Counting Links in Complete Graphs

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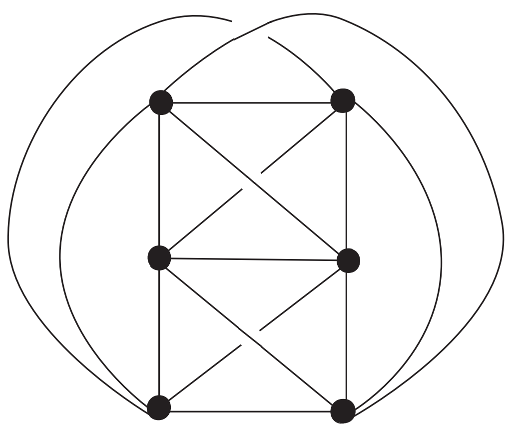
Sachs and Conway-Gordon showed that K_6 is intrinsically linked, and hence every embedding of K_6 contains at least one non-split link.

The embedding below contains exactly one link.



But what about K_7? It contains K_6, so it is clearly intrinsically linked, but how many links must it contain? There are 7 distinct copies of K_6, so it must have at least 7 triangle-triangle links, but is that all?

Many generalizations of intrinsic linking have been studied: n-linking, links with linking number $> n$, links with knotted components, and specified linking patterns (e.g. IC3L) are all ways to measure "how much" a graph is intrinsically linked.

These ideas focus on the kind of links a graph a must contain. For this talk, we will take a different approach and study the minimum number of non-split links (of any kind) any embedding of a graph must contain.

Here $mnl(G)$ is the minimum number of links in any embedding of G.

G	$mnl(G)$		G	$mnl(G)$
$K_{6,1}$	0		$K_{7,1}$	0
$K_{5,2}$	0		$K_{6,2}$	0
$K_{4,3}$	0		$K_{5,3}$	0
$K_{5,1,1}$	0		$K_{4,4}$	2
$K_{4,2,1}$	0		$K_{6,1,1}$	0
$K_{3,3,1}$	1		$K_{5,2,1}$	0
$K_{3,2,2}$	0		$K_{4,3,1}$	6
$K_{4,1,1,1}$	0		$K_{4,2,2}$	2
$K_{3,2,1,1}$	1		$K_{3,3,2}$	17
$K_{2,2,2,1}$	0		$K_{5,1,1,1}$	0
$K_{3,1,1,1,1}$	3		$K_{4,2,1,1}$	6
$K_{2,2,1,1,1}$	1		$K_{3,3,1,1}$	25
$K_{2,1,1,1,1,1}$	9		$K_{3,2,2,1}$	28
K_7	21		$K_{2,2,2,2}$	3
			$K_{4,1,1,1,1}$	12
			$K_{3,2,1,1,1}$	$34 \leq mnl(G) \leq 43$
			$K_{2,2,2,1,1}$	$30 \leq mnl(G) \leq 42$
			$K_{3,1,1,1,1,1}$	$53 \leq mnl(G) \leq 82$
			$K_{2,2,1,1,1,1}$	$54 \leq mnl(G) \leq 94$
			$K_{2,1,1,1,1,1,1}$	$111 \leq mnl(G) \leq 172$
			K_8	$217 \leq mnl(G) \leq 305$

Upper bounds for the larger cases are found by computer counts on specified embeddings.

Using Known Subgraphs

Rather than give full proofs of the results, we will instead look at the key techniques for the proofs.

The first approach is to examine subgraphs that are already understood.

For example, each distinct copy of K_6 gives a unique triangle-triangle link, and each $K_{4,4}$ yields two square-square links.

$K_{3,3,1}$

$K_{3,3,1}$ is a useful ally, as it contains a triangle square link, and the triangle must use the preferred vertex.

Partition the vertices of K_7 as (1)(246)(357). There is a triangle-square link where the triangle uses vertex 1 and some other vertices, say 2 and 3. Then by partitioning (7)(124)(356) we may find a new triangle-square link. Continuing this way, we can find 7 unique triangle-square links in K_7.

The $C \cup G$ Approach: Homology to the Rescue

Once we know that a graph contains a link of a certain type, we can often use that to find more links.

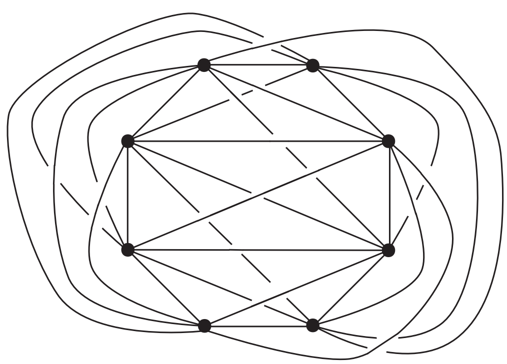
For example, if a cycle C has non-zero linking number with an other cycle that is contained in a K_4 disjoint from C, then C must link at least 4 cycles in K_4.

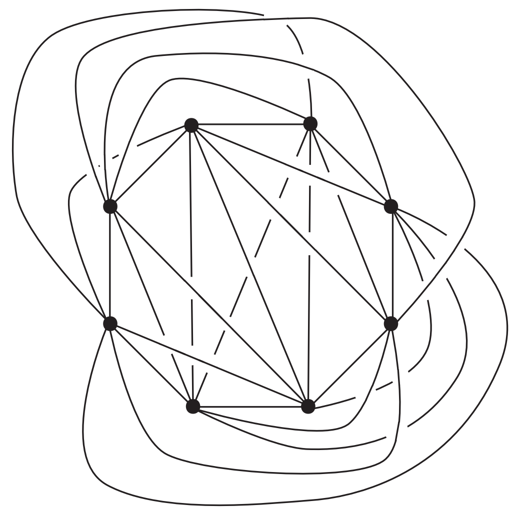
Further if C links a square in K_4 with odd linking number, then it links 2 squares in K_4.

