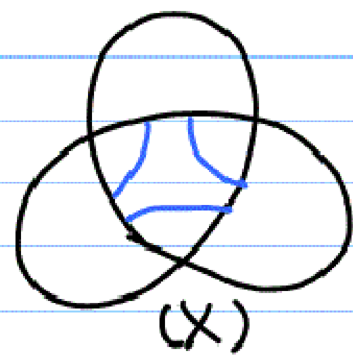
* Planarity of G derives some restrictions on connectors.

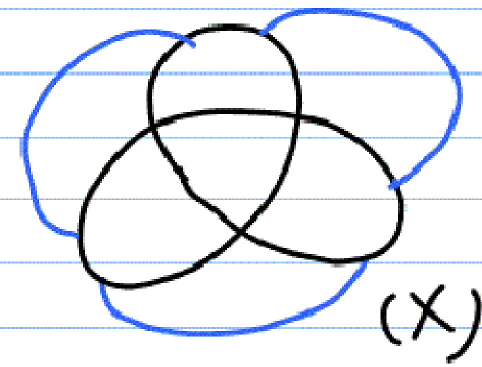
Observation

① $\exists$ at most two interm. regions which contain connectors.



②





③ ....

④ .....

disjoint three connectors of mutually diff types are Not allowed in Inner region.
(also Not in Outer region)

* Case enumeration:

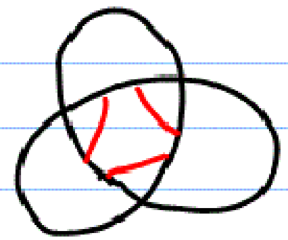
(# of types of conns in Inn Reg , # of " in Outer Reg)

= (3,3), (3,2), (3,1), (3,0), (2,2), (2,1), (2,0), (1,1), (1,0), (0,0).

(* Since $\varphi(G)$ is on S^2, we may assume :

For any (a,b), $a \geq b$)

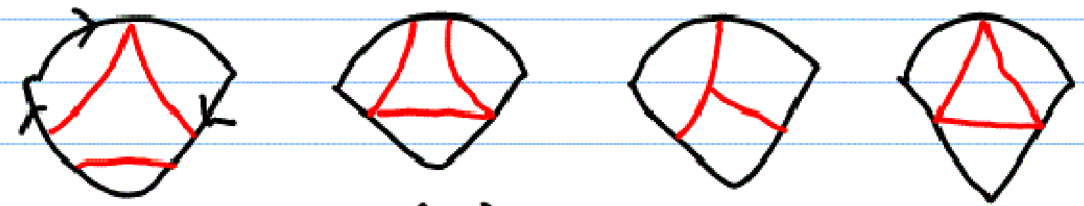
* (3,3) case :

obs② 

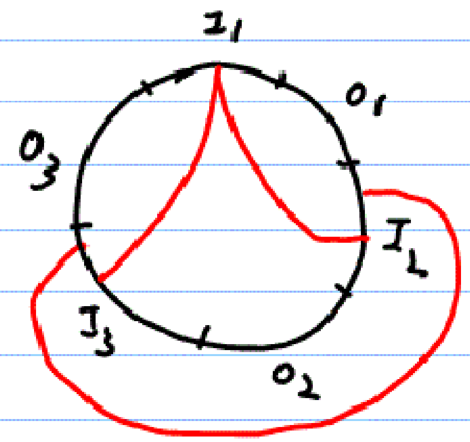
(X)

: Three conns that are mutually disj & (whose types are mutually diff) are Not allowed.

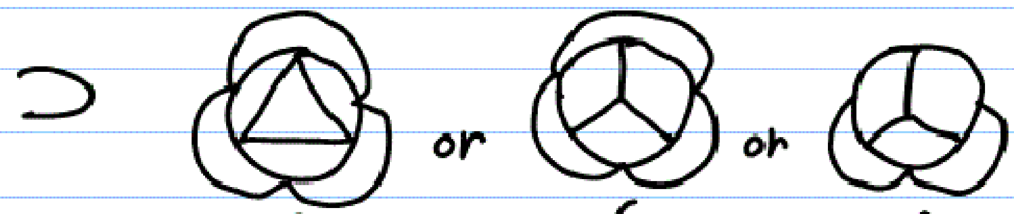
⇒ $\varphi(G)$ should contain



(X) (X)



[Separates O_2 from O_3 & O_1] ⇒⇐ (, 3)

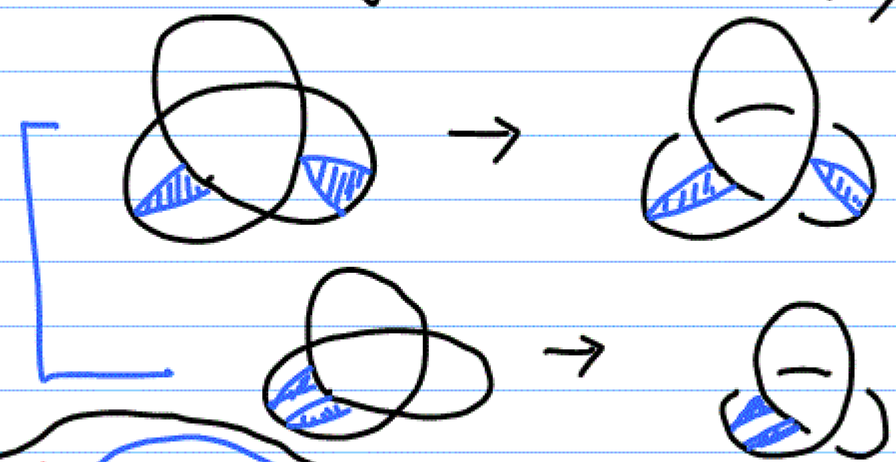
∴ $G \supset$ 

Because G is minimal, it should one of them itself.

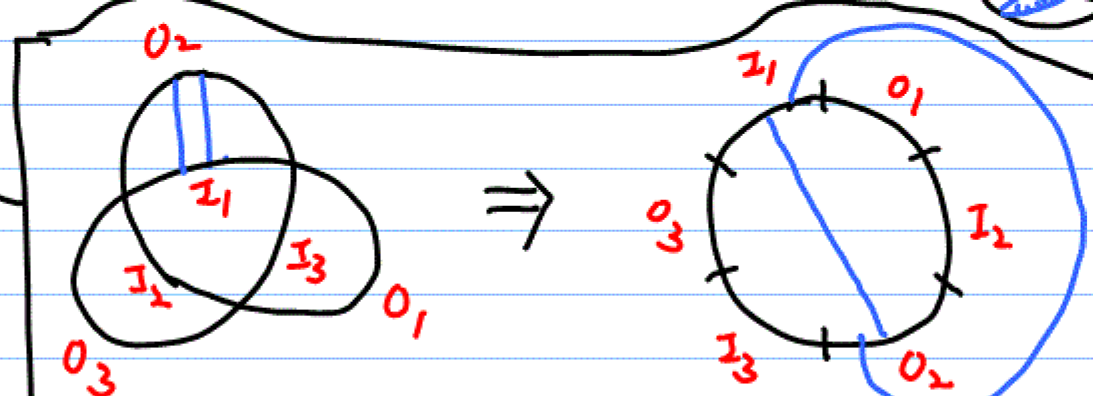
* The other nine cases can **Not** happen!

e.g.) (0,0) : Inner & Outer Regions are empty

obs① $\Longrightarrow$



trivial embedding $\not\Leftrightarrow$!!



the other Int. regions cannot contain any connector.