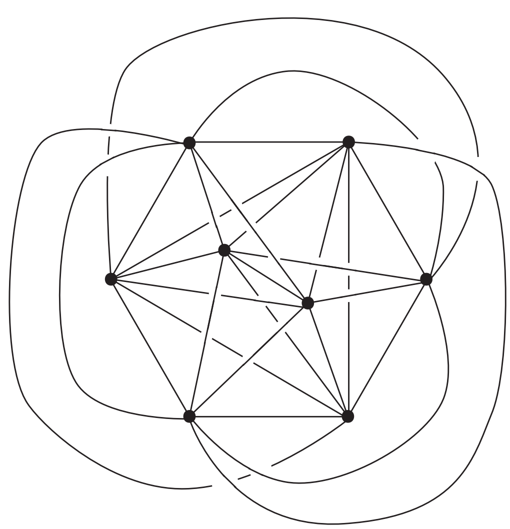
A similar argument maybe applied to other subgraphs. For example, if a cycle C links a cycle contained in $K_{2,1,1,1}$ then C must link at least 8 cycles from $K_{2,1,1,1}$.

The Curious Case of K_8

Here are three embeddings of K_8.







The Curious Case Continued

The first two emeddings have the same number of links: 305. However, they have a different distribution of links.

Embedding one has 28 triangle-triangle, 112 triangle-square, 109 triangle-pentagon, and 56 square-square.

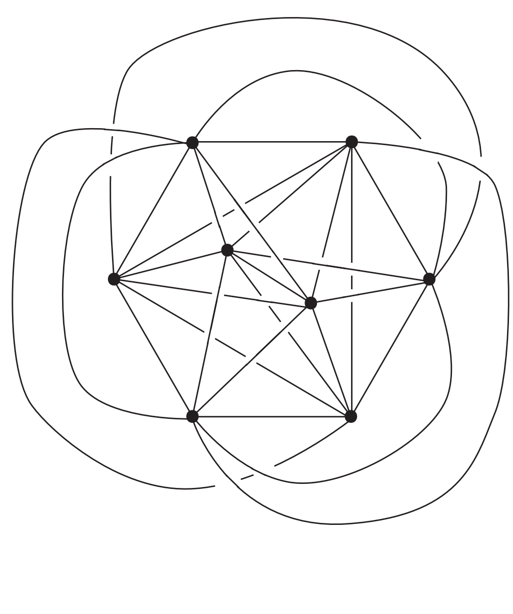
Embedding two has 28 triangle-triangle, 112 triangle-square, 112 triangle-pentagon, and 53 square-square.

Thus, there appear to be multiple ways to attain the minimum number of links in an embedding.

Curiouser and Curiouser

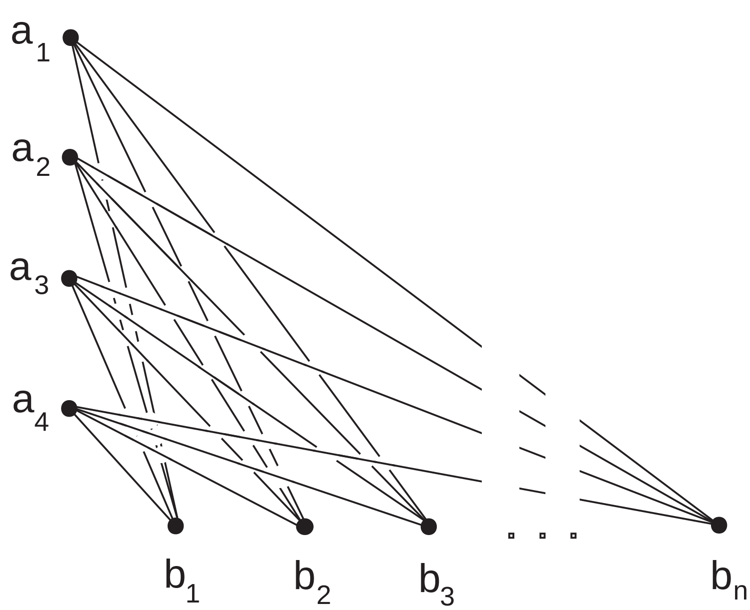
Both of the minimal embeddings on the previous slide contain a link of linking number 2. It is possible to embed K_8 so that it contains only Hopf links, but this seems to increase the number of links.

The embedding below contains 330 Hopf links (and no others).



$K_{4,n}$

Theorem. *The minimum number of links in an embedding of* $K_{4,n}$ *is* $2\binom{n}{4}$.



Note that this is a book embedding.

Questions

Do minimal book embeddings always realize the minimal number of links? (It is true for K_7, and it seems to be true for K_8)

K_8 shows it is possible to realize the minimum number of certain types of links in non-minimal embeddings. Can this apparent trade off between the number of links and the complexity of individual links be made precise?

What about the number of k-cycles linked with m-cycles? Is the minimum number of such links realized in an embedding with a minimum number of total links?